AN APPLICATION OF DIFFERENTIAL ALGEBRAIC METHODS AND LIOUVILLE'S THEOREM: UNIFORMIZATION OF GAUSSIAN BEAMS

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Abstract

Most charged particle beams under realistic conditions have Gaussian density distributions in phase space, or can be easily made so. However, for several practical applications beams with uniform distributions in physical space are advantageous or even required. Liouville's theorem and the symplectic nature of beam's dynamic evolution pose constraints on the feasible transformational properties of these density distribution functions. Differential Algebraic methods offer an elegant way to investigate the underlying freedom involving these beam manipulations. Here, we explore the theory, necessary and sufficient conditions, and practicality of the uniformization of Gaussian beams from a rather generic point of view.

INTRODUCTION

Several practical applications such as irradiation of targets for isotope production, uniform irradiation of detectors for improved efficiency, irradiation of biological samples and materials for testing require manipulation of beam density distributions. Typically, these applications require uniform spatial distributions at the target location. However, most beams delivered by accelerators to these targets are Gaussian. There are several approaches for uniformization of Gaussian beams. One such method, the so-called nonlinear focusing method, uses higher order multipoles to provide a material-less, elegant, purely optical solution. Prior work done in this direction, using nonlinear focusing methods, can be found in [**?**] and references therein. In this paper we present a new approach based on differential algebraic (DA) techniques to investigate the underlying freedom involving these beam manipulations.

Background

Detailed understanding of the beam dynamics requires the study of the motion of the reference particle as well as the motion of the particle in the relative coordinates. The position and momenta are usually sufficient to describe the motion. Usually the arclength s along the reference orbit is used as the independent variable. At each point on the reference orbit it is possible to define an unique orthogonal coordinate system, denoted by $(\hat{e}_x, \hat{e}_y, \hat{e}_s)$, satisfying a certain set of conditions [**?**, **?**]. In this coordinate system the motion of the particles in the beam can be described using relative coordinates, which are given by

$$\vec{z}(s) = \begin{pmatrix} x, a = \frac{p_x}{p_0}, y, b = \frac{p_y}{p_0}, \\ l = k(t - t_0), \delta = \frac{(E - E_0)}{E_0} \end{pmatrix}$$

where the position (x, y) describe the position of the particle in the local coordinate system, p_0 is a fixed momentum and E_0 and t_0 are the energy and the time of flight of the reference particle, a and b are the momentum slopes, E is the total energy, and k has a dimension of velocity which makes l a length like coordinate. The point $\vec{z} = 0$ corresponds to the reference particle.

Let position s_i, s_f be the initial and final position on the reference orbit. The transfer map or transfer function $\mathcal{M}$ relates initial conditions at s_i to the conditions at s_f via

$$\vec{z}(s_f) = \mathcal{M}(s_i, s_f)(\vec{z}(s_i)). \quad (1)$$

For weakly non-linear systems, like an accelerator system, the map can be expanded as a Taylor series. Implementation of such a map on a computer would require the map to be truncated at a certain order. A detailed discussion of the properties and use of the Taylor transfer maps can be found in [**?**].

Beam Phase Space Density Function

Beam production mechanism usually determines the phase space density function describing the distribution of particles in the beam. Let function $f(\vec{z}_i)$ be the initial phase space density function of the beam. According to Liouville's theorem, as long as the sytem can be considered a Hamiltonian system, the phase space distribution of the beam will stay constant along the trajectories. It also implies that the the volume of phase space occupied by the beam is conserved. Hence, it can be written that

$$f(\vec{z}_i) = g(\vec{z}_f), \quad (2)$$

where g is the final phase space density function at any point s_f along the reference orbit. In terms of the transfer map of the system (1), (2) becomes

$$g(x_f, a_f, y_f, b_f, \delta_f) = f \circ \mathcal{M}(s_i, s_f)^{-1}(x_f, a_f, y_f, b_f, \delta_f) \quad (3)$$

where $(x_f, a_f, y_f, b_f, \delta_f)$ are the initial and final phase space coordinates. The function $g(x_f, a_f, y_f, b_f, \delta_f)$ is the new phase space density function, $\mathcal{M}(s_i, s_f)$ is the transfer map of the system. For most practical application the

quantity that is more useful is the density function in position variables. This can be obtained by integrating the phase space density function, g, with respect to the momentum and energy spread variables,

$$\rho\left(x_f, y_f\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(x_f, a_f, y_f, b_f, \delta_f\right)\cdot da_f \cdot db_f \cdot d\delta_f$$

