Instabilities and Space Charge

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- Introduction
- Modeling the Transverse Force
- Beam Transfer Functions
- Data and Comparison with simulations

The Transverse Force

The direct space charge force is not trivial.

$$\rho(x,y,z,t) \qquad \vec{J}(x,y,z,t) \rightarrow Full \;\; Maxwell \;\; Equations$$

One boosts to the center of mass frame.

In this frame motion is usually non-relativistic (bends?)

Then, one assumes electrostatics

$$\vec{J}(x,y,z-vt) = v\vec{z}\rho(x,y,z-vt)$$

Now "easy", solve the Poisson equation.

$$E + v \times B = \frac{E}{\gamma^2}$$

How many updates per betatron oscillation?

How many longitudinal bins along the bunch?

How many macroparticles?

Smoothing length and effective IBS rates. (LANL PSR ecloud10)

Transverse Nonlinearity

Consider the 1D system

$$\frac{d^2 x_j}{d\theta^2} + Q_j^2 x_j - C_O x^3 = 2\kappa \sum_{k=1}^{N_p} \left(x_j - x_k\right)\left(1 - \frac{4\left(x_j - x_k\right)^2}{3\sigma^2}\right) + W\bar{x} + \alpha \frac{d\bar{x}}{dt}$$

This can be solved in Vlasov limit with the single side band approximation. The coherent tune for the perturbed distribution is ν and is related to the coherent tune for no damping ν_0 via:

$$\nu_0 = -\kappa + \left(\frac{2\kappa}{\sigma^2}\right)^2 I_3(\nu) - \frac{\left[1 - \frac{2\kappa}{\sigma^2} I_2(\nu)\right]^2}{I_1(\nu)}$$

$$I_k(\nu) = \int_0^\infty dJ\, J^k \frac{d\hat{F}_0(J)}{dJ} \int_{-\infty}^{\infty} d\Delta \frac{\rho(\Delta)}{\nu - \Delta - \frac{dU_0}{dJ}}$$

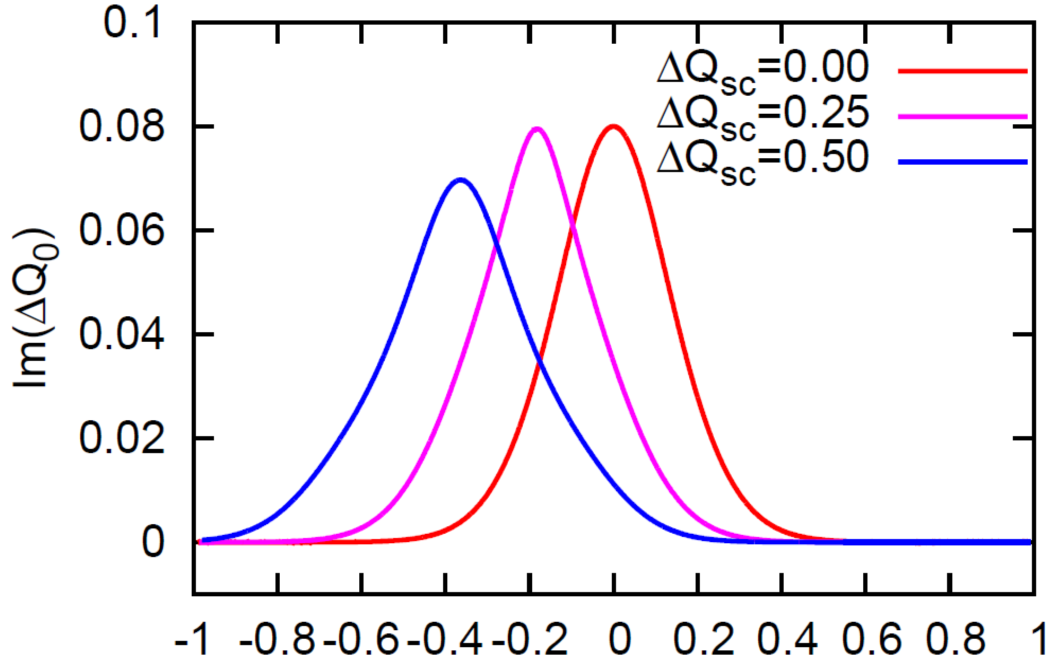
The stability threshold can be obtained by parametrically plotting ν_0 as a function of real ν.

