Rigorous Fixed Point Enclosures and an Application to Beam Transfer Maps

Alexander Wittig, Martin Berz

Department of Physics and Astronomy, Michigan State University
East Lansing, MI, 48824

ICAP 2012, Warnemünde

Transfer Maps

Typical workflow:

1. Computation of transfer map
   - Using differential algebra (DA) techniques, e.g. COSY INFINITY
   - High order computations
   - Yielding functional relationship between initial and final coordinates
2. Particle tracking
   - Pick initial conditions
   - Apply transfer map, correct symplectic errors, repeat
3. Analysis
   - Study tracking picture
   - Determine "dynamic aperture"

Dynamic Aperture

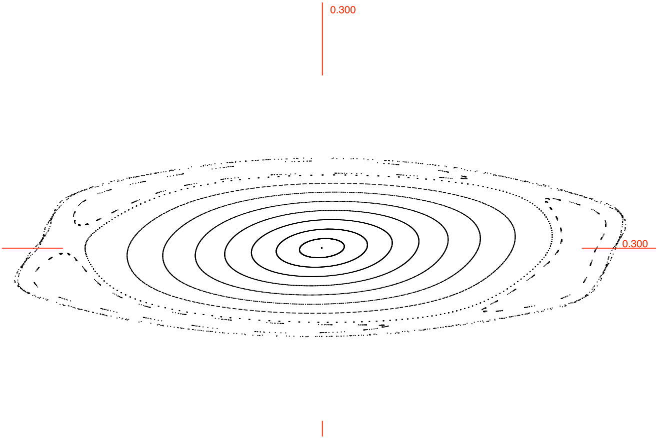
Not a (mathematically) well defined concept!

- Only finitely many initial conditions tracked
- Only a finite number of revolutions tracked

Essentially educated guessing from pictures

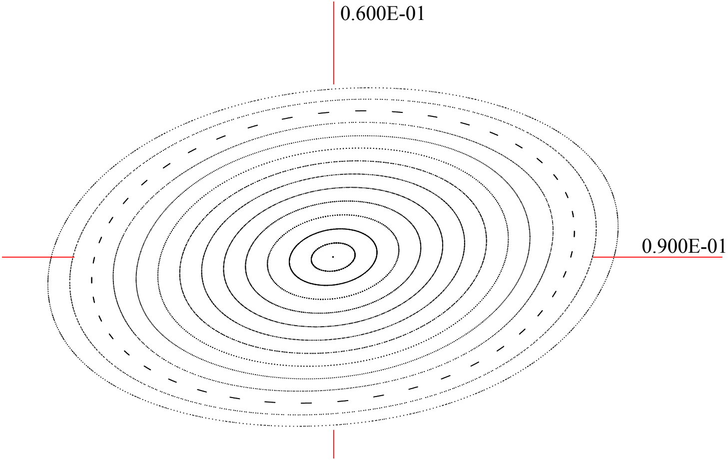
BUT: Pictures can be misleading!

Dynamic Aperture



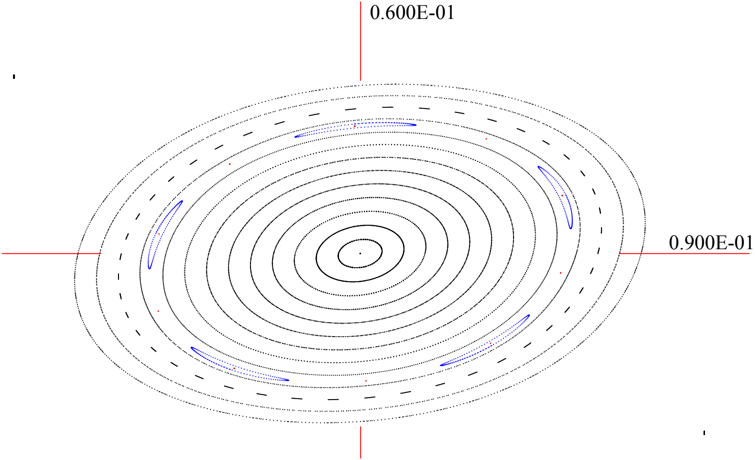
x-a tracking picture of real Tevatron map: nice interior with bad fringes?

Dynamic Aperture



Zoom in on the “nice” part of the map

Dynamic Aperture



Surprise!