Gaussian Density Function Most charged particle beams under realistic conditions have Gaussian density distributions in phase space, or can be easily made so. To keep the discussion reasonably simple, we restrict ourselves to an initial uncorrelated Gaussian distribution. This is without loss of generality, since the formalism can be applied essentially without change to the general, correlated Gaussian situation. The Gaussian density function in phase space, $f\left(x, a, y, b, \delta\right)$, is given by

$$f\left(x, a, y, b, \delta\right) = K\exp\left(\begin{array}{c} -\frac{(x-\mu_x)^2}{2\sigma_x^2} - \frac{(a-\mu_a)^2}{2\sigma_a^2} \\ -\frac{(y-\mu_y)^2}{2\sigma_y^2} - \frac{(b-\mu_b)^2}{2\sigma_b^2} - \frac{(\delta-\mu_\delta)^2}{2\sigma_\delta^2}\end{array}\right)$$

where μ_x,μ_a,μ_y,μ_b,μ_δ and σ_x,σ_a,σ_y,σ_b,σ_δ are the means and variances in x, a, y, b, δ respectively, and the factor $K = 1/\left((2\pi)^{5/2}\sigma_x\sigma_a\sigma_y\sigma_b\sigma_\delta\right)$ is a normalization constant.

Computation of Beam Density Function Using Differential Algebra

For a beam with initial Gaussian density distribution in phase space a DA based technique has been developed to compute the density function at end of a beam-optics system, given by a transfer map $\mathcal{M}$. For simplicity we describe the technique for a one dimensional case. The initial phase space density function at point s_i is given by,

$$f\left(x_i, a_i\right) = K_2\exp\left(-\frac{\left(x_i-\mu_{x_i}\right)^2}{2\sigma_{x_i}^2} - \frac{\left(a_i-\mu_{a_i}\right)^2}{2\sigma_{a_i}^2}\right), \tag{4}$$

where $K_2 = 1/\left(2\pi\sigma_{x_i}\sigma_{a_i}\right)$. Without loss of generality we can assume $\mu_{x_i} = \mu_{a_i} = 0$. We denote the exponential in the above equation as $h\left(x, a\right)$,

$$h\left(x_i, a_i\right) = -\frac{1}{2}\left(\frac{x_i^2}{\sigma_{x_i}^2} + \frac{a_i^2}{\sigma_{a_i}^2}\right).$$

To compute the density function, $\rho\left(x_f\right)$, at point s_f on the reference orbit, we need to solve the integral

$$\begin{aligned}\rho\left(x_f\right) &= \int_{-\infty}^{\infty} K\exp\left(h\right)\circ\mathcal{M}\left(s_i, s_f\right)^{-1}\left(x_f, a_f\right)da_f \\ &= \int_{-\infty}^{\infty} K\exp\left(h\circ\mathcal{M}\left(s_i, s_f\right)^{-1}\right)\left(x_f, a_f\right)da_f.\end{aligned} \tag{5}$$

Since the integration limits tend to infinity the above integral is difficult to solve directly using numerical techniques. By changing the form of the integral to a Gaussian integral and applying a perturbation method to evaluate the the integral term-by-term, one can use the following closed form solution for each term [**?**]:

$$\int_{-\infty}^{\infty} z^n\exp\left(-\alpha z^2 + \beta z + \gamma\right)dz = \sqrt{\frac{\pi}{\alpha}}\exp\left(\frac{\beta^2}{4\alpha} + \gamma\right)\sum_{k=0}^{|n/2|}\frac{n!}{k!\left(n-2k\right)!}\frac{\left(2\beta\right)^{n-2k}}{\left(4\alpha\right)^{n-k}}. \tag{6}$$

Below we describe the steps to change the form of the integral in equation 5:

1. Compute the inverse map, $\mathcal{M}\left(s_i, s_f\right)^{-1}$, to a pre selected truncation order N [**?**].

2. Compute $h\circ\mathcal{M}\left(s_i, s_f\right)^{-1}$ and arrange the resulting polynomials in powers of a_f by collecting its coefficients A_n. We express this as a sum of second order polynomial and remainder term $\mathcal{R}_3$,

$$h\left(x_i, a_i\right)\circ\mathcal{M}\left(s_i, s_f\right)^{-1} = A_0\left(x_f\right) + A_1\left(x_f\right)a_f + A_2\left(x_f\right)a_f^2 + \mathcal{R}_3\left(x_f, a_f\right)$$
$$\mathcal{R}_3\left(x_f, a_f\right) = \sum_{n=3}^{N} A_n\left(x_f\right)a_f^n.$$

