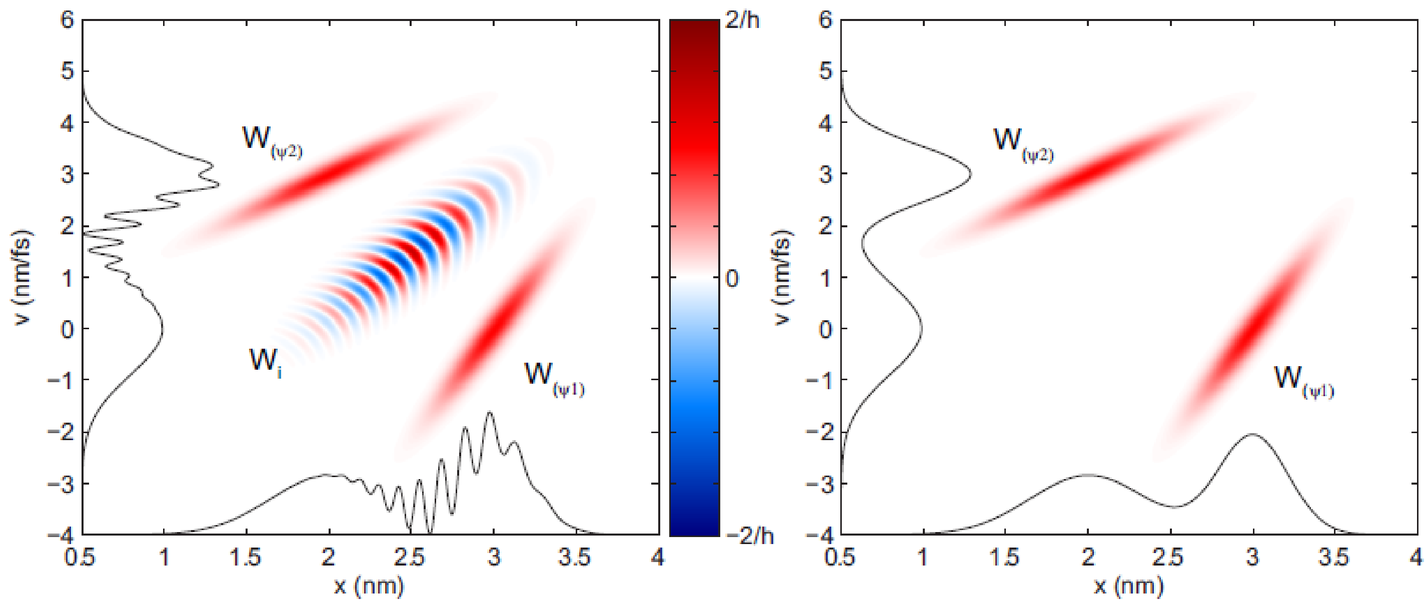
CLASSE

Synchrotron Radiation Representation in Phase Space

Ivan Bazarov and Andrew Gasbarro

Cornell University



phase space of coherent (left) and incoherent (right) 2-state superposition

Motivation for this work

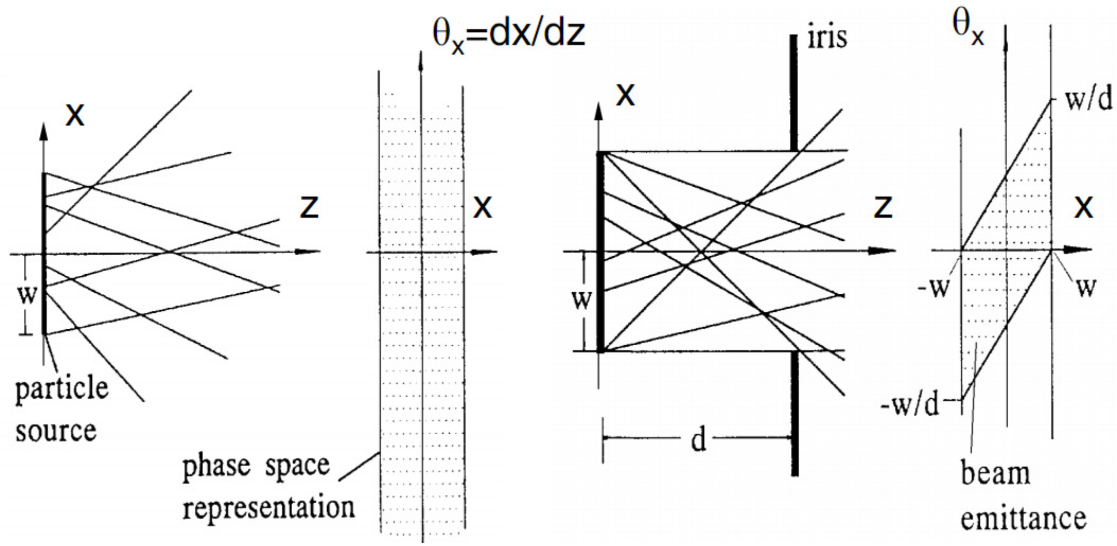
CLASSE

- **Do we understand physics of x-ray brightness?**
- **Is diffraction limit same as full transverse coherence?**
- **How to account for non-Gaussian beams (both e^- and γ)?**

This talk is a brief introduction, refer to **PRSTAB 15 (2012) 050703**

Brightness: geometric optics

- Rays moving in drifts and focusing elements



- Brightness = particle density in phase space (2D, 4D, or 6D)

Phase space in classical mechanics

- Classical: particle state (x, p)
- Evolves in time according to $\dot{p} = -\frac{\partial \mathcal{H}}{\partial x}$, $\dot{x} = \frac{\partial \mathcal{H}}{\partial p}$
- E.g. drift:

  $$\mathcal{H} = \frac{p^2}{2m}$$

  $$\dot{p} = 0, \quad \dot{x} = \frac{p}{m}$$

  linear restoring force:

  $$\mathcal{H} = \frac{p^2}{2m} + \frac{kx^2}{2}$$

  $$\dot{p} = -kx, \quad \dot{x} = \frac{p}{m}$$

- Liouville's theorem: phase space density stays const along particle trajectories

Phase space in quantum physics

- Quantum state:

Position space		momentum space
$\psi(x)$	or	$\phi(p)$
Position space	$\leftarrow \mathcal{FT} \rightarrow$	momentum space

- If either $\psi(x)$ or $\phi(p)$ is known – can compute anything. Can evolve state using time evolution operator: $\exp(-\frac{i\mathcal{H}t}{\hbar})$
- $|\psi(x)|^2 dx$ - probability to measure a particle with $(x, x+dx)$
- $|\phi(p)|^2 dp$ - probability to measure a particle with $(p, p+dp)$

Wigner distribution

$$W(x,p) \equiv \int_{-\infty}^{+\infty} \langle \psi | x + \tfrac{x'}{2} \rangle \langle x + \tfrac{x'}{2} | p \rangle \langle p | x - \tfrac{x'}{2} \rangle \langle x - \tfrac{x'}{2} | \psi \rangle \, dx'$$

