Saint-Petersburg State University
Faculty of Applied Mathematics and
Control Processes

A. Ivanov, S. Andrianov

Matrix Formalism for long-term evolution of charged particle and spin dynamics in electrostatic fields

speaker:
Andrei Ivanov

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The aims are

- to derive the equations for numerical implementation of matrix formalism for ODE solving;
- to develop tool for Taylor series expansion and building solution of ODE in matrix form;
- to apply the approach to beam dynamics modeling;

Matrix Formalism

is an integration method based on map building in 2-dim matrix form

$$\frac{dX(t)}{dt} = F(t, X)$$

$$P^{1k}(t) = \frac{1}{(k)!} \frac{\partial^k F(t, X_0)}{\partial (X^{[k]})^T}$$

$$\frac{dX(t)}{dt} = \sum_{k=1}^{\infty} P^{1k}(t) X^{[k]}$$

A solution can be found as a series with respect to the powers of initial point

$$X(t) = \sum_{k=1}^{\infty} R^{1k}(t) X_0^{[k]}$$

An example

Orbital motion of a particle in cylindrical deflector can be described by *

$$x'' + \frac{2}{R_{eq}^2}x - \frac{1}{R_{eq}^3}x^2 = 0,$$
$$y'' = 0,$$

$$\frac{d}{ds}\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} x' \\ -2/R_{eq}^2\, x + 1/R_{eq}^3\, x^2 \end{pmatrix}.$$

In matrix form we have

$$\frac{d}{ds}\begin{pmatrix} x \\ x' \end{pmatrix} = \sum_{\mathbf{k}=0}^{\infty} \mathbb{P}^{1k}(s)\begin{pmatrix} x \\ x' \end{pmatrix}^{[k]},$$

$$\mathbb{P}^{11} = \begin{pmatrix} 0 & 1 \\ -2/R_{eq}^2 & 0 \end{pmatrix},\ \mathbb{P}^{12} = \begin{pmatrix} 0 & 0 & 0 \\ 1/R_{eq}^3 & 0 & 0 \end{pmatrix},\ \mathbb{P}^{1k} = \mathbb{O},\ k = \overline{3,\infty}.$$

*** Senichev Yu., Moeller S.P. Beam Dynamics in Electrostatic Rings.**

An example

Following the algorithm for building matrices R *

$$\mathbf{X}(s) = \mathbb{R}^{11}(s)\,(x_0, x_0')^T + \mathbb{R}^{12}(s)\,\left(x_0^2, x_0 x_0', {x_0'}^2\right)^T$$

where

$$\mathbb{R}^{11}(s|0) = \begin{pmatrix} \cos(bs) & \frac{1}{b}\sin(bs) \\ -b\sin(bs) & \cos(bs) \end{pmatrix},$$

$$\mathbb{R}^{12}(s|0) = \begin{pmatrix} R^{12}_{11} & R^{12}_{12} & R^{12}_{13} \\ R^{12}_{21} & R^{12}_{22} & R^{12}_{23} \end{pmatrix},$$

$$R^{12}_{11} = \frac{b^2}{2^{3/2}}\left(-\frac{(\cos(2bs)-3)}{6b} - \frac{\cos(bs)}{3b}\right), \quad R^{12}_{21} = \frac{b^3}{2^{3/2}}\left(\frac{\sin(2bs)}{3b} + \frac{\sin(bs)}{3b}\right),$$

$$R^{12}_{12} = \frac{b}{2^{1/2}}\left(\frac{\sin(bs)}{3b} - \frac{\sin(2bs)}{6b}\right), \quad R^{12}_{22} = \frac{b^2}{2^{1/2}}\left(\frac{\cos(bs)}{3b} - \frac{\cos(2bs)}{3b}\right),$$