3. Expand the exponential of the remainder term $\mathcal{R}_3\left(x_f, a_f\right)$ in Taylor series and arrange the resulting polynomial in powers of a_f by collecting its coefficients C_j, $\exp\left(\mathcal{R}_3\left(x_f, a_f\right)\right) = \sum_{j=3}^{N} C_j\left(x_f\right)a_f^j$. The equation 5 can now be rewritten as,

$$\rho\left(x_f\right) = K\sum_{j=3}^{N} C_j\left(x_f\right)\int_{-\infty}^{\infty} a_f^j\exp\left(A_0\left(x_f\right) + A_1\left(x_f\right)a_f + A_2\left(x_f\right)a_f^2\right)da_f. \tag{7}$$

The equation is the desired form where we can utilize the Gaussian integral formula, equation 6. The same procedure, essentially without any change, will work for the 5D case too, where the steps shown above are performed one-by-one for each integration variable a, b and δ.

APPLICATIONS

We consider a system with an octupole magnet and a quadrupole magnet, separated by a 20cm drift. The setup is followed by a 110 cm drift to the final image. Each magnet is 25cm in length and 20cm full aperture. Both the transverse beam optics, in coordinates (x, a, y, b), and kinetic energy spread are considered for the simulation. A

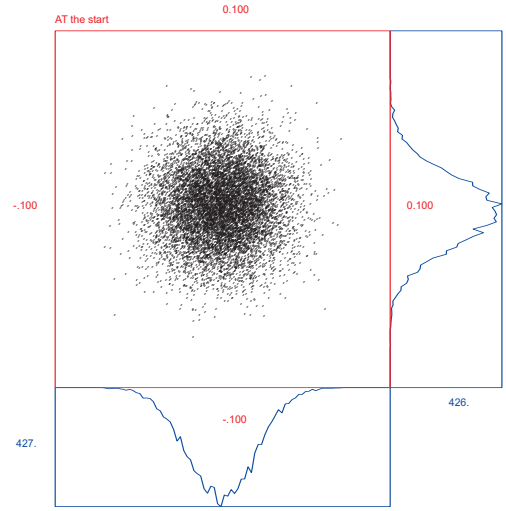

Figure 1: XY distribution plot and the histogram plot of 10000 particles at the entrance of the system.The beam has Gaussian density distribution in phase space.

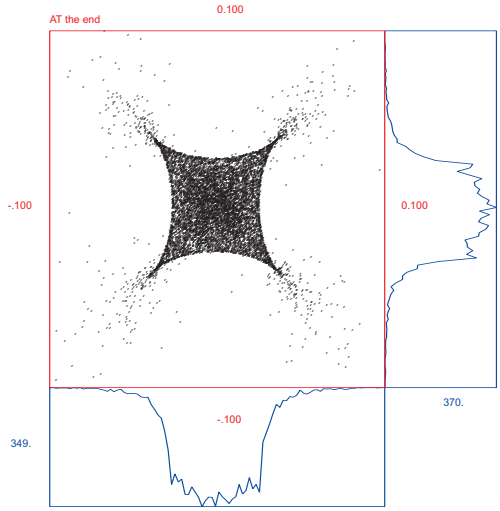

Figure 2: XY distribution for 10000 particles at the final image.

100MeV proton beam with RMS emittance, $\varepsilon_x = \varepsilon_y = 2 \times 10^{-5}$ and 5% kinetic energy spread is considered. Using the DA technique described in the section , the beam density function at the final image is computed as a Taylor expansion in position variables. By using the poletip field of the octupole magnet as a fit parameter, the lower order coefficients in the Taylor expansion of the beam density function are minimized leading to a near uniform beam density distribution with resonable octupole strengths. Figures 1 and 2 show the $x - y$ distribution and histogram in x and y for the beam at the start and end of the system. It can be noticed from the figures that the beam density function changes from a Gaussian distribution to a near uniform one. The uniformity can be improved in principle by the use of even higher order multipoles, if necessary.

SUMMARY

We showed that DA-based methods are powerfull and very general methods for applications to beam uniformization of Gaussian beams. The main ingredients of the method are DA-based methods that allow transfer map computation and inversion of arbitrary lattice maps, Liouville's theorem, and a novel application of closed form Gasussian integrals to this setting. Unlike previous methods, our method is free of assumed correlations in the initial beam distributions, it is as easy to work with in 5D as it is in 2D, takes into account arbitrary lattice nonlinearities to any desired order, and the optimization of the system can be automated. The method has been implemented in the code COSY Infinity [**?**, **?**]. More detailed studies and applications, including the available freedom in phase space manipulations will be published in forthcoming papers.

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