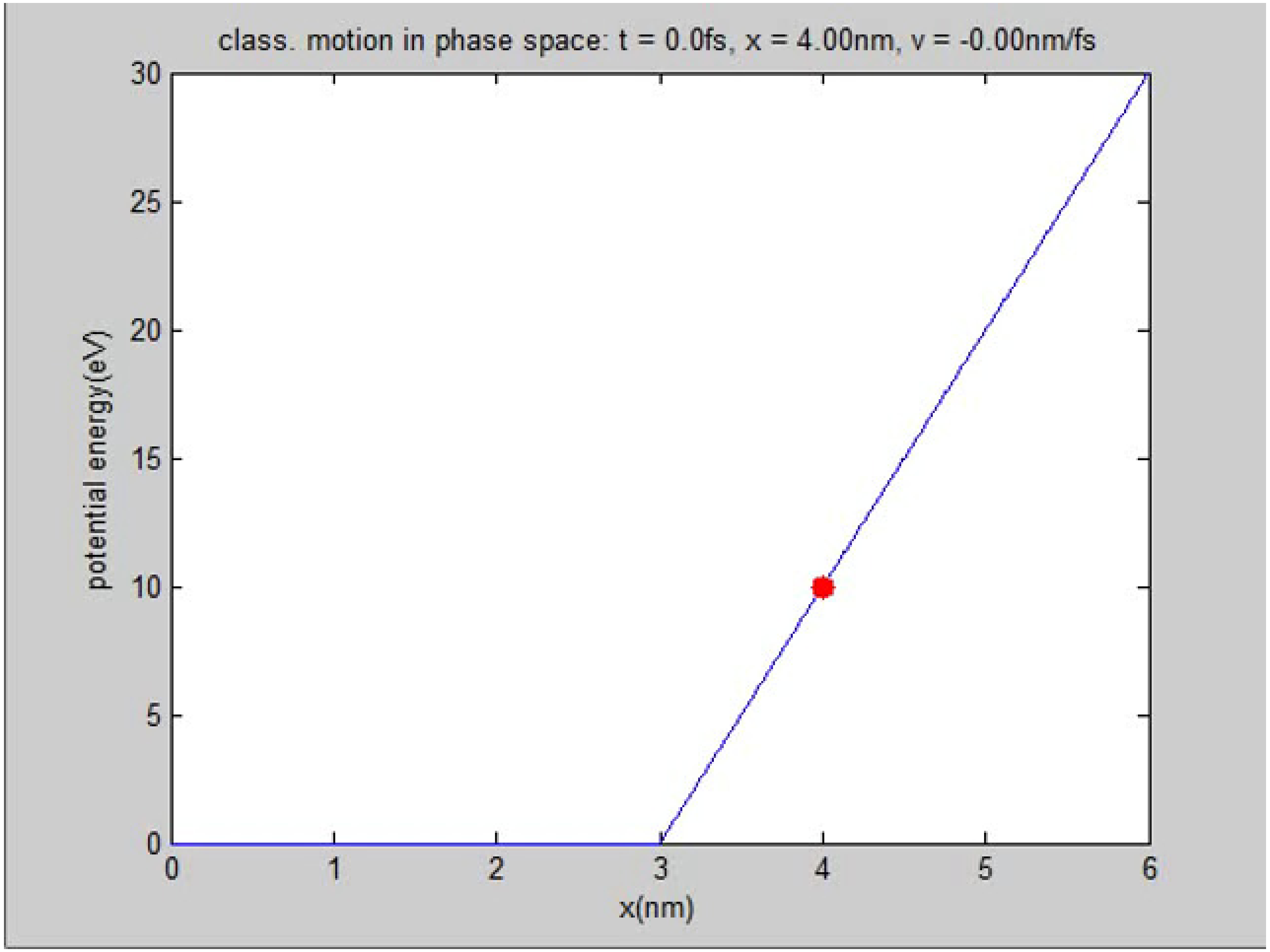
$$= \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} \psi^*(x + \frac{x'}{2}) \psi(x - \frac{x'}{2}) e^{ipx'/\hbar} dx'$$

$$= \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} \phi^*(p + \frac{p'}{2}) \phi(p - \frac{p'}{2}) e^{-ip'x/\hbar} dp'$$

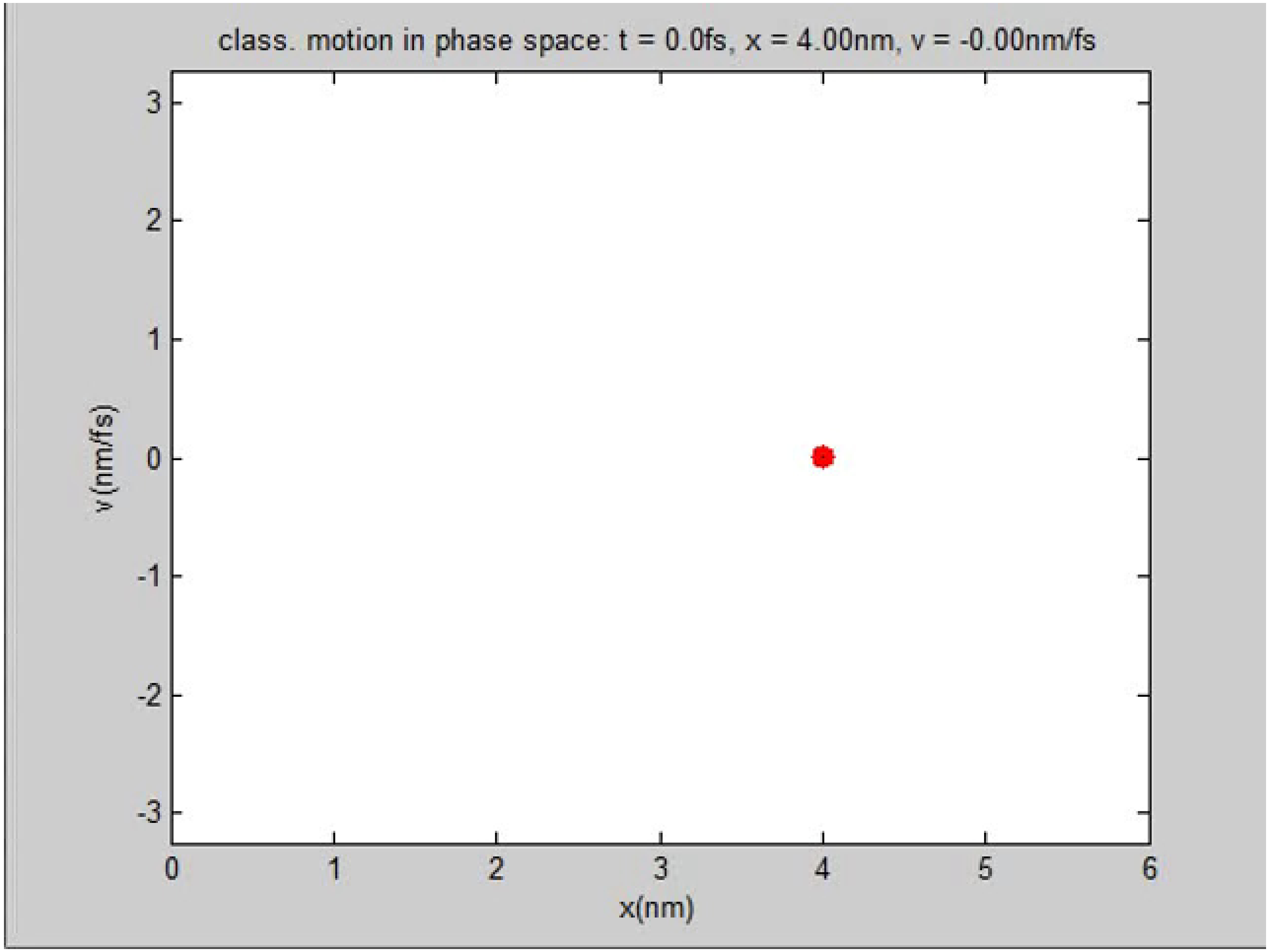
- $W(x,p)dxdp$ – (quasi)probability of measuring quantum particle with $(x, x+dx)$ and $(p, p+dp)$