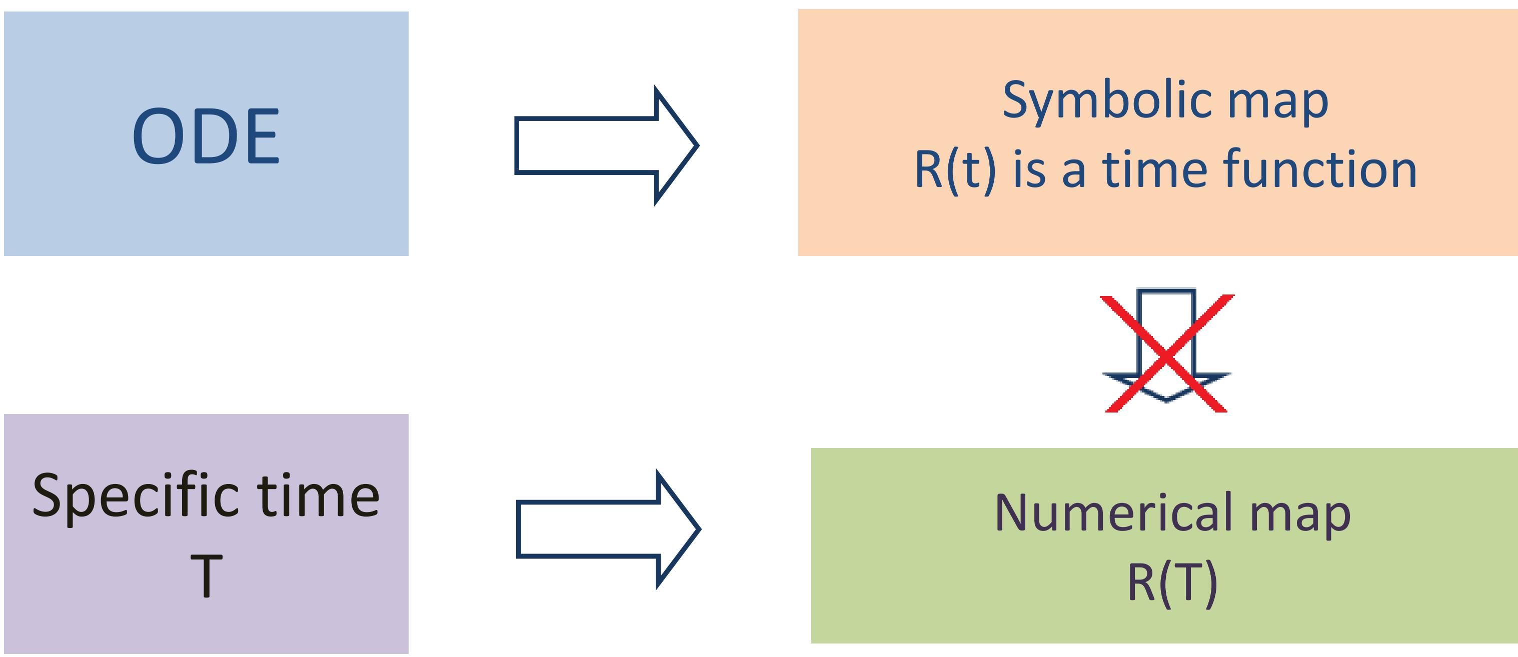
$$R^{12}_{13} = \frac{b^2}{2^{3/2}}\left(\frac{(\cos(2bs)+3)}{6b} - \frac{2\cos(bs)}{3b}\right), \quad R^{12}_{23} = \frac{b^2}{2^{3/2}}\left(\frac{2\sin(bs)}{3b} - \frac{\sin(2bs)}{3b}\right).$$

* Andrianov S. Dynamic modeling of beam control systems.

Numerical implementation

There are 2 ways for map building:

- symbolic mode;
- numerical mode.



Numerical implementation

According to the initial equation for system of ODE and its solution

$$X(t)=\sum_{k=1}^{\infty} R^{1k}(t)\,X_0^{[k]}$$

$$\frac{dX(t)}{dt}=\sum_{k=1}^{\infty} P^{1k}(t)\,X^{[k]}$$

$$\sum_{k=1}^{\infty} \frac{dR^{1k}(t)}{dt} X_0^{[k]}=\sum_{k=1}^{\infty} P^{1k}(t)\,X^{[k]}$$

By partial deriving of this equation we can obtain:

$$\frac{dR^{1k}(t)}{dt}=P^{1k}(t)\frac{\partial X^{[k]}}{\partial \left(X_0^{[k]}\right)^T} \quad (*)$$

Using a step-by-step integration method for solving equation (*) we can evaluate matrices R

Map building

- desciption of the equation of the motion and spin dynamics in analytical form (Newton-Lorentz and BMT equation);

- Taylor series expansion of these equation;

- buildin ODE with corresponds to the matrices R following to the algorithm that described above;

- evaluating elements of matrices R by a numerical integration;

Map building

$$S_x' = S_s/R + \frac{Q}{m_0c^2}\left(G + \frac{1}{1+\gamma}\right)((h_sE_x - x'E_s)S_s -$$
$$-(x'E_y - y'E_x)S_y),$$

