THIRD-ORDER APOCHROMATIC DRIFT-QUADRUPOLE BEAMLINE

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Abstract

In this paper we present the design of a straight drift-quadrupole system which can transport certain beam ellipses (apochromatic beam ellipses) without influence of the second and of the third order chromatic and geometric aberrations of the beamline transfer map.

INTRODUCTION

A straight drift-quadrupole system can not be designed in such a way that a particle transport through it will not depend on the difference in particle energies and this dependence can not be removed even in first order with respect to the energy deviations. Nevertheless, the situation will change if instead of comparing the dynamics of individual particles one will compare the results of tracking of monoenergetic particle ensembles through the system or will look at chromatic distortions of the betatron functions appearing after their transport through the system. From this point of view, as it was proven in [1], for every drift-quadrupole system there exists an unique set of Twiss parameters (apochromatic Twiss parameters), which will be transported through that system without first order chromatic distortions. In this paper we continue the study of the apochromatic properties of straight drift-quadrupole systems and present the design of a beamline which can transport apochromatic beam ellipses without influence of the second and of the third order chromatic and geometric aberrations of the beamline transfer map.

VARIABLES, MAPS AND APOCHROMATS

We consider the beam dynamics in a magnetostatic system and, as usual, take the path length along the reference orbit τ to be the independent variable. We use a complete set of symplectic variables $\boldsymbol{z} = (x, p_x, y, p_y, \sigma, \varepsilon)^\top$ as particle coordinates. Here x, y measure the transverse displacements from the ideal orbit and p_x, p_y are transverse canonical monenta scaled with the constant kinetic momentum of the reference particle p_0. The variables σ and ε which describe longitudinal dynamics are

$$\sigma = c\,\beta_0\,(t_0 - t), \qquad \varepsilon = (\mathcal{E} - \mathcal{E}_0)\,/\,(\beta_0^2\,\mathcal{E}_0), \tag{1}$$

where $\mathcal{E}_0$, β_0 and $t_0 = t_0(\tau)$ are the energy of the reference particle, its velocity in terms of the speed of light c and its arrival time at a certain position τ, respectively.

We represent particle transport from the location $\tau = 0$ to the location $\tau = l$ by a symplectic map $\mathcal{M}$ and assume that the point $\boldsymbol{z} = \boldsymbol{0}$ is the fixed point and that the map $\mathcal{M}$ can be Taylor expanded in its neighborhood. Additionally we assume that the transverse motion is dispersion free (always true for the drift-quadrupole systems) and uncoupled in linear approximation, which is a restriction on the form of the 6×6 symplectic matrix M which represents the linear part of the map $\mathcal{M}$.

Let g_0 be some function of the variables $\boldsymbol{z}$ given at the system entrance. Then its image g_l at the system exit under the action of the map $\mathcal{M}$ is given by the following relation

$$\forall \boldsymbol{z} \qquad g_l(\boldsymbol{z}) = g_0(\mathcal{M}^{-1}(\boldsymbol{z})), \tag{2}$$

which symbolically we will write as $g_l = :\mathcal{M}:^{-1} g_0$.

Let us consider some Courant-Snyder quadratic forms

$$\begin{cases} I_x(\tau) = \gamma_x(\tau)\,x^2 + 2\alpha_x(\tau)\,x\,p_x + \beta_x(\tau)\,p_x^2 \\ I_y(\tau) = \gamma_y(\tau)\,y^2 + 2\alpha_y(\tau)\,y\,p_y + \beta_y(\tau)\,p_y^2 \end{cases} \tag{3}$$

We say that the map $\mathcal{M}$ is an m-order ($m \geq 2$) apochromat with respect to the incoming Courant-Snyder quadratic forms $I_x(0)$ and $I_y(0)$ if

$$:\mathcal{M}:^{-1} I_{x,y}(0) - :M:^{-1} I_{x,y}(0) = O\left(|\boldsymbol{z}|^{m+2}\right), \tag{4}$$

i.e. if the chromatic and geometric distortions to the shapes of the ellipses $I_{x,y}(l)$ after the system passage come only from nonlinear map aberrations which are of the order $m+1$ and higher. We call the Twiss parameters that enter the Courant-Snyder quadratic forms satisfying (4) apochromatic Twiss parameters.