Classical electron motion in potential

CLASSE

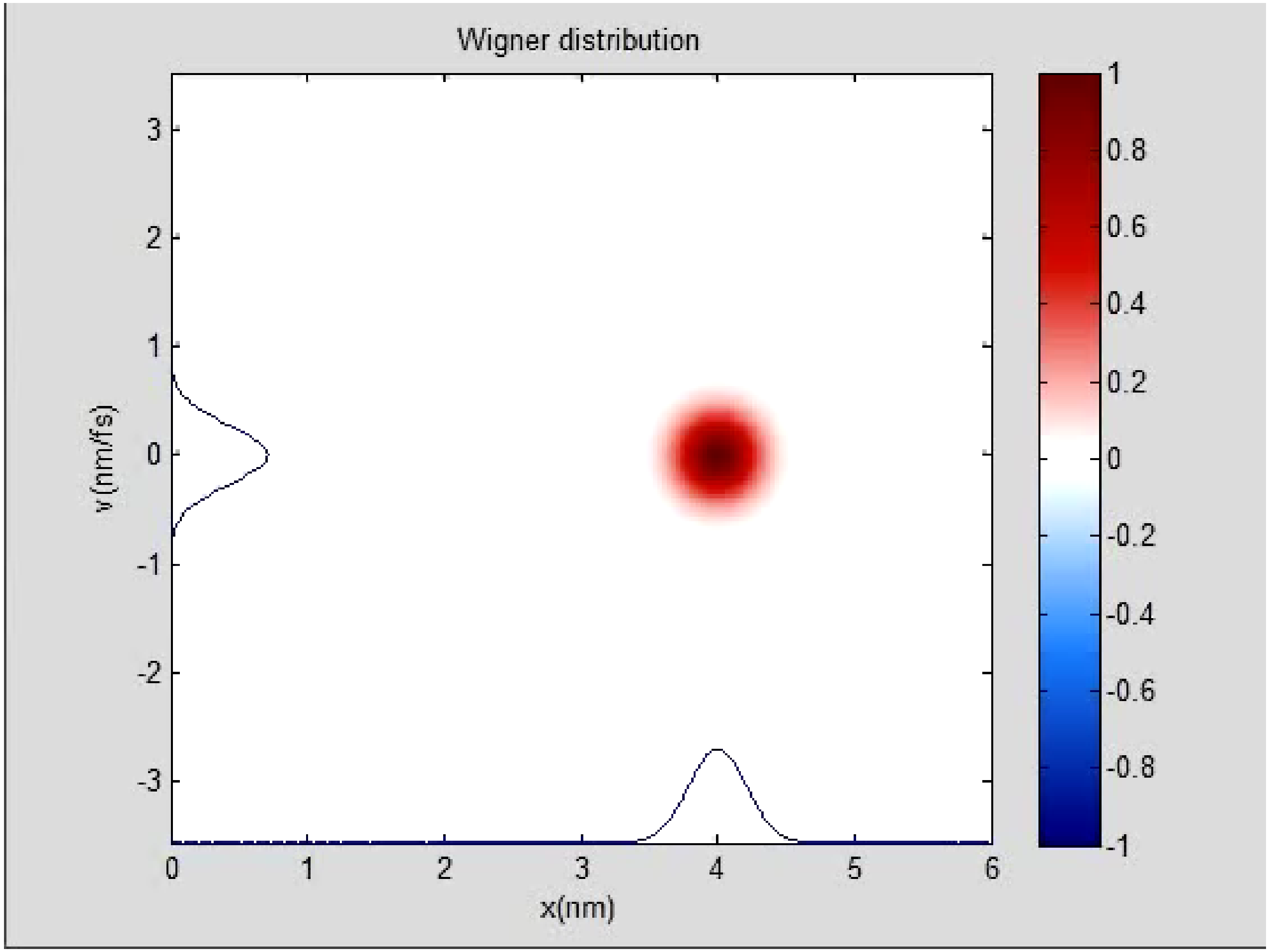


Same in phase space…



Going quantum in phase space…

CLASSE



+2/h

–2/h

Some basic WDF properties

- $W(x,p) \in \mathbb{R}$ (can be negative)
- $\iint W(x,p)\,dx\,dp = 1$
- $\int W(x,p)\,dp = |\psi(x)|^2$
- $\int W(x,p)\,dx = |\phi(p)|^2$
- Time evolution of $W(x,p)$ is *classical* in absence of forces or with linear forces

Connection to light

- **Quantum** – $\psi(x)$
- Linearly polarized **light** (1D) – $E(x)$
- Measurable $|\psi(x)|^2$ – **charge density**
- Measurable $|E(x)|^2$ – photon **flux density**
- **Quantum**: momentum representation $\phi(p)$ is FT of $\psi(x)$
- **Light**: far field (angle) representation $\mathcal{E}(\theta)$ is FT of $E(x)$

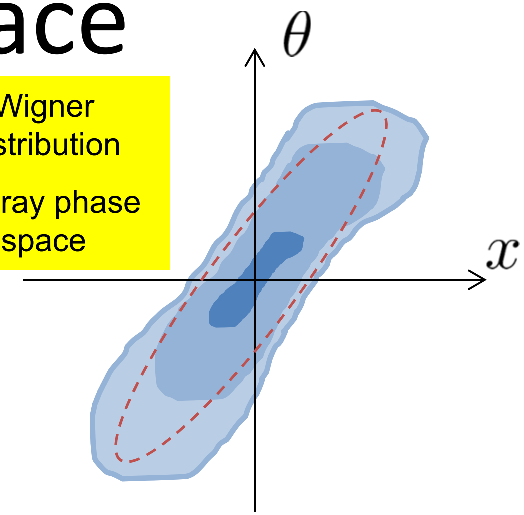
Connection to classical picture

- **Quantum**: $h \to 0$, recover classical behavior
- **Light**: $\lambda \to 0$, recover geometric optics
- $W(x,p)$ or $W(x,\theta)$ – phase space density (=brightness) of a quantum particle or light
- Wigner of a quantum state / light propagates classically in absence of forces or for linear forces
- **Wigner density function = brightness**

Extension of accelerator jargon to x-ray (wave) phase space

Wigner distribution

= x-ray phase space



- Σ -matrix

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle x\theta \rangle \\ \langle \theta x \rangle & \langle \theta^2 \rangle \end{pmatrix}$$