Lattice elements

$$\Phi(x,y) = -V + \frac{2V}{\ln\frac{R_2}{R_1}}\ln\frac{r}{R_1}$$

Matrix Formalism Tools

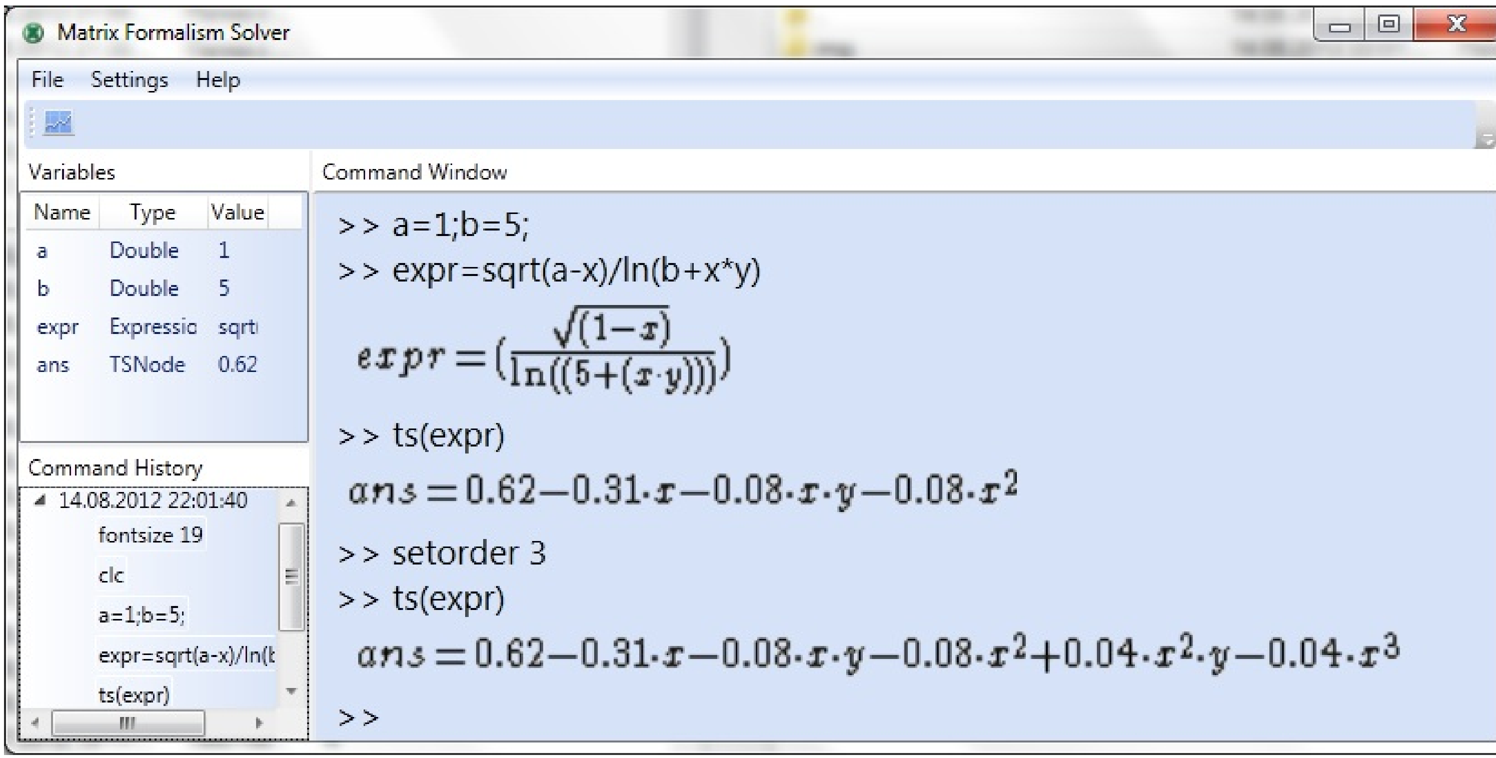
- Taylor series expansion;
- Building ODE's for matrices form;
- Numerical integration;

Data Base

quadrupole lenses, cylindrical deflectors, drifts, RF, etc

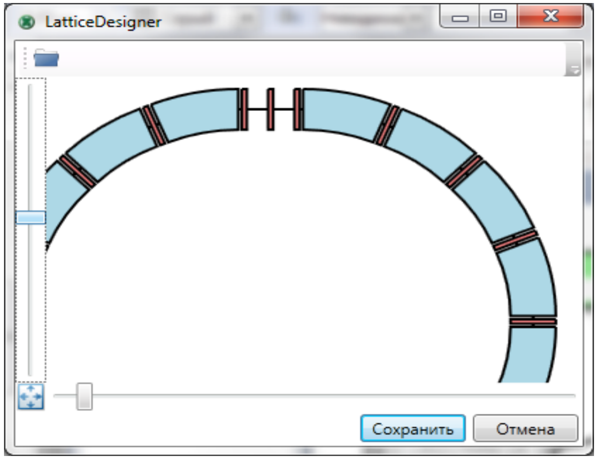
Taylor series expansion

The libraries for automatically expansion of a nonlinear function to corresponded Taylor series up to the necessary order and symbolic interpretator have been implemented.