Up to any predefined order m the aberrations of the map $\mathcal{M}$ can be represented through a Lie factorization as

$$:\mathcal{M}: =_m \exp(:\mathcal{F}_{m+1} + \ldots + \mathcal{F}_3:)\,:M:, \tag{5}$$

where each of the functions $\mathcal{F}_k$ is a homogeneous polynomial of degree k in the variables $\boldsymbol{z}$ and the symbol $=_m$ denotes equality up to order m (inclusive) when maps on both sides of (5) are applied to the phase space vector $\boldsymbol{z}$.

Using the representation (5) we can state that the magnetostatic system is an m-order apochromat with respect to $I_x(0)$ and $I_y(0)$ if, and only if, all homogeneous polynomials $\mathcal{F}_k$ in the Lie exponential representation (5) of the system transfer map can be expressed as functions of $I_x(0)$, $I_y(0)$ and ε only.

NONLINEAR ABERRATIONS OF STRAIGHT DRIFT-QUADRUPOLE CELL

The straight drift-quadrupole cell is a magnetostatic system which is symmetric about both, the horizontal midplane $y = 0$ and the vertical midplane $x = 0$. It means that in the Lie exponential representation of the cell map

$$:\mathcal{M}_c: =_m \exp(:\mathcal{F}_{m+1}^c + \ldots + \mathcal{F}_3^c:)\,:M_c: \tag{6}$$

the polynomials $\mathcal{F}_k^c$ do not depend on the variable σ and are even functions of the variables (y, p_y) and of the variables (x, p_x). If the drift-quadrupole cell is not a pure drift space, then the structure of the polynomial $\mathcal{F}_3^c$, which is responsible for the second-order map aberrations, can be further clarified using the concept of the apochromatic Twiss parameters. Let $\beta_{x,y}^a(\tau)$, $\alpha_{x,y}^a(\tau)$ and $\gamma_{x,y}^a(\tau)$ be

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 ISBN 978-3-95450-115-1

the cell apochromatic Twiss parameters and let $I^a_{x,y}(\tau)$ be the corresponding Courant-Snyder quadratic forms. Then the polynomial $\mathcal{F}_3^c$ can be written as follows

$$\mathcal{F}_3^c = -(\varepsilon\,/\,2)\cdot\big(\xi_x(\beta_x^a)\cdot I_x^a(0) \,+\, \xi_y(\beta_y^a)\cdot I_y^a(0) \\ -\, l_c\cdot(\varepsilon\,/\,\gamma_0)^2\big), \quad (7)$$

where l_c is the cell length, γ_0 is the Lorentz factor of the reference particle, and $\xi_x(\beta_x^a)$ and $\xi_y(\beta_y^a)$ are the cell chromaticities calculated for the apochromatic Twiss parameters (see more details in [1, 3]).

CONCEPTUAL SOLUTION FOR THIRD-ORDER APOCHROMAT

Let us consider a system constructed by a repetition of n identical drift-quadrupole cells and let us assume that the horizontal and vertical transfer matrices of the n-cell system are equal to the plus or minus identity matrices, but the horizontal and vertical cell focusing matrices are not equal to the plus or minus identity matrices. Then, as it was shown in [2], the shape distorting effects of the second-order aberrations of the cell map on the transport of the cell periodic Courant-Snyder quadratic forms are washed out at the exit of the n-cell system by averaging. In other words, it means that under the assumptions made the n-cell system is a second-order apochromat with respect to the cell periodic Courant-Snyder quadratic forms. And, as we will show below, the idea of the usage of averaging provided by repetition of n identical cells for automatic cancellation of undesired effects can also be applied to the design of a third-order apochromat if the cell periodic Twiss parameters coincide with the cell apochromatic Twiss parameters.

So, let us start with the assumption that the cell transfer matrix allows periodic beam transport and that the cell periodic Twiss parameters are, at the same time, the cell apochromatic Twiss parameters. Then the map of the repetitive n-cell system can be expressed as follows

$$:\mathcal{M}_{nc}:=_4 \exp(:\mathcal{S}_3+\mathcal{S}_4+\tilde{\mathcal{S}}_5+\hat{\mathcal{S}}_5:) : M_c^n :, \quad (8)$$

where the homogeneous polynomials (aberration functions) $\mathcal{S}_3$, $\mathcal{S}_4$, $\tilde{\mathcal{S}}_5$, $\hat{\mathcal{S}}_5$ are given by the formulas

$$\mathcal{S}_3(\boldsymbol{z}) = n\cdot\mathcal{F}_3^c(\boldsymbol{z}), \quad (9)$$