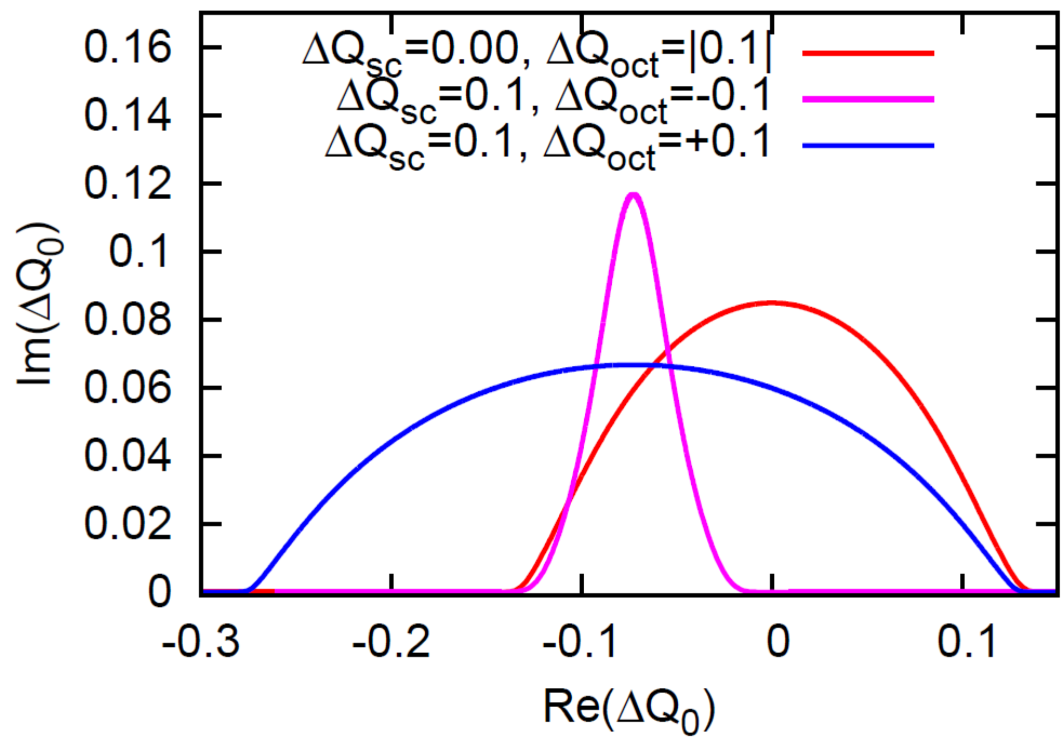
Suppose a beam with no damping would have a complex tune shift ΔQ_0.

Then with damping a beam would be stable if it had a tune shift below the curve.

Tune spread matches a gaussian beam.

Is this real?





Simulations in 2 transverse variables.

Used FFT convolution algorithm with 128x128 grid on beam

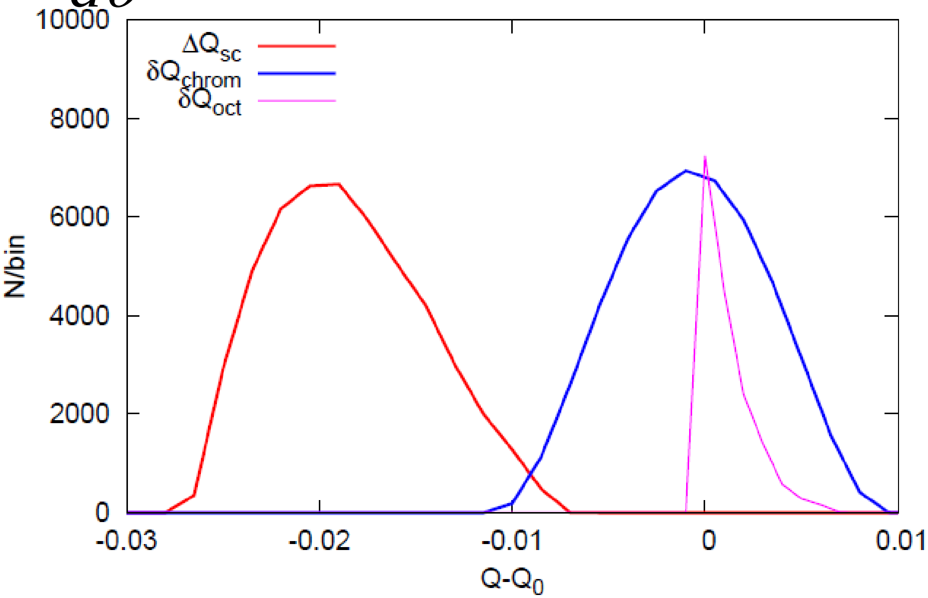
50,000 particles and 30 updates per betatron oscillation.