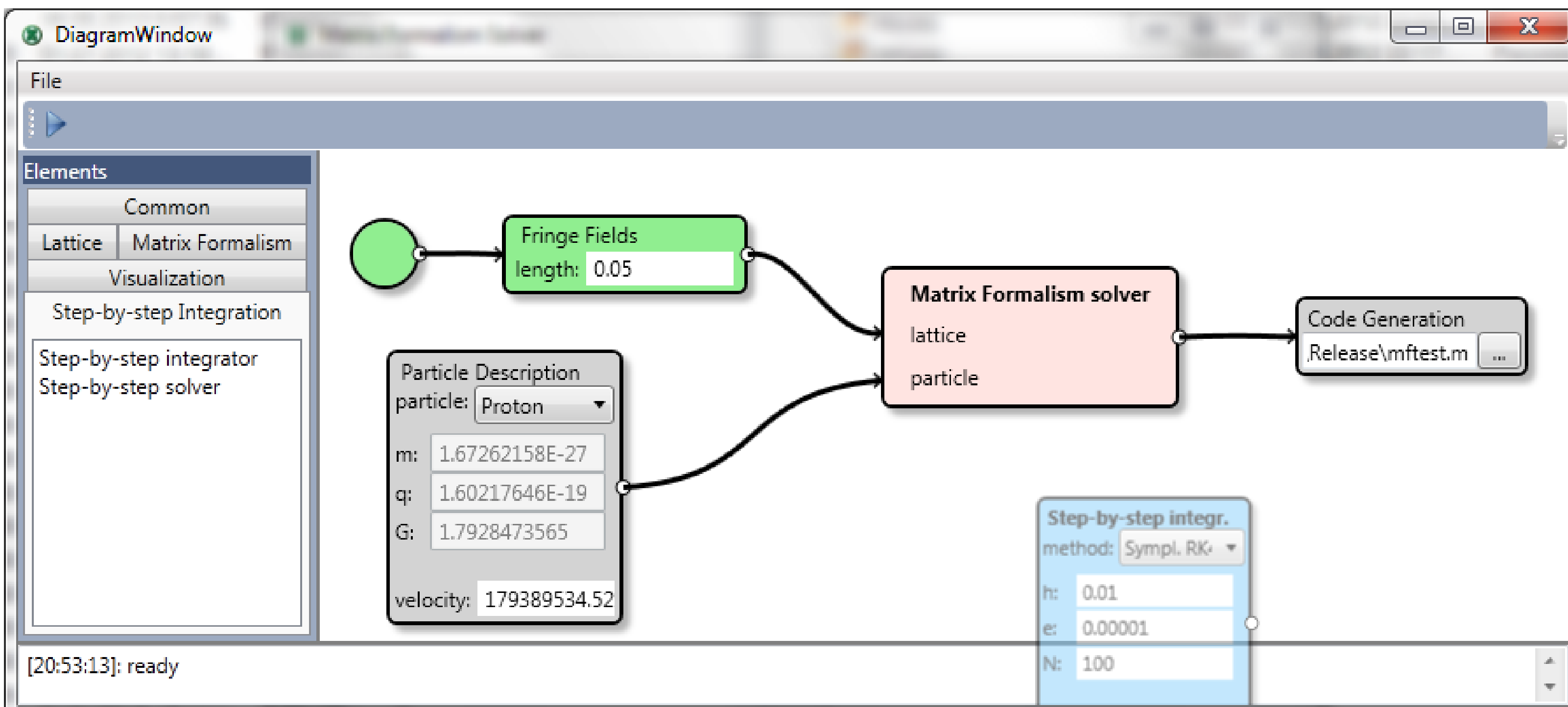


The function may be a composition of elementary functions (sin(x), cos(x), tan(x), exp(x), sqrt(x), ln(x)) and operators +, -, *, /.