Periodic Point Finder

Goal: Ensure we avoid such surprises!

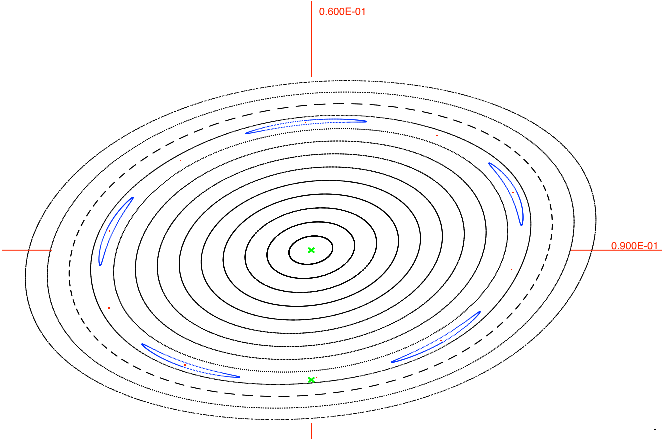
- Need the "right" initial conditions for tracking
- Catch *all* possibly problematic points

Island structures form around elliptic periodic points (resonances)
$\Rightarrow$Try to find periodic points!

Note: The following are equivalent problems

- fixed point: $f(x) = x$
- periodic point: $f^n(x) = x$
- root finding: $f(x) = 0$

Periodic Point Finder

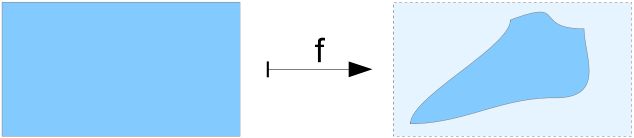


Naive Newton's method fails, because origin is too "strong"

Attractive Fixed Point (Existence)

Theorem (Schauder's Theorem)

Let $K \subset \mathbb{R}^n$ be a non-empty, compact, and convex set. Then, any continuous map $f : K \to K$ has a fixed point in K.



(Super-Brief) Intro to Taylor Model Methods

Taylor Model Methods (Makino, Berz) for verified numerics:

- Extension of differential algebra (DA) techniques
- Automatic rigorous calculation of truncation errors
  DA:
  $f(x_0 + \delta x) \approx f(x_0) + f'(x_0) \cdot \delta x + \cdots + \frac{1}{n!} f^{(n)}(x_0) \cdot \delta x^n$
  TM:
  $f(x_0 + \delta x) \in f(x_0) + f'(x_0) \cdot \delta x + \cdots + \frac{1}{n!} f^{(n)}(x_0) \cdot \delta x^n + [-\varepsilon, \varepsilon]$
- Polynomial bounders (e.g. LDB) provide highly accurate bounds for images of sets under f

Attractive Fixed Point (Uniqueness)

Theorem (Banach's Theorem)

Let K be a non-empty, complete metric space. Then, any contraction $f : K \to K$ has a unique fixed point in K.

Reminder: A map f is a contraction if $\exists\ 0 \leqslant K < 1$ such that

$$|f(x) - f(y)| < K\,|x - y|$$

$\forall\ x, y \in K$

Combining both theorems:

Corollary

Let $K \subset \mathbb{R}^n$ be a convex, compact set, and $f : K \mapsto K$ be continuously differentiable with $|Df(x)| < 1\ \forall x \in K$.
Then f has a unique fixed point in K.

Verification procedure:

1. Find small box B around assumed fixed point and show it is mapped into itself ($\Rightarrow$existence)
2. Bound $|Df|$ over B and show it is less than 1 ($\Rightarrow$uniqueness)

MICHIGAN STATE UNIVERSITY

Goals

Global fixed point finder

1. works for all types of fixed points (repelling, hyperbolic, elliptic, attracting)
2. find verified fixed point enclosures automatically
3. find *all* fixed points in given area

MICHIGAN STATE UNIVERSITY

Preconditioning

For any regular matrix A:

$$f(x) = x \;\Leftrightarrow\; A \cdot f(x) = A \cdot x$$

As fixed point problem:

$$f(x) = x \;\Leftrightarrow\; A \cdot (f(x) - x) + x = x$$

Idea

Choose A so $A \cdot (f(x) - x) + x$ has a strongly contracting fixed point x_0.