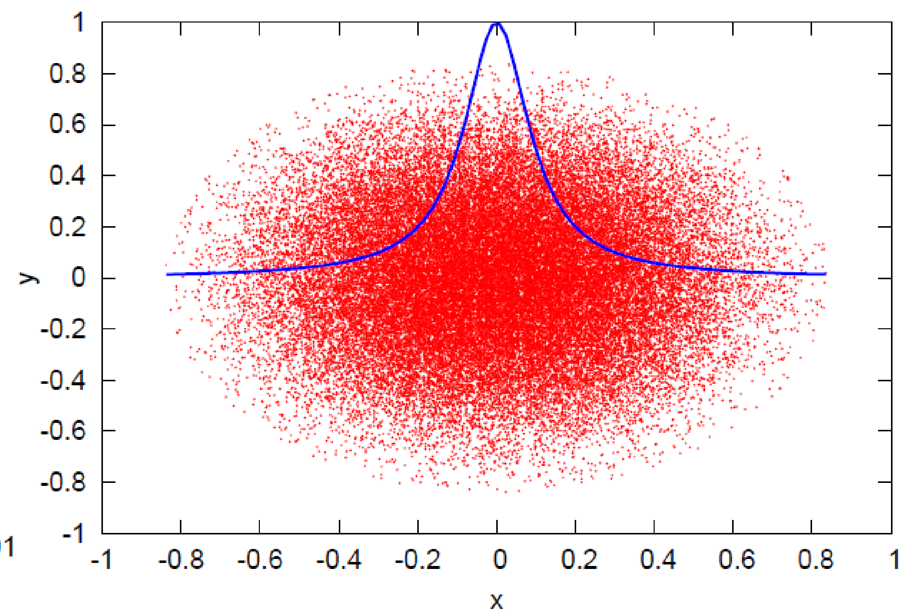
Insensitive to factor of 5 changes in update rate or Np.

$$\frac{dp_j}{d\theta} + Q_x x_j = C_o x_j (x_j^2 + p_j^2) - \delta Q_j x_j$$

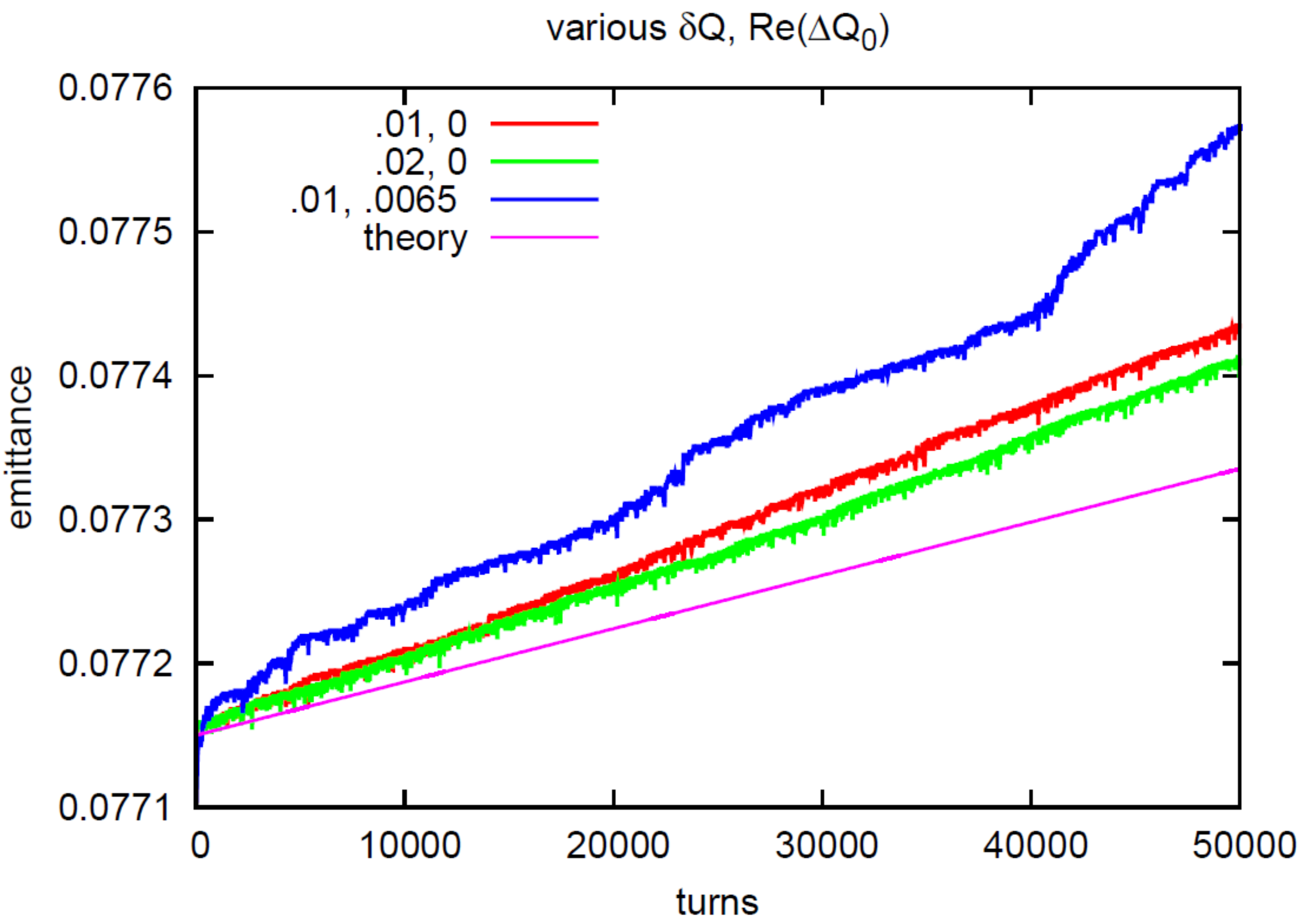
$$+ C_{sc} \sum_{m=1}^{N_p} \frac{x_j - x_m}{\varepsilon^2 + (x_j - x_m)^2 + (y_j - y_m)^2} + 2\,\mathrm{Im}(\Delta Q_0)\,\overline{p} - 2\,\mathrm{Re}(\Delta Q_0)\,\overline{x}$$