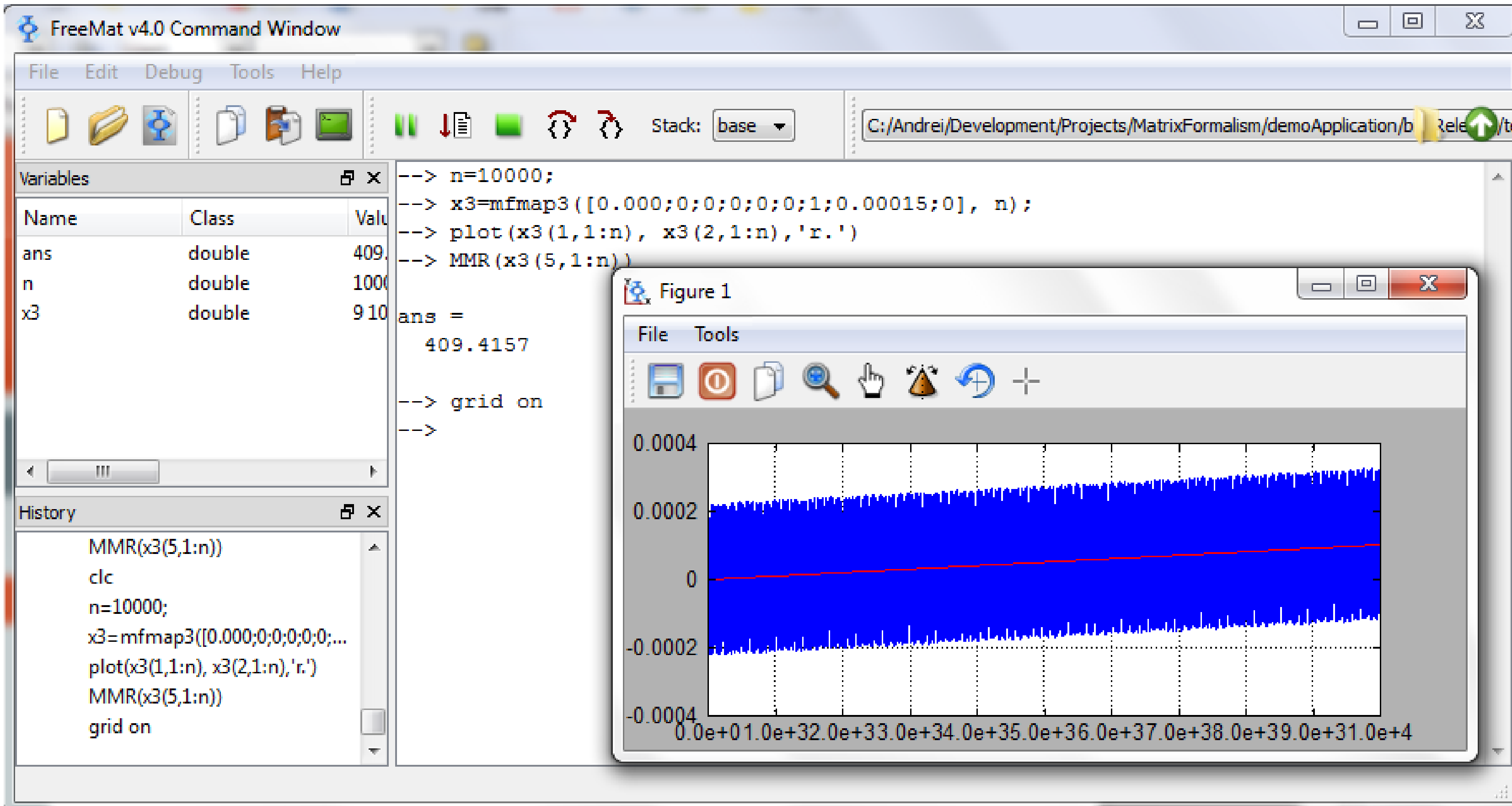
Map building for a lattice



You can specify any field distribution (both electric and magnetic) in an analytical form for automatically map bilding. Although computational code generation is available.



Particle simulation: demo



X = (x, x', y, y, Sx, Sy, Sz, dv, t)

Code generation

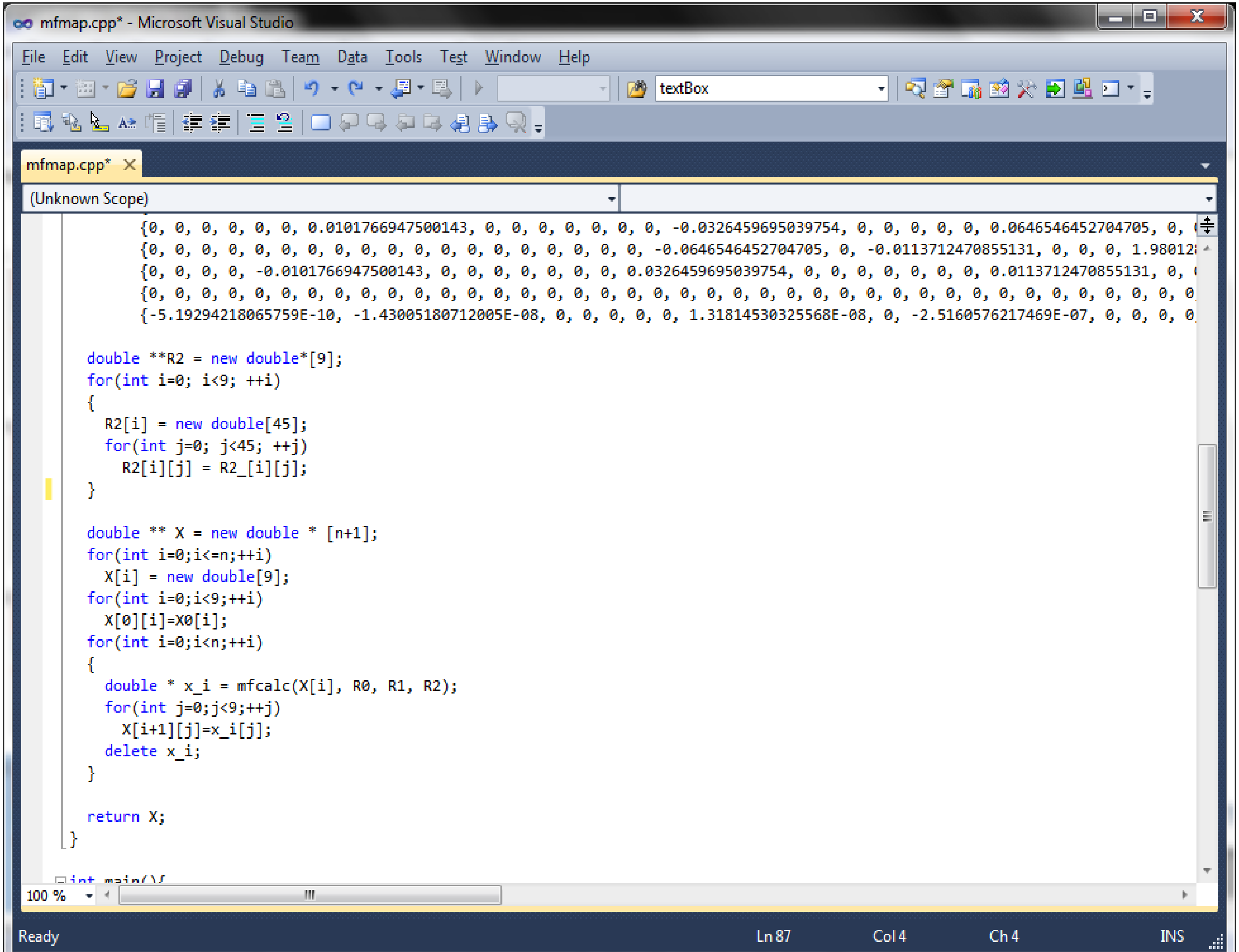
mfmap3.m* (C:/Andrei/Development/Projects/MatrixFormalism/demoApplication/bin/Release/test/) - FreeMat v4.0 Editor