- Twiss (equivalent ellipse) and emittance

$$\Sigma = \epsilon \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \epsilon \mathbf{T}$$

with $\det(\mathbf{T}) = 1$ and $\epsilon = \det(\Sigma)$ or

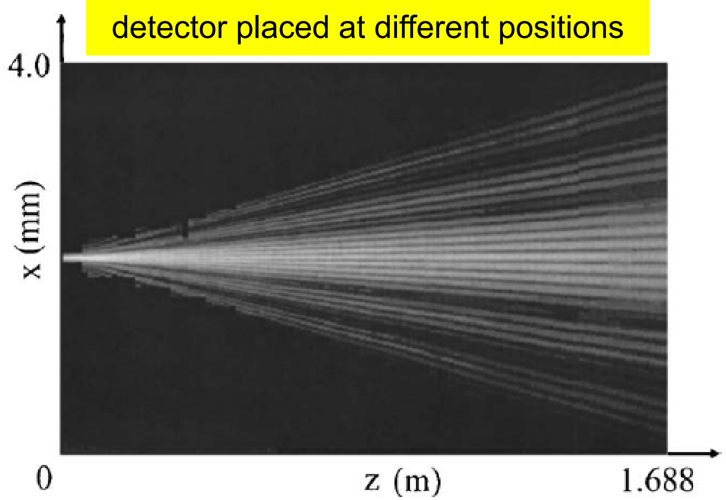
$$\epsilon = \sqrt{\langle x^2 \rangle \langle \theta^2 \rangle - \langle x\theta \rangle^2}$$

***K.J. Kim, NIMA 246* (1986) 71**

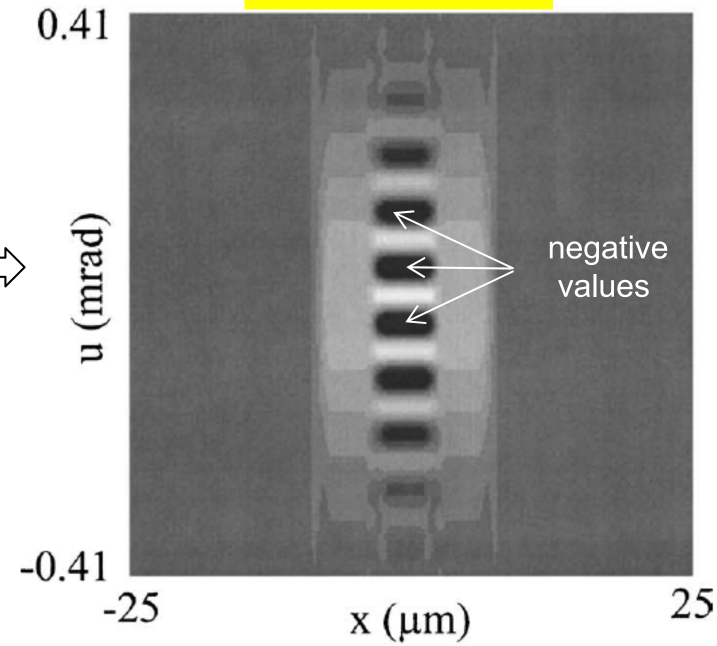
X-ray phase space can be measured using tomography

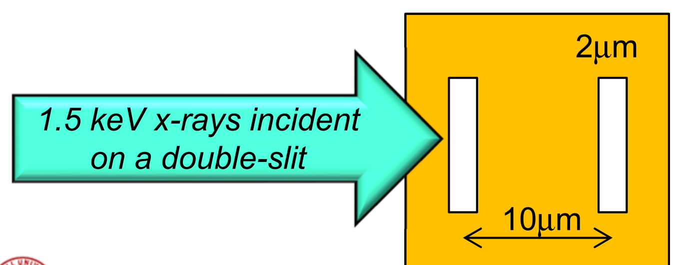
CLASSE

- Same approach as phase space tomography in accelerators
- Except the phase space is now allowed to be locally negative



Tomography





C.Q. Tran et al., JOSA A 22 (2005) 1691

Diffraction limit vs. coherence

CLASSE

- Diffraction limit (same as uncertainty principle)

$$\sigma_p \sigma_x \geq \hbar/2 = h/4\pi \ \text{(QM)} \qquad \sigma_\theta \sigma_x \geq \lambda/4\pi \ \text{(light)}$$

$$M^2 = \frac{\epsilon_{\text{light}}}{\lambda/4\pi}$$

$M^2 \geq 1$ (ability to focus to a small spot)

  – a classical counterpart exists (= e-beam emittance)

- Coherence (ability to form interference fringes)
  – Related to visibility or *spectral degree of coherence*

$$\mu_{12}(\omega) = \frac{\langle E_1(\omega) E_2^*(\omega) \rangle}{|E_1(\omega)|\,|E_2(\omega)|}$$

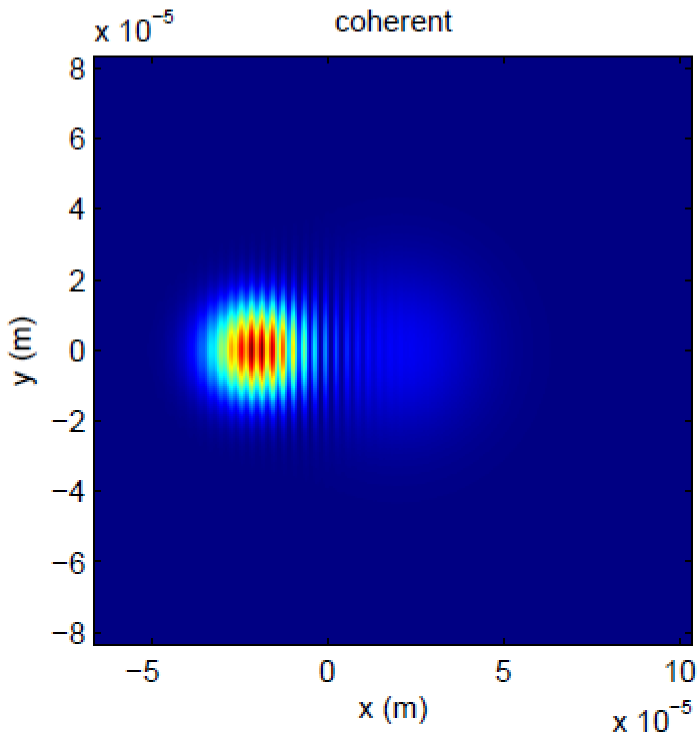
$0 \leq |\mu_{12}| \leq 1$

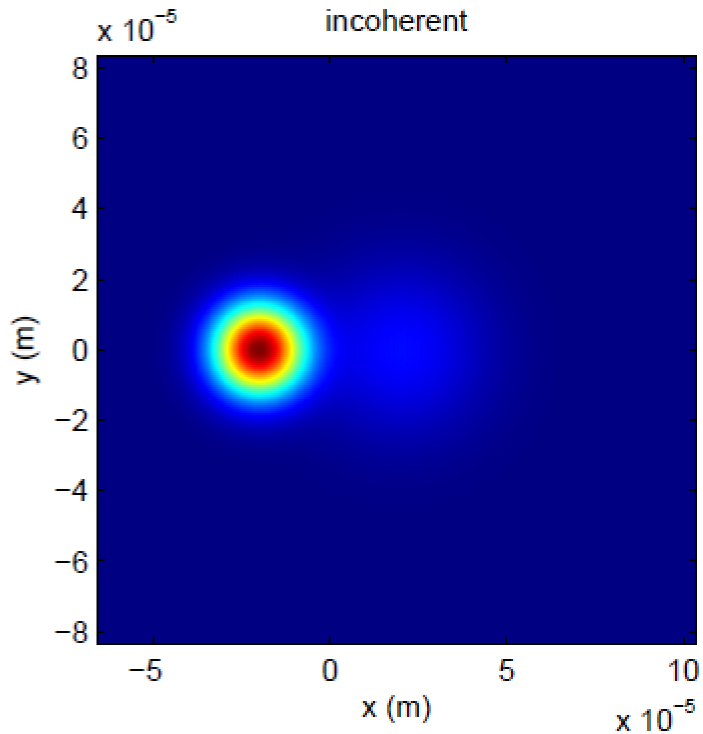
  – quantum/wave in its nature – no classical counterpart exists

- Wigner distribution contains info about both!