$$\mathcal{S}_4(\boldsymbol{z}) = n\cdot\boldsymbol{\mathcal{R}}(\mathcal{F}_4^c(\boldsymbol{z})),\;\; \tilde{\mathcal{S}}_5(\boldsymbol{z}) = n\cdot\boldsymbol{\mathcal{R}}(\mathcal{F}_5^c(\boldsymbol{z})), \quad (10)$$

$$\hat{\mathcal{S}}_5(\boldsymbol{z}) = (1\,/\,2)\{\mathcal{F}_3^c(\boldsymbol{z}), \mathcal{W}_n(\boldsymbol{z})\}, \quad (11)$$

the binary operation $\{*,*\}$ is the Poisson bracket,

$$\boldsymbol{\mathcal{R}}\big(f(\boldsymbol{z})\big) \;=\; \frac{1}{n}\sum_{m=0}^{n-1} f(M_c^m \boldsymbol{z}), \quad (12)$$

$$\mathcal{W}_n(\boldsymbol{z}) \;=\; \sum_{m=1}^{[n\,/\,2]} (n+1-2m) \\ \cdot\big(\mathcal{F}_4^c(M_c^{n-m}\boldsymbol{z}) \,-\, \mathcal{F}_4^c(M_c^{m-1}\boldsymbol{z})\big), \quad (13)$$

and the symbol $[k]$ denotes the biggest integer which is smaller or equal to k.

Our second assumption is that the periodic cell phase advances μ_x^c and μ_y^c satisfy

$$\mu_{x,y}^c \;=\; 2\pi\, q_{x,y}\,/\,n \pmod{2\pi} \quad (14)$$

for some integer or half integer q_x and q_y which are smaller than n and are such that the resonances

$$2\,\mu_{x,y}^c,\quad 4\,\mu_{x,y}^c,\quad 2\,\mu_x^c \,\pm\, 2\,\mu_y^c \quad (15)$$

are avoided. From this assumption it follows that the functions $\mathcal{S}_4$ and $\tilde{\mathcal{S}}_5$ (as a result of averaging provided by the operator $\boldsymbol{\mathcal{R}}$) can be expressed as functions of the variables $I^a_{x,y}(0)$ and ε only, and thus the n-cell beamline becomes a third-order apochromat with respect to the cell periodic Courant-Snyder quadratic forms $I^a_{x,y}(0)$.

Note that the minimal number of cells required for the construction of the third-order apochromatic beamline following the above recipe is five, and that the first distortions to the shapes of the Courant-Snyder ellipses $I^a_{x,y}(0)$ after the system passage are due to forth-order aberrations of the map $\mathcal{M}_{nc}$ generated by the function $\hat{\mathcal{S}}_5$.

PROOF-OF-PRINCIPLE NUMERICAL EXAMPLE

To complete the third-order apochromat design we need a drift-quadrupole cell with the coinciding periodic and apochromatic Twiss parameters and with the phase advances satisfying the conditions (14) and (15). What is the minimum number of quadrupoles which are required for such a cell to exist? Using thin-lens approximation for quadrupole focusing one can show that this number can not be smaller than four, and with four quadrupoles we have many examples (found by direct numerical search using hard-edge quadrupole models) of the cells which satisfy all needed requirements. The parameters of one such cell are given in the Table 1 and its periodic (and apochromatic) betatron functions can be seen in Fig.1. The horizontal and vertical phase advances of this cell are equal to 144° and 108° respectively, which means that the sequence of five such cells gives a third-order apochromatic beamline.

Table 1: Parameters of apochromatic four quadrupole cell.

k_1	0.208766	l_4	2.211502
k_2	-0.192846	l_5	7.825868
k_3	0.158699	β_x	15.243567
k_4	-0.145879	α_x	-0.464978
l_{quad}	1.000000	μ_x^c	144°
l_1	6.922985	β_y	22.770092
l_2	0.892888	α_y	0.525805
l_3	18.011646	μ_y^c	108°

The total number of quadrupoles in our third-order apochromatic beamline is twenty, and it is interesting to compare its beam transfer properties with the beam transport through the chain of ten FODO cells which also has twenty quadrupoles and which is designed to have the same length and about the same average betatron functions.

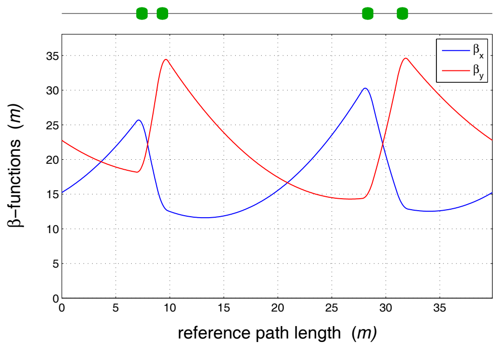

Figure 1: Periodic betatron functions along apochromatic four quadrupole cell with $\mu_x^c = 144°$ and $\mu_y^c = 108°$.