File Edit Tools Debug Help

mfmap3.m*

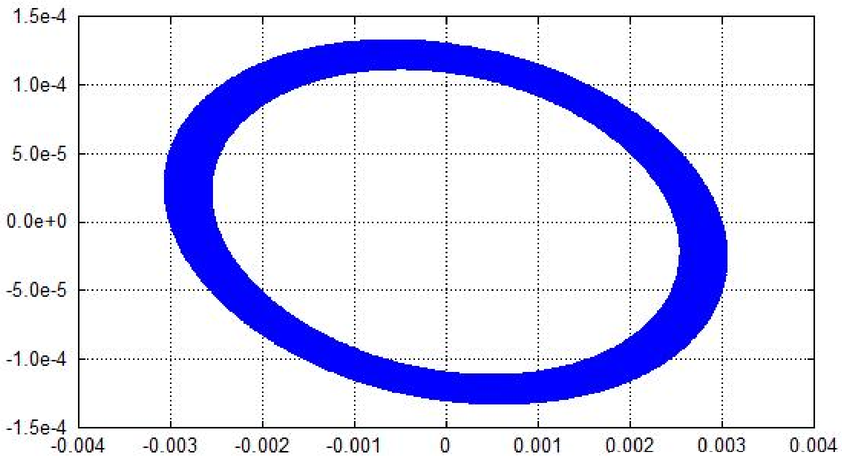
```
%code generation
%mapping function
function X = mfmap(X0, n)
%map generation:
    R0=[0;0;0;0;0;0;0;0;1.06369637954473E-06;];
    R1=[-0.670513915242833 14.3255631639029 0 0 0 0 0 35.3231449743497 0;
        -0.0270879171669592 -0.912653250594104 0 0 0 0 0 0.332988869053756 0;
        0 0 -0.970853028677094 62.9527681188037 0 0 0 0 0;
        0 0 -0.0113712470855131 -0.292679232191561 0 0 0 0 0;
        0 0 0 0 1 0 0 0 0;
        0 0 0 0 0 1 0 0 0;
        0 0 0 0 0 0 1 0 0;
        0 0 0 0 0 0 0 1 0;
        3.19743598523805E-09 1.61311298409876E-07 0 0 0 0 0 -2.80213794351364E-07 1;];
    R2=...      R3=...
    X = zeros(length(X0), n+1);
    X(:,1) = X0;
    for i=1:n
        X(:, i+1) = mfcalc(X(:,i), R0, R1, R2, R3);
    end
end
%one-tune solution
function X = mfcalc(X0, R0, R1, R2, R3)
%initial state:
    x0=X0(1);x1=X0(2);x2=X0(3);x3=X0(4);x4=X0(5);x5=X0(6);x6=X0(7);x7=X0(8);x8=X0(9);
    %calculation of kronecker pows:
    X1=[x0;x1;x2;x3;x4;x5;x6;x7;x8];
    X2=[x0*x0;x0*x1;x0*x2;x0*x3;x0*x4;x0*x5;x0*x6;x0*x7;x0*x8;x1*x1;x1*x2;x1*x3;x1*x4;x1*x5;x
    X3=[x0*x0*x0;x0*x0*x1;x0*x0*x2;x0*x0*x3;x0*x0*x4;x0*x0*x5;x0*x0*x6;x0*x0*x7;x0*x0*x8;x0*x
%solution:
    X=R0+R1*X1+R2*X2+R3*X3;
end
```

Code generation



Matrix Formalism

for long-term evalution



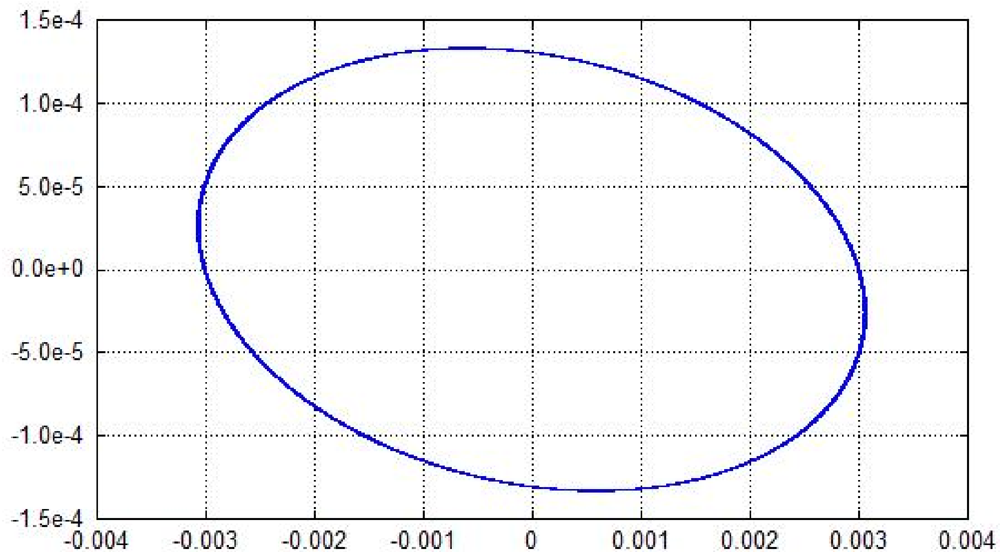
Using a symplectic integration algorithm to build a map does not guarantee the symplectic map (left: phase plane for 100000 turns, symplectic condition violation)