Example of combining sources (coherent vs incoherent)

two laser Gaussian beams

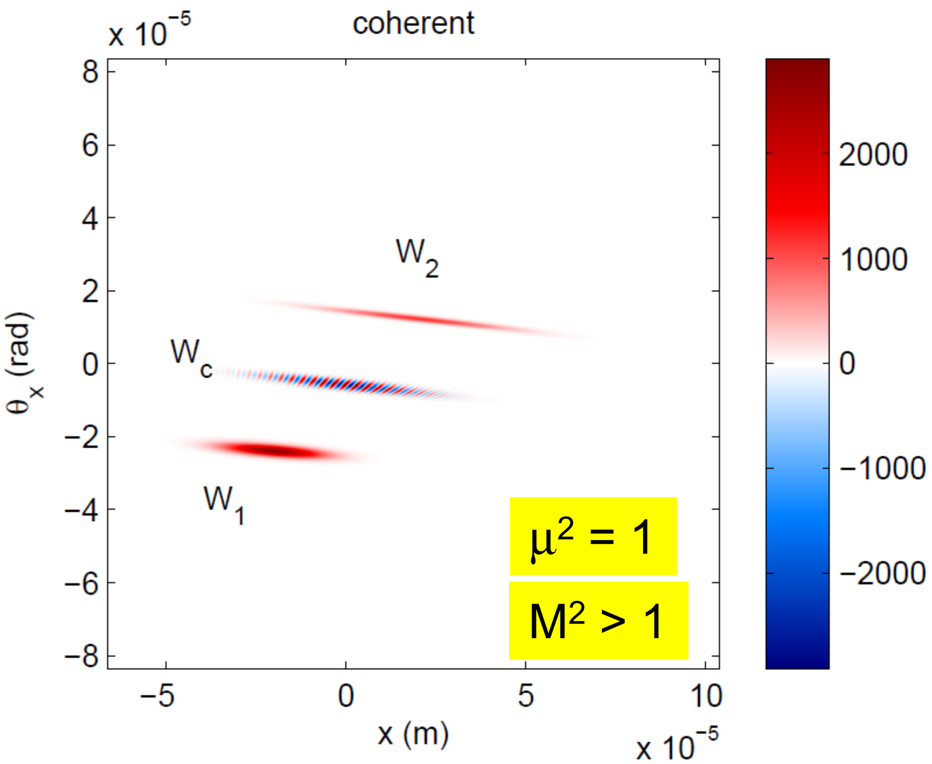


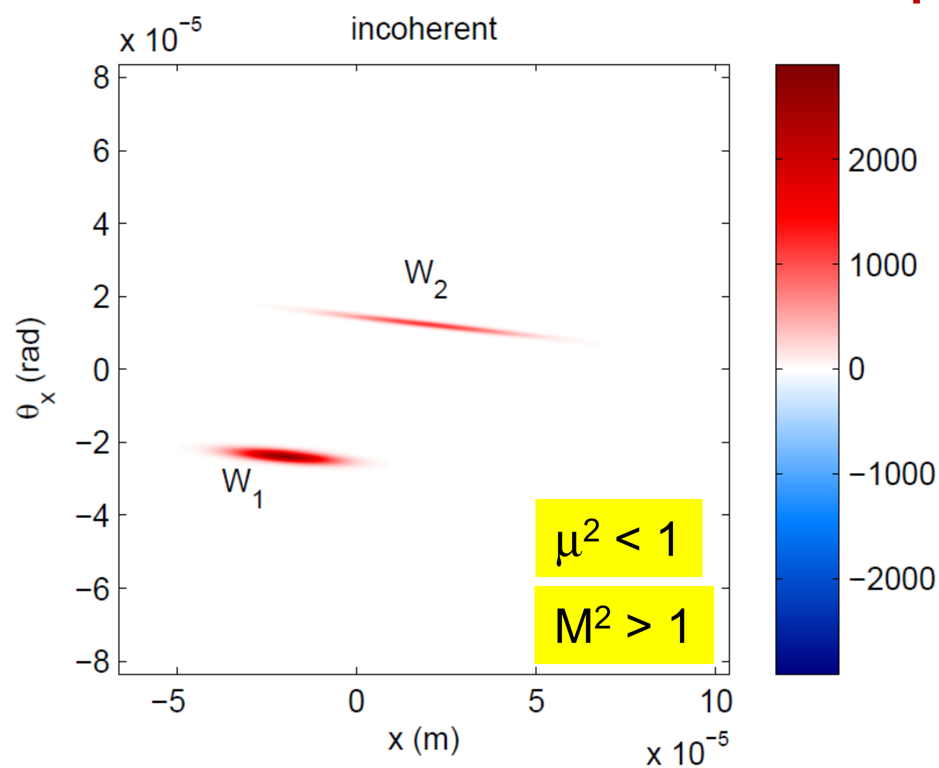


Same picture in the phase space

CLASSE

two laser Gaussian beams





Facts of life

CLASSE

- **Undulator radiation (single electron) is fully coherent ($\mu^2 = 1$)**

$$\mu^2 \equiv \lambda^2 \frac{\int W^2 d^2r d^2\theta}{(\int W d^2r d^2\theta)^2}$$

