RECONSTRUCTION OF VELOCITY FIELD*

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Abstract

In this paper we suppose that the distribution density of particles in phase space is known. Using Liouville's equations the problem of finding velocity field is considered as a minimization problem. Thus the problem of determination of velocity field is reduced to solving of elliptic system of Euler-Lagrange equations.

INTRODUCTION

Different inverse problems of electrodynamics have been the subject of attention of many researchers. In particular, solving of inverse problems of electrodynamics, where by pre-assigned motions (given velocity field) electromagnetic fields were determined, had been investigated in works of G.A.Grinberg, A.R.Lucas, B.Meltzer, V.T.Ovcharov, V.I.Zubov , E.D. Kotina [1-9]. It shoud be noted, that the problem of determination of velocity field is a separate task. In particular, the task of determination of velocity field could be considered as the problem of the optimal control theory [10]. In this case it is needed to find the velocity field securing necessary beam dynamics. In this paper we suppose that the distribution density of particles in phase space is known. The problem of finding the velocity field is considered as a minimization problem. Similar problem is widely discussed in the literature for image processing based on the so-called optical flow. This approach was also used for the motion correction for radionuclide tomographic studies [11]. In this work the problem of determining the velocity field in solving the problem of charged particle beam formation in a stationary magnetic field is also considered.

PROBLEM STATEMENT

Let the dynamical system be given by a differential equation

$$\frac{dx}{dt} = f(t,x), \tag{1}$$

where t is independent variable further referred to as time, x is $n-$vector of the phase coordinates $x_1, x_2, ..., x_n$, , f is $n-$ dimensional vector function. We assume that $f(t,x)$ is defined and, together with the partial derivates $\partial f/\partial x$ it is continuous with respect to the variables t,x on $T_0 \times R^n$, where $T_0 = [t_0, T] \subset R^1$, the number t_0, T are fixed. We assume that solution $x = x(t, t_0, x_0)$ is defined on the entire interval T_0 for any $x_0 \in R^n$.

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Let us consider together equation (1) the following equation:

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x} f + \rho \cdot \mathrm{div}_x f = 0, \tag{2}$$

under initial condition

$$\rho(t_0, x) = \rho_0(x), \tag{3}$$

where $\rho_0(x) -$ given function and $\rho = \rho(t,x)$.

We will consider arbitrary region $G_{t0} \subset R^n$. Suppose that, by the system (1), G_t is the image set of G_{t0}. Using the equation (2) we obtain

$$\int_{G_t} \rho(t, x_t) dx_t = \int_{G_{t0}} \rho_0(x_0) dx_0, \qquad t \in T_0. \tag{4}$$

If for some nonnegative scalar function $\rho(t,x)$ the equality (4) holds for any $G_{t0} \subset R^n$, than let us say that the system (1) has integral invariant of order n. The function $\rho(t,x)$ is called the kernel or the density of the integral invariant [7]. From physical point of view the equality (4) may be treated as conservation of the particle (charge) mass along the trajectories of the system (1). The equation (2) is called the transport equation[10]. Note that the equation (2) is also called the generalized Liuville equation or Liuville's equation when the system (1) is given in canonical conjugate variables, here $\mathrm{div}_x f(t,x) = 0$.

Suppose that the phase trajectory family equation (1) corresponds to the set of random initial values of the coordinates with the density of probability distribution (3) and moreover $\int_{M_0} \rho_0(x_0) dx_0 = 1$. The function $\rho(t,x)$ may be treated as the variation of the density of probability distribution in time, and in the phase space of coordinates of the dynamical system (1). In this case the equation (2) is called the Fokker-Planck-Kolmogorov equation.

Further we suppose that given function $\rho(t,x)$ satisfies the transport equation (2). The problem is to restore the velocity field of the system (1), i.e. to find function $f(t,x)$. In common case, this is ill-posed problem [12]. So we will be use the method of regularization. Let us fix some moment t and formulate the problem of determination function $f(t,x)$ as a minimization problem. We will now introduce the functional

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$$J(f) = \int_M (\varphi_1 + \alpha^2 \varphi_2) dx , \qquad (5)$$

where M – region in R^n of a nonzero measure, α^2 – parameter of regularization,

$$\varphi_1 = \left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} f + \rho \cdot \mathrm{div}_x f \right)^2 , \ \varphi_2 = \sum_{i,j=1}^{n} \left(\frac{\partial f_j}{\partial x_i} \right)^2 .$$

Thus we find the velocity field in the class of smooth enough functions.

EULER-LAGRANGE EQUATIONS

Let us introduce the following operator:

$$A(f) = -\alpha^2 \Delta f - \rho \,\mathrm{graddiv}(\rho f). \qquad (6)$$

Here $f = (f_1, f_2, ..., f_n)^T$, $\Delta f = (\Delta f_1, \Delta f_2, ..., \Delta f_n)^T$. Euler-Lagrange equations for functional (5) can be represented as operator equation

$$A(f) = g , \qquad (7)$$

where $g = \rho \,\mathrm{grad} \rho_t$.

Further we presuppose that the set M has smooth enough boundary Γ and

$$f(x) = 0 , \quad x \in \Gamma . \qquad (8)$$

Operator $A(f)$ is positively definite differential operator. Indeed, if we consider corresponding scalar product than we can obtain the following:

$$(A(f), f) = -\alpha^2 \int_M (\Delta f, f) dx - \int_M (\rho \ \mathrm{graddiv}(\rho f), f) dx =$$
$$\alpha^2 \int_M \varphi_2 dx + \int_M (\mathrm{div}(\rho f))^2 dx > 0, \quad \alpha \neq 0, \quad f \neq 0.$$

The system of equations, defined by (7) is the system of strongly elliptic differential equations. It is well-known that strongly elliptic systems behave themselves as single elliptic equation from the point of view their solvability [13, 14].