Symplectification is only matrix elements corrections:

$$M^* J M = J, \forall X_0,$$

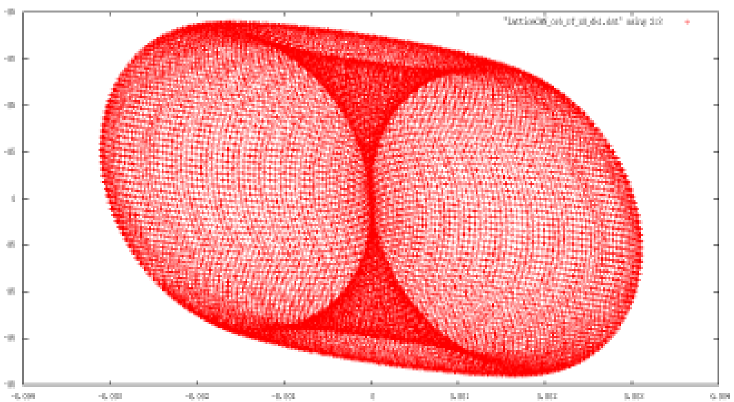
$$J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}.$$

1 000 000 turns:



Comparison:

orbital motion in transverse plane (RF ON)



COSY INFINITY

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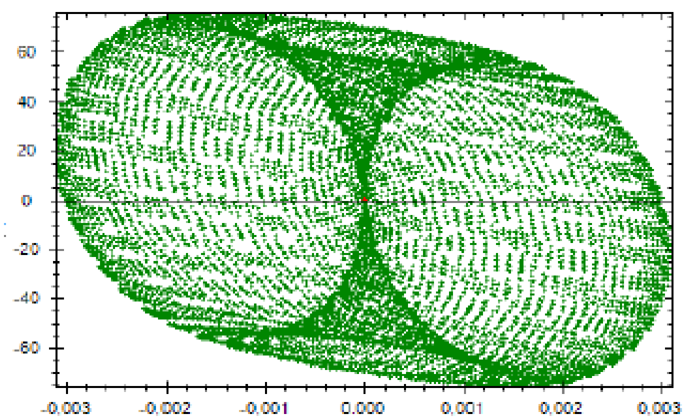
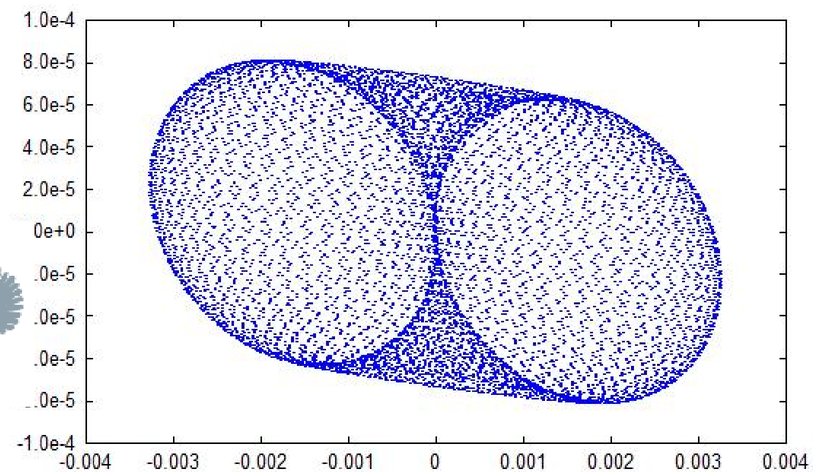
Thanks to D. Zyuzin for help with COSY INFINITY calculation