$$\frac{dx_j}{d\theta} - Q_x p_j = -C_o p_j (x_j^2 + p_j^2) + \delta Q_j p_j$$

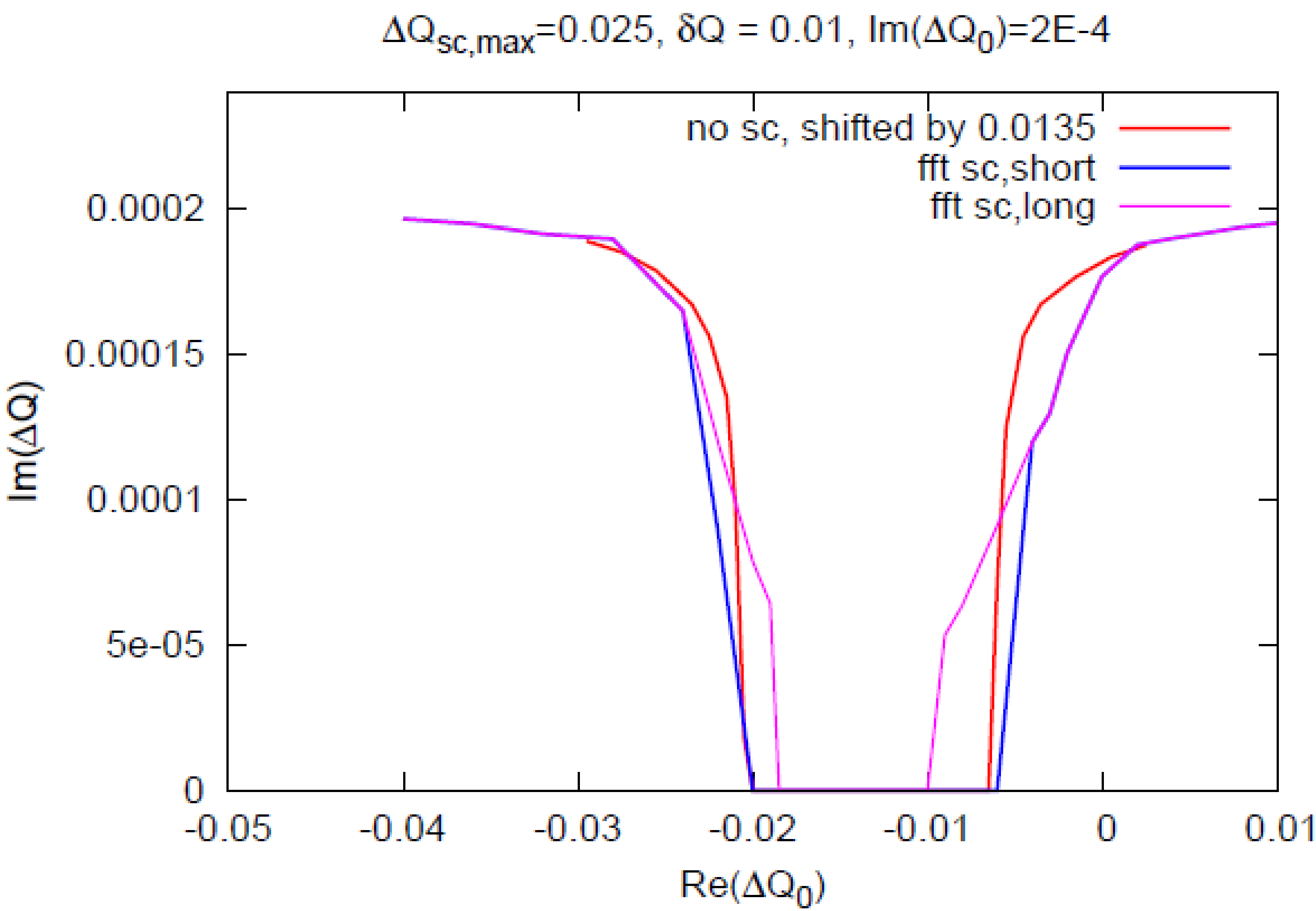




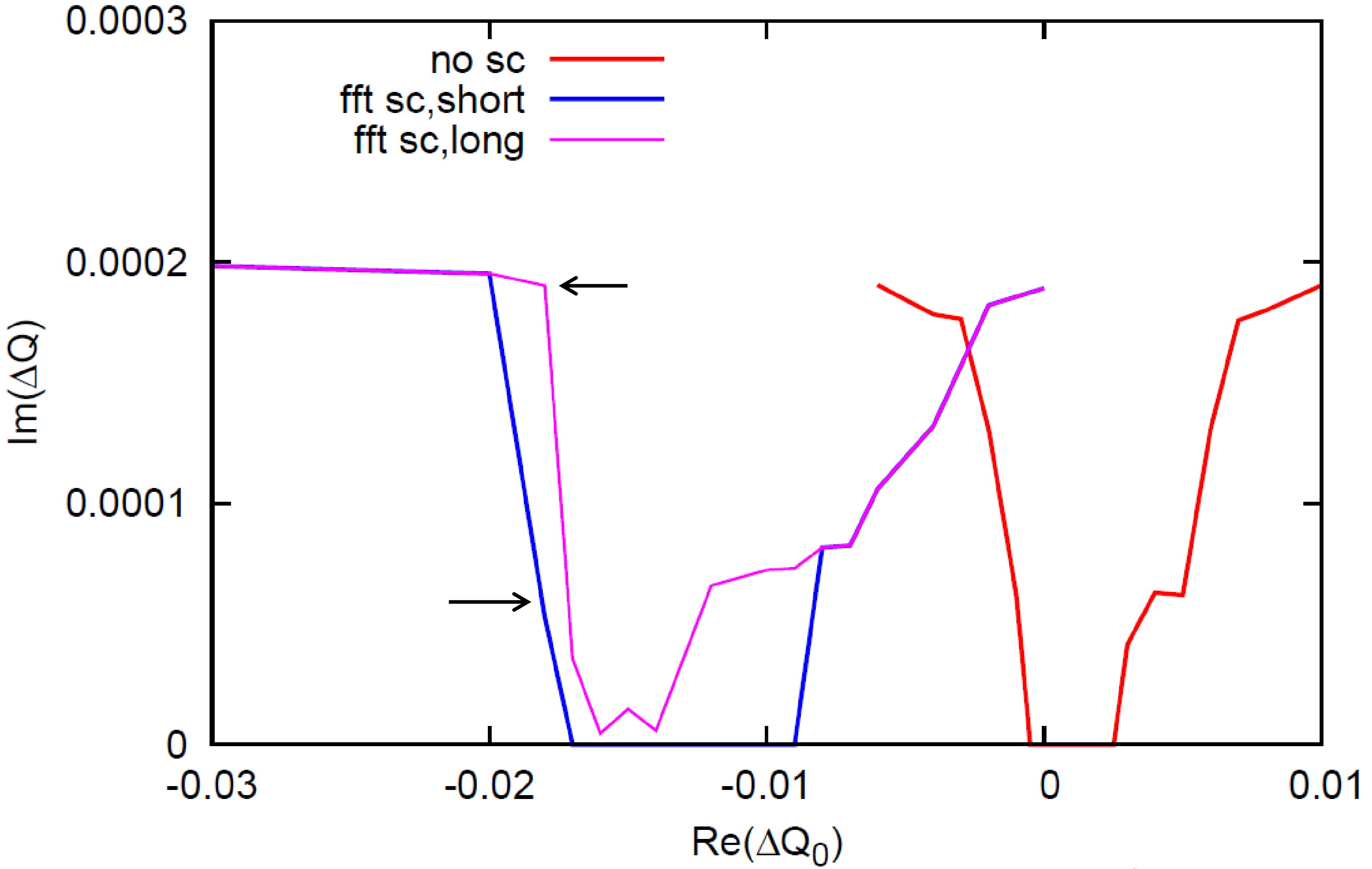
Purely linear first. Stochastic cooling theory says to expect $\mathrm{Im}(Q) \approx \mathrm{Im}(\Delta Q_0)/N_p$ for stable beams. Blue curve is just stable.