Preconditioning

Idea

Choose A so $A \cdot (f(x) - x) + x$ has a strongly contracting fixed point x_0.

$$A = -\left(Df(x_0) - I\right)^{-1}$$

Derivative of

$$A \cdot (f(x) - x) + x$$

at fixed point x_0:

$$A \cdot (Df(x_0) - I) + I = -\left(Df(x_0) - I\right)^{-1} \cdot (Df(x_0) - I) + I = -I + I = 0$$

MICHIGAN STATE UNIVERSITY

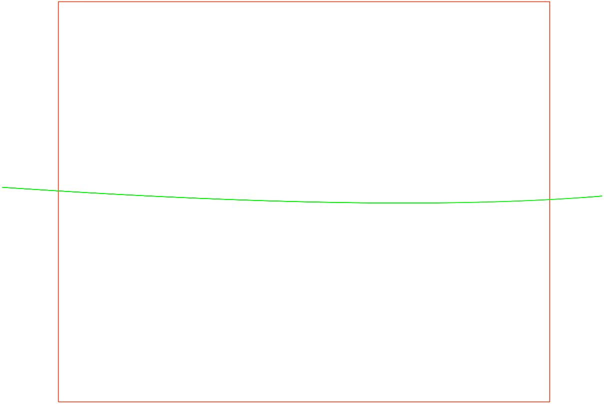
Preconditioning

Idea

Choose A so $A \cdot (f(x) - x) + x$ has a strongly contracting fixed point x_0.

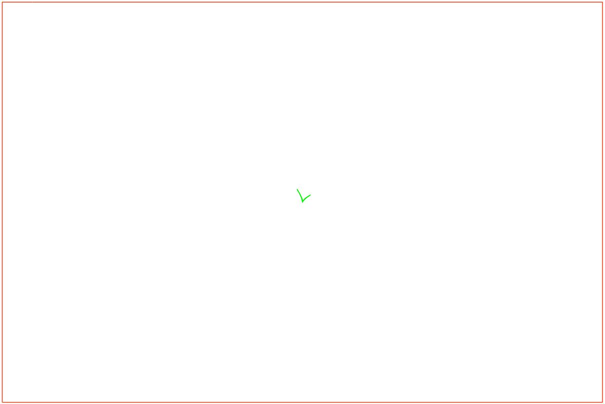
- Similar to Newton method applied to $f(x) - x$,
- A does not have to be rigorous, any regular A will do,
- A above is best choice yielding superlinear contraction.

Effect of Preconditioning



without preconditioning

Effect of Preconditioning



with preconditioning

Rigorous Root & Fixed Point Finder

Basic algorithm for Global Fixed Point Finder:

1. Start with region of interest on stack
2. Test top box on stack for fixed point
   - No FP: discard
   - FP found: keep box as result (or split if enclosure too big)
   - Unknown: split box
3. While stack not empty: continue with 2.

Yields verified enclosures of all fixed points in area of interest.

Invariant Manifolds

Theorem (Invariant Manifold Theorem)

Hyperbolic fixed points of diffeomorphisms have invariant manifolds.
Stable manifold of fixed point p:

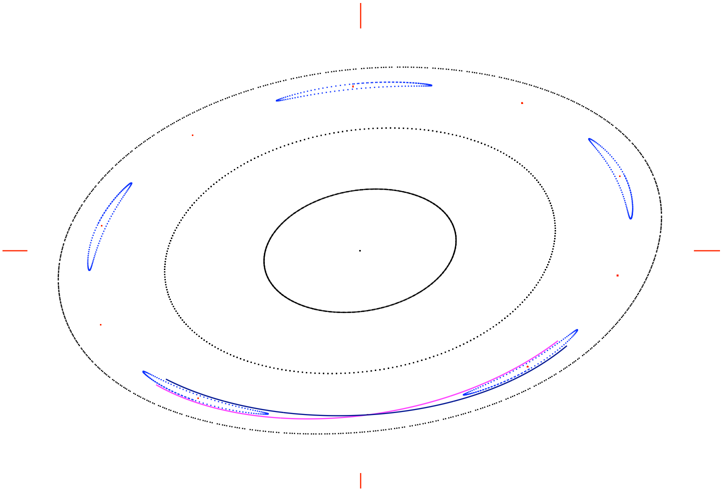
$$W_p^s = \left\{ x \in \mathbb{R}^n | \lim_{n\to\infty} f^n(x) = p \right\}$$