- Step-by-step integration
- Matrix Formalism

Comparison:
fringe fields influence

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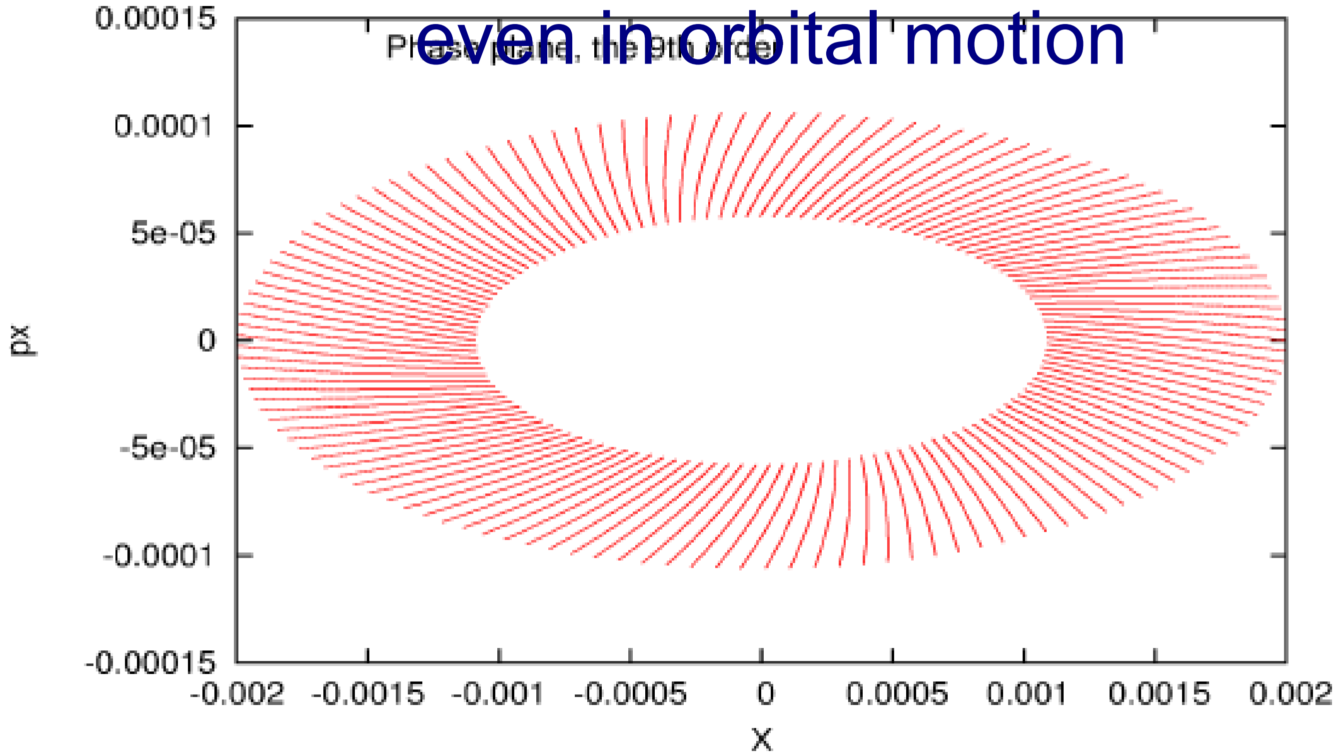
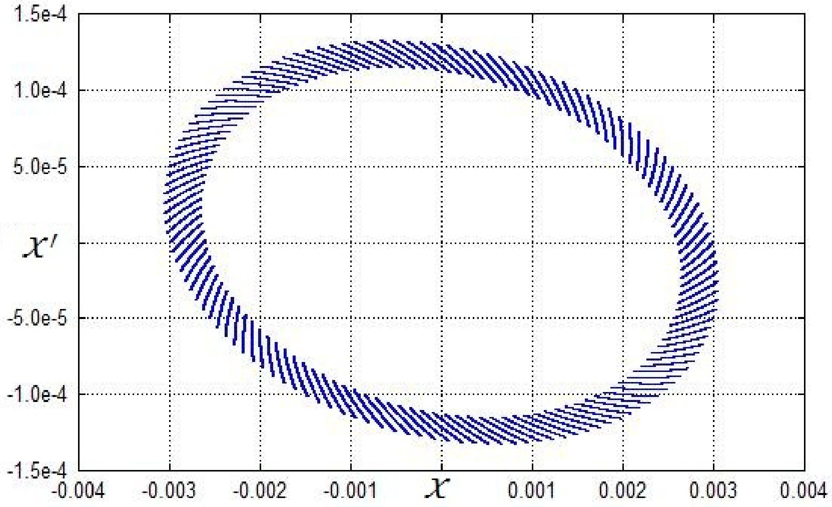
Strange behavior not only in spin dynamics
but
even in orbital motion



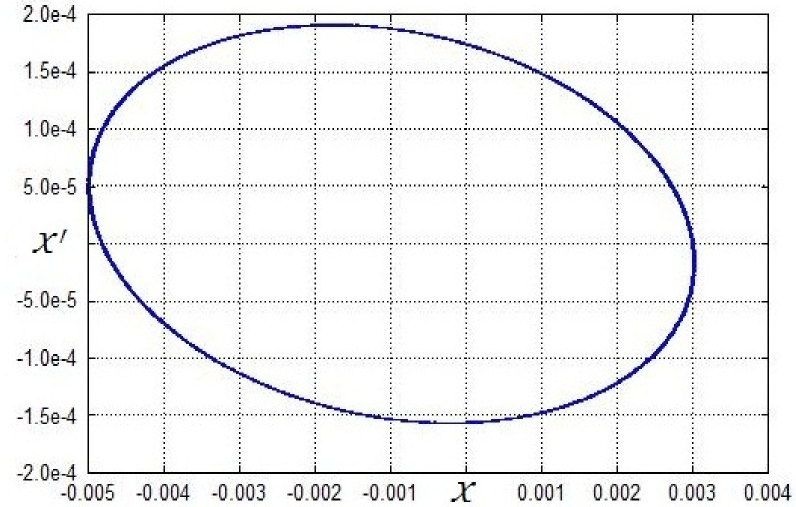
It seems that all particles shrink in time to the reference particle

Matrix Formalism:

fringe fields influence



a)



b)

a) transverse plane with fringe fields without energy conservation

b) energy conservation corrections, when reference orbit can be found by following way

$$X_{ref} = \sum_{i=0}^{k} R^{1i}(t) X_{ref}^{[i]}. \qquad X_{ref} = (E - R^{11})^{-1} R^{10}.$$

Conclusion

Results

- ✓ matrix formalism as a high-performance mapping approach for solving of ODE is implemented in numerical mode

Laptop Intel Core i5

3 order, 9-dim state
1 000 000 turns for
1 point per **3.2 sec**

!

- ✓ sofware for beam dynamics (spin-orbital motion) simulation is developed

Further development

- implementation of matrix formalism in symbolic mode;
- research on EDM project and spin dynamics investigation.

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Thank you for your attention