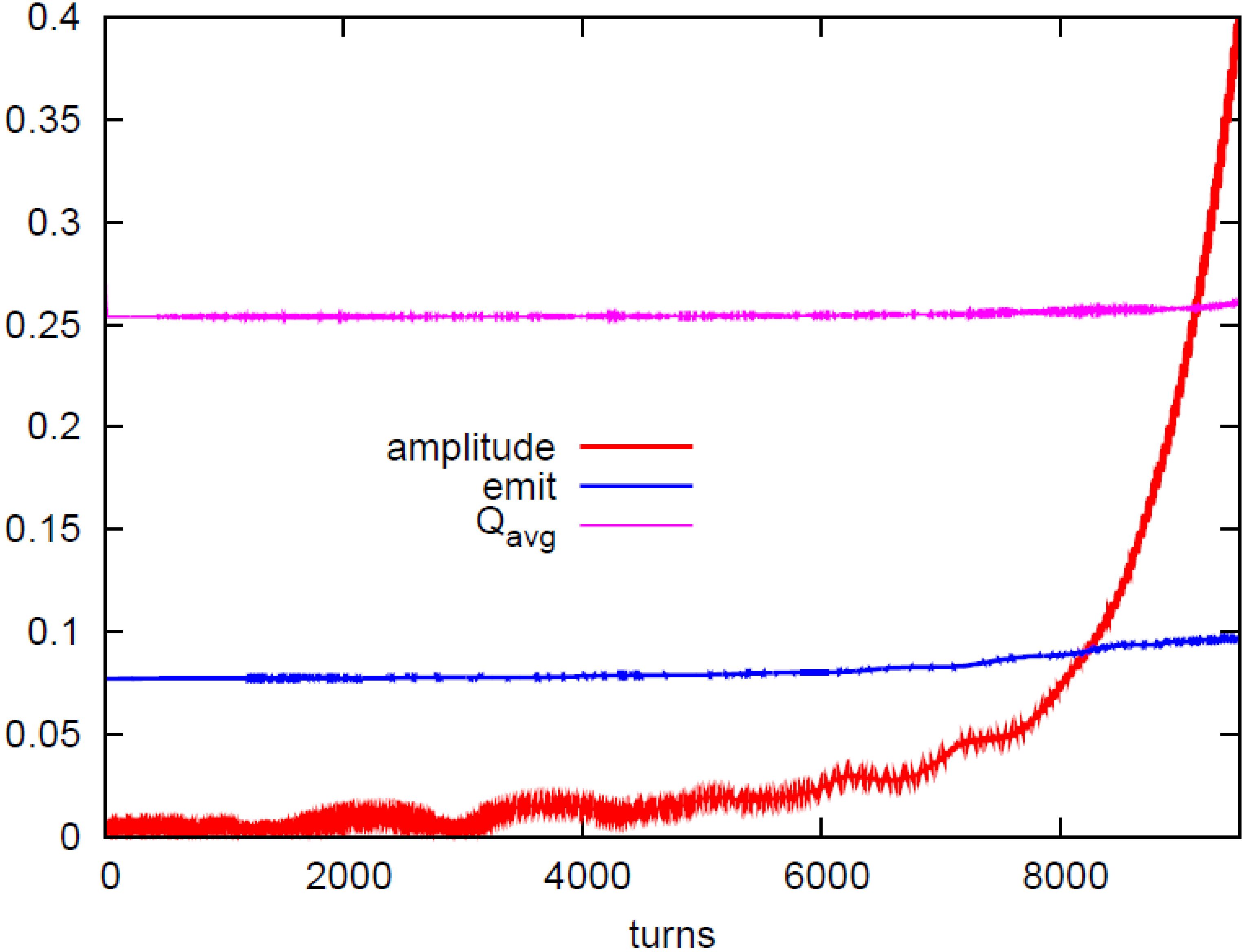
- Growth rate as a function of wall induced tune shift for space charge and tune spread due to chromaticity.
- Growth rates with 5000 and 50,000 turns compared with shifted cure and no space charge