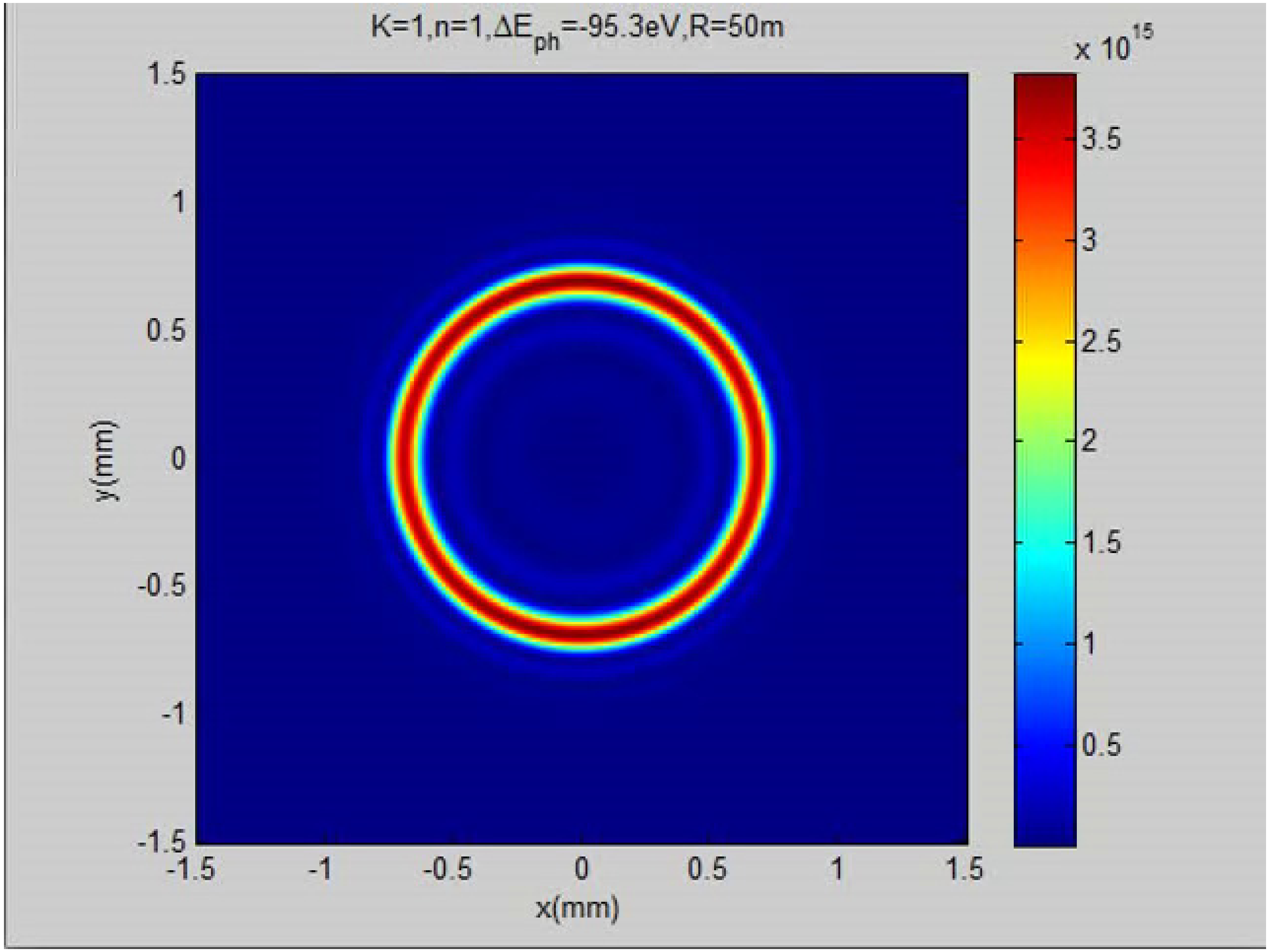
- **But is not diffraction limited $M^2 > 1$**
- **X-ray phase space of undulator's central cone is not Gaussian**
- **Old (Gaussian) metrics are not suitable for (almost) fully coherent sources**

But the undulator radiation in central cone is Gaussian… or is it?

CLASSE

animation: scanning around 1st harm. ~6keV (zero emittance)

Spectral flux (ph/s/0.1%BW/mm^2) at 50m from undulator (5GeV, 100mA, λ_p = 2cm)