It should be noted that functional (5) is the quadratic functional. It differs only by a constant from the following functional

$$J(f) = (A(f), f) - 2(g, f). \qquad (9)$$

Since operator $A(f)$ is positively definite differential operator, the solution equation (7) is the solution of minimization problem of functional (9) or functional (5) [15]. Moreover under these assumptions there exists a unique generalized solution equation (7) with boundary conditions (8). In this connection under enough smoothness coefficient of the equation (7) there exists unique classic solution of it due to the corresponding embedding theorems [15 - 17].

Let us consider the three-dimensional case, i.e., we consider the density as a function $\rho = \rho(t, x, y, z)$, and search the velocity field (1) as

$$\dot{x} = u(t, x, y, z),$$
$$\dot{y} = v(t, x, y, z),$$
$$\dot{z} = w(t, x, y, z).$$

Euler-Lagrange equations in this case have the form

$$-\alpha^2 \begin{pmatrix} u_{xx} + u_{yy} + u_{zz} \\ v_{xx} + v_{yy} + v_{zz} \\ w_{xx} + w_{yy} + w_{zz} \end{pmatrix} - \rho^2 \begin{pmatrix} u_{xx} + v_{xy} + w_{xz} \\ u_{xy} + v_{yy} + w_{yz} \\ u_{xz} + v_{yz} + w_{zz} \end{pmatrix} -$$
$$\rho \begin{pmatrix} 2\rho_x & \rho_x & \rho_x \\ \rho_y & 2\rho_y & \rho_y \\ \rho_z & \rho_z & 2\rho_z \end{pmatrix} \begin{pmatrix} u_x \\ v_y \\ w_z \end{pmatrix} -$$
$$\rho \begin{pmatrix} \rho_y & 0 & 0 \\ 0 & \rho_x & 0 \\ 0 & 0 & \rho_x \end{pmatrix} \begin{pmatrix} v_x \\ u_y \\ u_z \end{pmatrix} - \rho \begin{pmatrix} \rho_z & 0 & 0 \\ 0 & \rho_z & 0 \\ 0 & 0 & \rho_y \end{pmatrix} \begin{pmatrix} w_x \\ w_y \\ v_z \end{pmatrix} -$$
$$\rho \begin{pmatrix} \rho_{xx} & \rho_{xy} & \rho_{xz} \\ \rho_{xy} & \rho_{yy} & \rho_{yz} \\ \rho_{xz} & \rho_{yz} & \rho_{zz} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \rho \begin{pmatrix} \rho_{tx} \\ \rho_{ty} \\ \rho_{tz} \end{pmatrix}.$$

These equations may be used if we consider, for example, the distribution density of particles in configuration space.

THE BEAM FORMATION IN MAGNETIC FIELD

Here particular case of reconstruction of velocity field for solving of problem charged particle beam formation in stationary magnetic field is considered. In this case velocity field is determined in analytical form by given density distribution of charged particles.

Further the approach suggested by V. I. Zubov is used. Let us describe charged particle motion in stationary magnetic field by the system of differential equations

$$\frac{dX}{dt} = Y, \qquad \frac{d(mY)}{dt} = eY \times B, \qquad (10)$$

where we have $X = (x_1, x_2, x_3)$ as coordinate vector of particle in the Cartesian coordinate system,

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$Y = (y_1, y_2, y_3)$ as particle velocity vector, m is particle mass, e is particle charge, B is vector function, determining magnetic induction , t is time.
Let us denote η - velocity field in configuration space, then

$$\dot{X} = \eta(t, X). \tag{11}$$

In V.I. Zubov's works it has been shown, that stationary magnetic field (i.e. vector function B which satisfies Maxwell equation div $B = 0$) in which charged particles motion accords to the velocities field (11), can be represented in the form $B = -\frac{m}{e}\operatorname{rot}\eta$. This means that in configuration space charged particles trajectories determined by system (10) under the same initial conditions will coincide with trajectories of the system (11).

However, this representation does not cover all possibilities, and for practical realization has been more natural to seek the magnetic field in the form [8, 9]

$$B = -\frac{m}{e}\operatorname{rot}\eta + h\eta \,, \tag{12}$$

where $h = h(x_1, x_2, x_3)$ is an arbitrary function satisfying the following condition $\operatorname{div}(h\eta) = 0$.
Let us consider stationary axially symmetric magnetic field and cylindrical system of coordinates (r, θ, z) . The radial motion of particles in this case is described by the equation

$$\frac{dr}{dz} = f(r, z). \tag{13}$$

Equation (2) in this case has the form

$$\frac{\partial\rho(r,z)}{\partial z} + \frac{\partial\rho(r,z)}{\partial r} f(r,z) + \rho(r,z)\frac{\partial f(r,z)}{\partial r} = 0. \tag{14}$$

It is necessary to find a function $f = f(r,z)$, satisfying the equation (14).

We define the function of the density distribution in the form of a normal distribution

$$\rho(r,z) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(\frac{(r - \bar{r}(z))^2}{2\sigma^2}\right).$$

Then let $\bar{r}(z) = r_0\cos(z_0 z) + a_0$. Solving equation (14), we can show that the velocity field in this case can be represented as

$$f = r_0 z_0 \sin z - r_0 z_0 \sin(z_0 z) \times \exp\left(\frac{r^2 - 2(r_0\cos(z_0 z) + a_0)}{2\sigma^2}\right).$$

Thus, specifying the required density distribution we can determine the velocity field on which to build further magnetic field according to the algorithm described in the works [8,9].

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