- Growth rate with octupolar detuning and space charge
- 5000 and 50,000 turns
- Growth rates calculated before significant emittance increase observed.

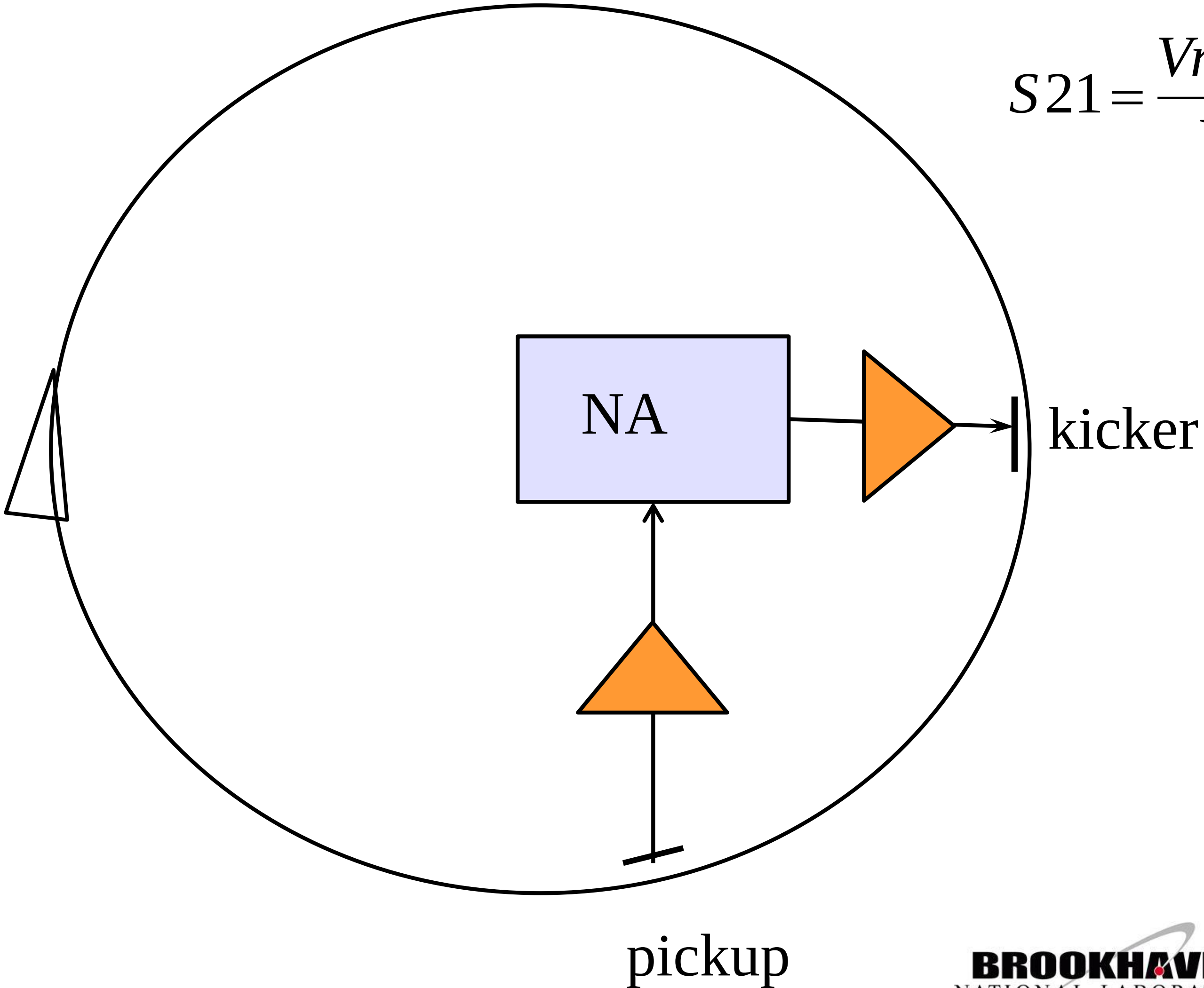


- Simulation used for -0.018 points on previous curves
- 1st 5000 turns give Im(Q)=0.53E-4, last 1000 Im(Q)=1.9E-4