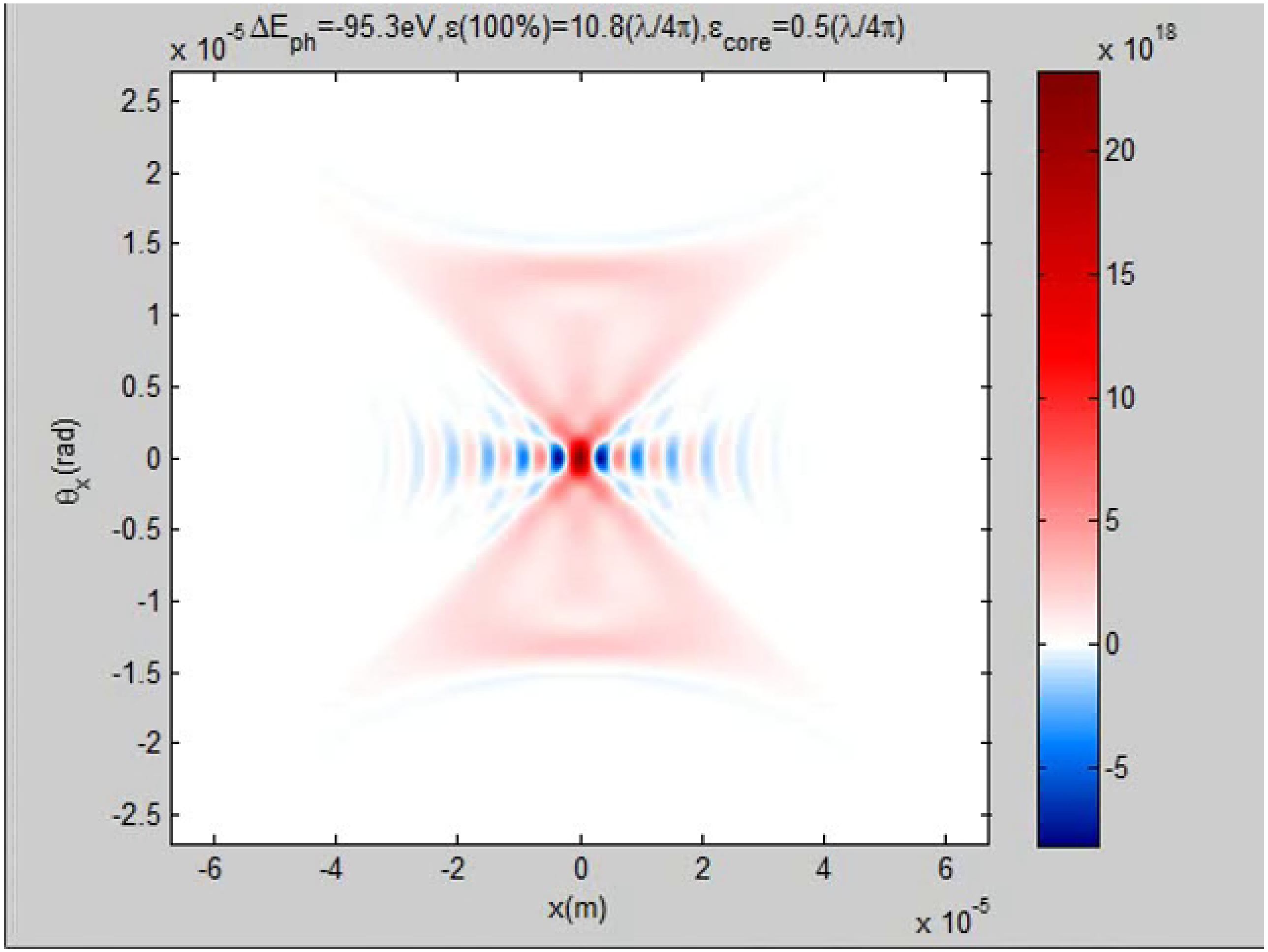


Light in phase space

CLASSE

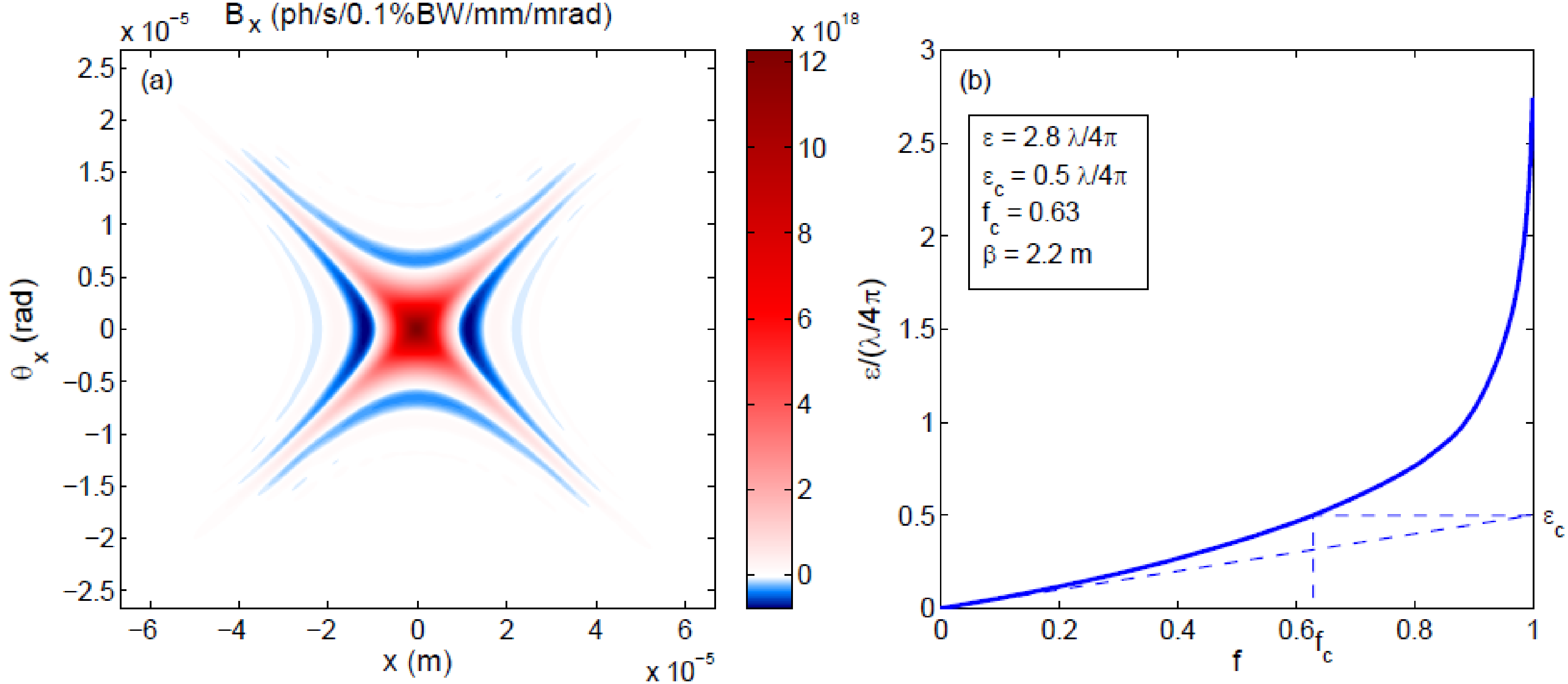
animation: scanning around 1st harm. ~6keV (zero emittance)

Phase space near middle of the undulator (5GeV, 100mA, λ_p = 2cm)



Emittance vs. fraction for light:

100% rms emittance, core emittance, and core fraction



- **Change clipping ellipse area from ∞ to 0, record emittance vs. beam fraction contained**
- **Smallest M^2 ~ 3 of x-ray undulator cone (single electron)**
- **Core emittance of any coherent symmetric mode is always λ/8π**

Example of accounting for realistic spreads in the electron beam (Cornell ERL)

CLASSE

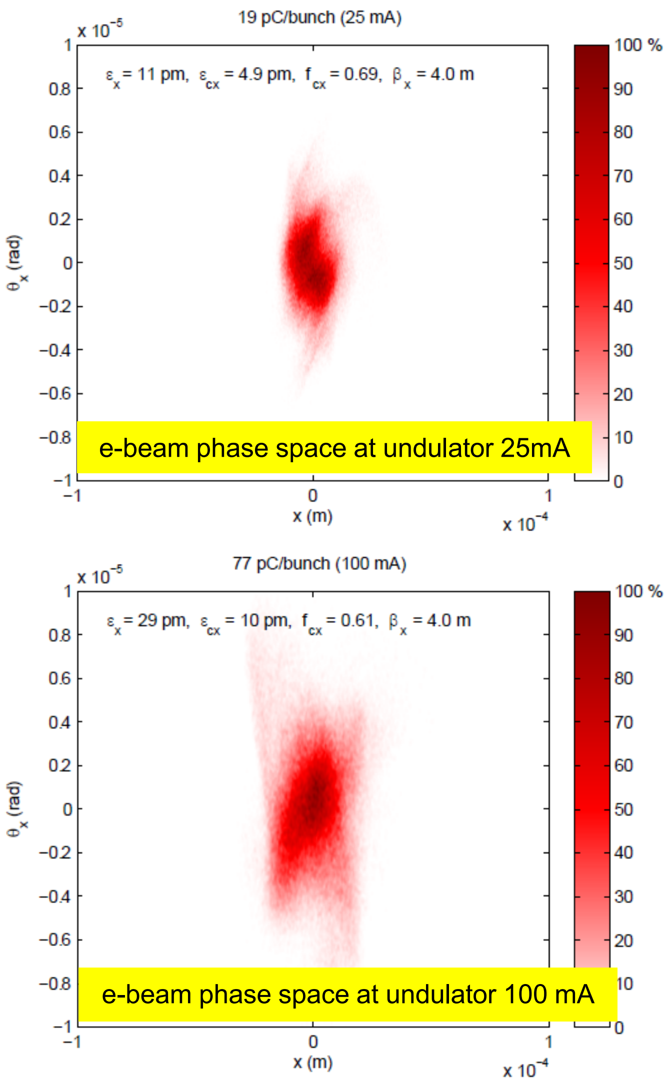
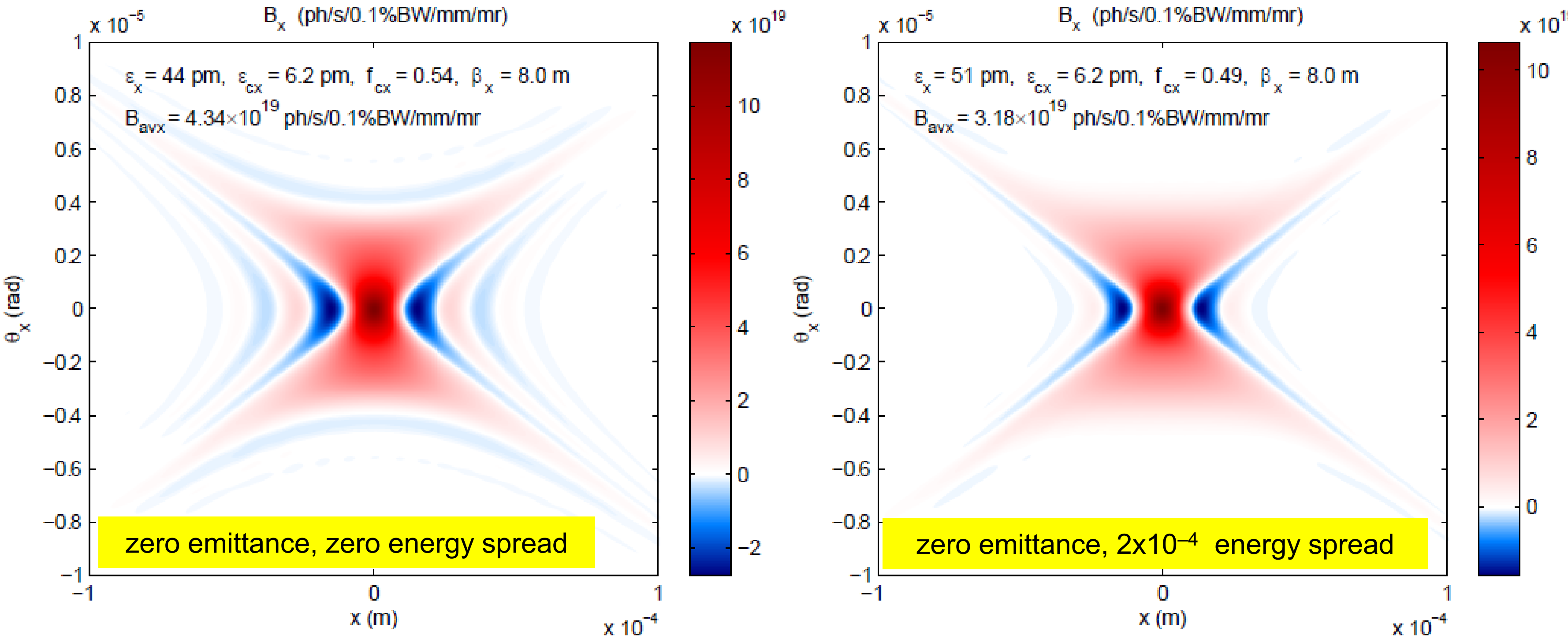