Unstable manifold of fixed point p:

$$W_p^u = \left\{ x \in \mathbb{R}^n | \lim_{n\to\infty} f^{-n}(x) = p \right\}$$

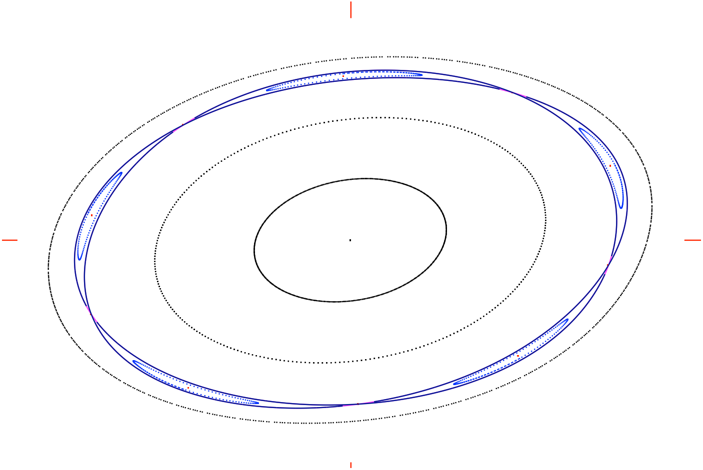
Hyperbolic fixed points in transfer maps have invariant manifolds!

Invariant Manifolds in Transfer Maps



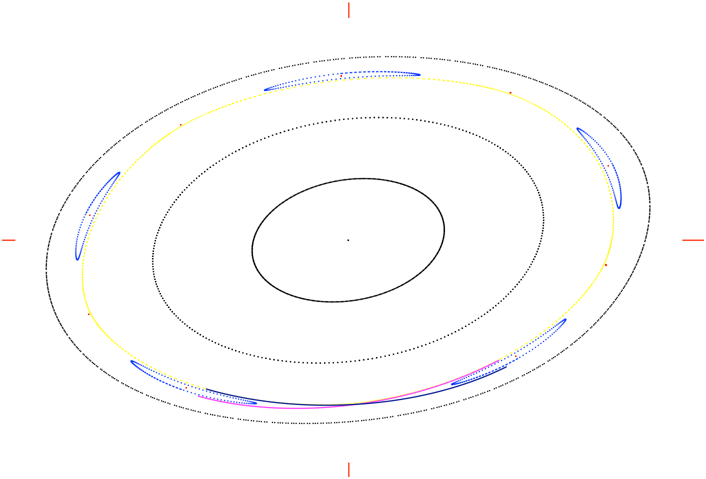
stable (blue) and unstable (magenta) manifolds

Invariant Manifolds in Transfer Maps



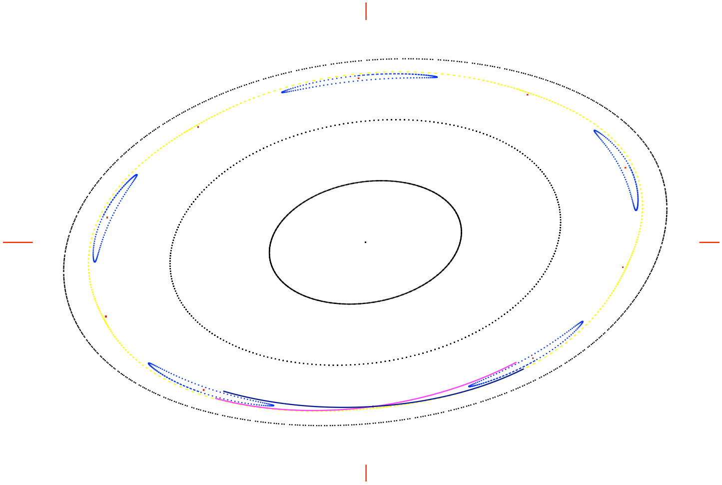
stable and unstable manifolds of all periodic points