We chose for such comparison not one but three different FODO beamlines with the cell phase advances equal to 54°, 60° and 72° (see, for example Fig.2). The first and the third FODO beamlines have the phase advances which are equal to the half of the vertical and of the horizontal phase advances of our apochromatic four quadrupole cell respectively and, at the same time, are second-order apochromats. The second FODO beamline is taken because, from one side, its phase advance 60° lies in between the phase advances of the two other beamlines, but, from the other side, its overall phase advance is not multiple of 180° and, therefore, it is not a second-order apochromat.

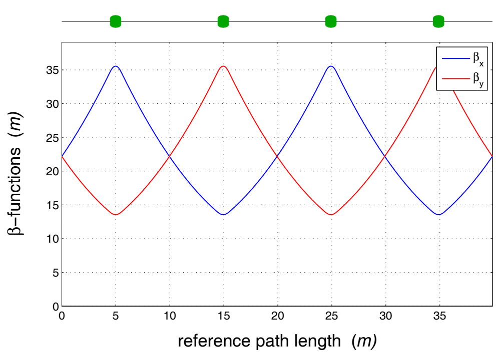

Figure 2: Betatron functions along two 54° FODO cells.

Because it is not easy to find a single quantity which will characterize in a clear and general way the effect of aberrations which are nonlinear in the transverse variables, we will restrict our comparison to the study of the beam dynamics provided by the Hamiltonian

$$H = \varepsilon + \sqrt{(1+\varepsilon)^2 - p_x^2 - p_y^2 - (\varepsilon\,/\,\gamma_0)^2} + (k\,/\,2)\cdot(x^2 - y^2) \qquad (16)$$

after its linearization with respect to p_x^2 and p_y^2, i.e. we will look only at the effect of the energy deviation.[1]

For each beamline, for each transverse plane and for each energy offset we track the nominal beamline Twiss parameters (Twiss parameters which are periodic for the nominal energy) through the system and, at the system exit, characterize the accumulated chromatic effects by calculating first the mismatch parameter m_p and then the mismatch amplitude

$$a = m_p + \sqrt{m_p^2 - 1}, \qquad (17)$$

which is a measure of the created optics distortion. The results of these calculations are presented in Fig.3, where a_x and a_y denote the horizontal and vertical mismatch amplitudes of the third-order apochromat and for each FODO chain only one curve is presented, because for the FODO beamlines the horizontal and vertical mismatch amplitudes coincide. One sees, as expected, that our third-order apochromatic beamline demonstrates the best performance and the next comes the second-order apochromats with 54° and 72° cell phase advances.

Note that our theory is an asymptotic theory and uses Taylor expansion with respect to the phase space variables. It certainly can fail at large energy deviations, that is demonstrated in Fig.4.

[1] If only chromatic aberrations are of concern and the influence of the geometric effects can be ignored, then the minimal number of cells required for automatic cancellation of aberrations in our solution can be reduced from five to two because in this case only the resonances $2\mu_{x,y}^c$ in (15) must be avoided.

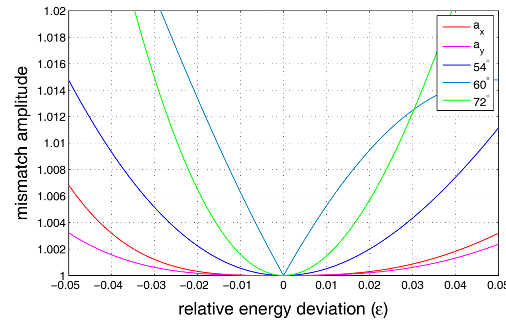

Figure 3: Mismatch amplitudes for different beamlines.

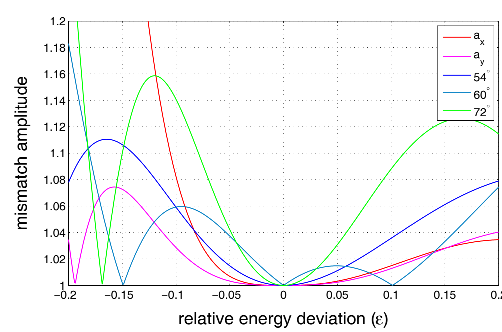

Figure 4: Mismatch amplitudes for different beamlines.

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ISBN 978-3-95450-115-1