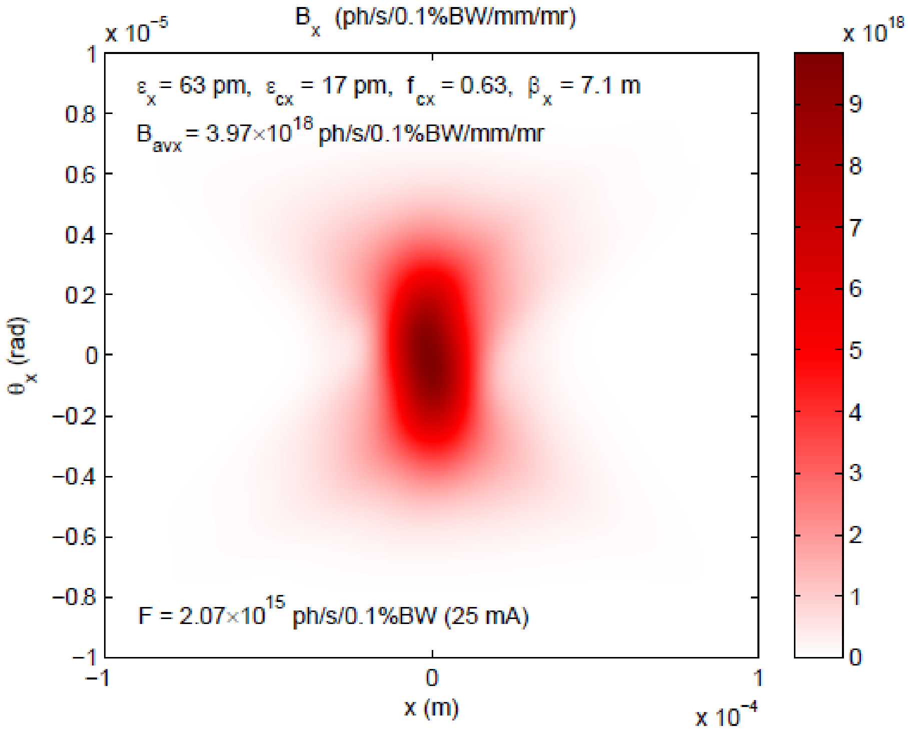
TABLE II. Parameters used in computing the radiation phase space.

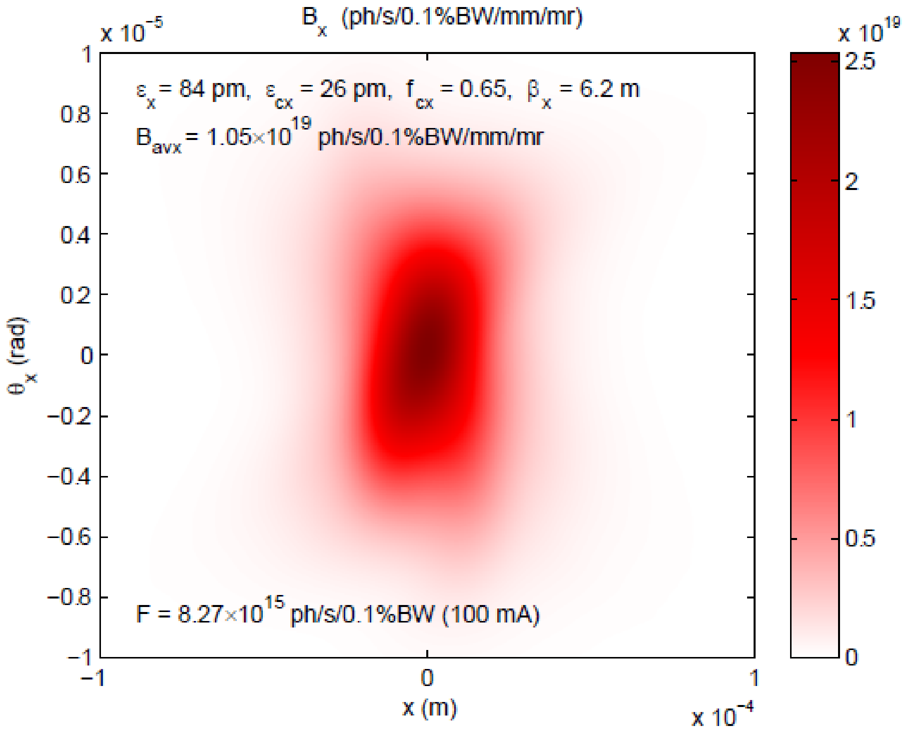
Number of periods, $N_u = 1250$
Undulator period, $\lambda_u = 2\,\mathrm{cm}$
Harmonic number, $n = 1$
Resonant photon energy, $\hbar\omega = 8\,\mathrm{keV}$
Detuning radiation frequency, $\Delta\omega = -0.75\omega/N_u$
Beam energy, $E = 5\,\mathrm{GeV}$
Electron energy spread, $\sigma_{\delta_e} = 2 \times 10^{-4}$
Electron emittance, $\epsilon_x = 11, 29\,\mathrm{pm}$
Average current, $I = 25, 100\,\mathrm{mA}$
β-function, $\beta_x = 4\,\mathrm{m}$

Accounting for energy spread (phase space of x-rays)



And finite emittance…
(phase space of x-rays)





Summary

CLASSE

- Wigner distribution function: natural framework for describing partially to fully coherent x-ray sources
- Distinction between diffraction limit and coherence must be made
  - Fully coherent synchrotron radiation is not diffraction limited
  - Gaussian approximation does not work well when describing diffraction limited sources
  - Refined metrics must be used for comparison
- There is much interconnection between classical wave optics in phase space and quantum mechanics waiting to be explored further