Invariant Manifolds in Transfer Maps



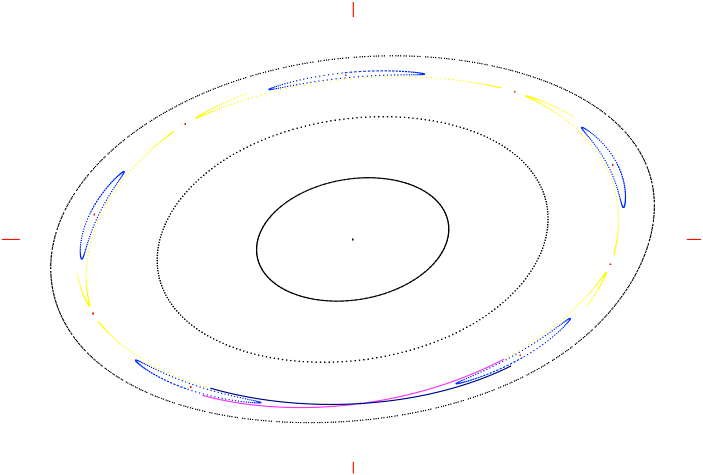
stable (blue) and unstable (magenta) manifolds

Invariant Manifolds in Transfer Maps



stable (blue) and unstable (magenta) manifolds

Invariant Manifolds in Transfer Maps



stable (blue) and unstable (magenta) manifolds

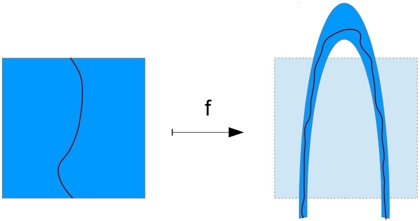
Invariant Manifolds and Chaos

Invariant Manifolds can prove chaotic behavior

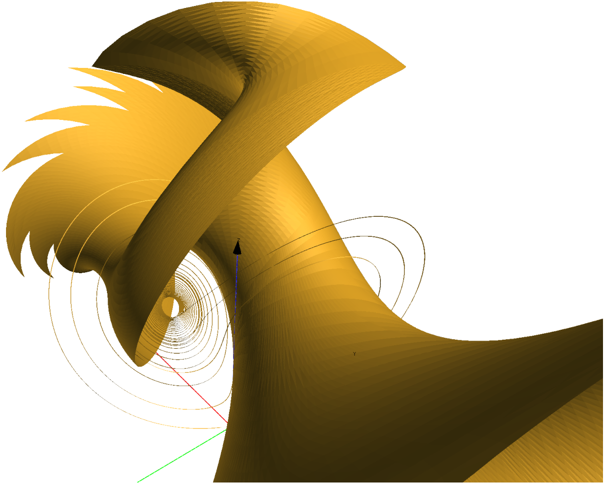
Theorem (Poincaré-Birkhoff-Smale)

If the stable and unstable manifold of a hyperbolic fixed point intersect transversely, the map exhibits chaos (has positive topological entropy).

Chaos is induced by the construction of a Smale horseshoe from higher iterates of the manifolds:

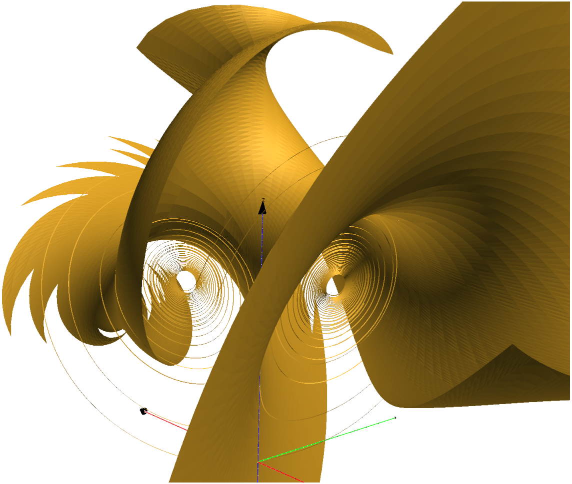


Invariant Manifolds in Lorenz



stable (2D) and unstable (1D) manifold of the Lorenz system

Invariant Manifolds in Lorenz



stable (2D) and unstable (1D) manifold of the Lorenz

Conclusion

A rigorous fixed point finder

- can automatically identify fixed points in area of interest
- utilizes functional dependence in transfer maps
- is guaranteed to find *all* periodic points
- is mathematically fully rigorous

Invariant Manifolds

- can mathematically rigorously prove existence of chaos
- can provide bounds on the island region of the transfer map

Thank You

Thank you for your attention.

Questions?