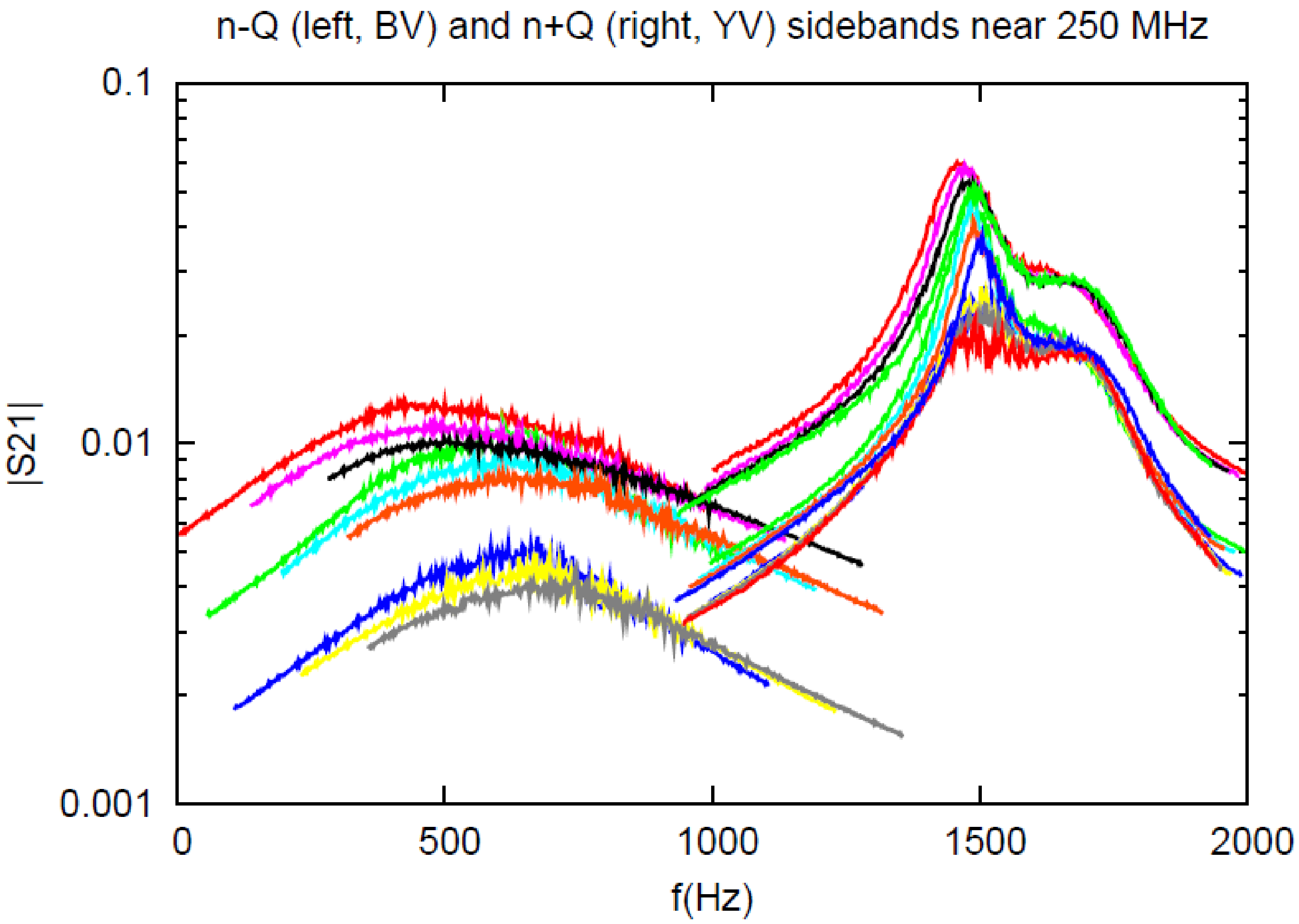
- Basic idea behind beam transfer functions (BTFs)
- The network analyzer (NA) steps through frequencies
- Amplitude and phase of received signal are recorded

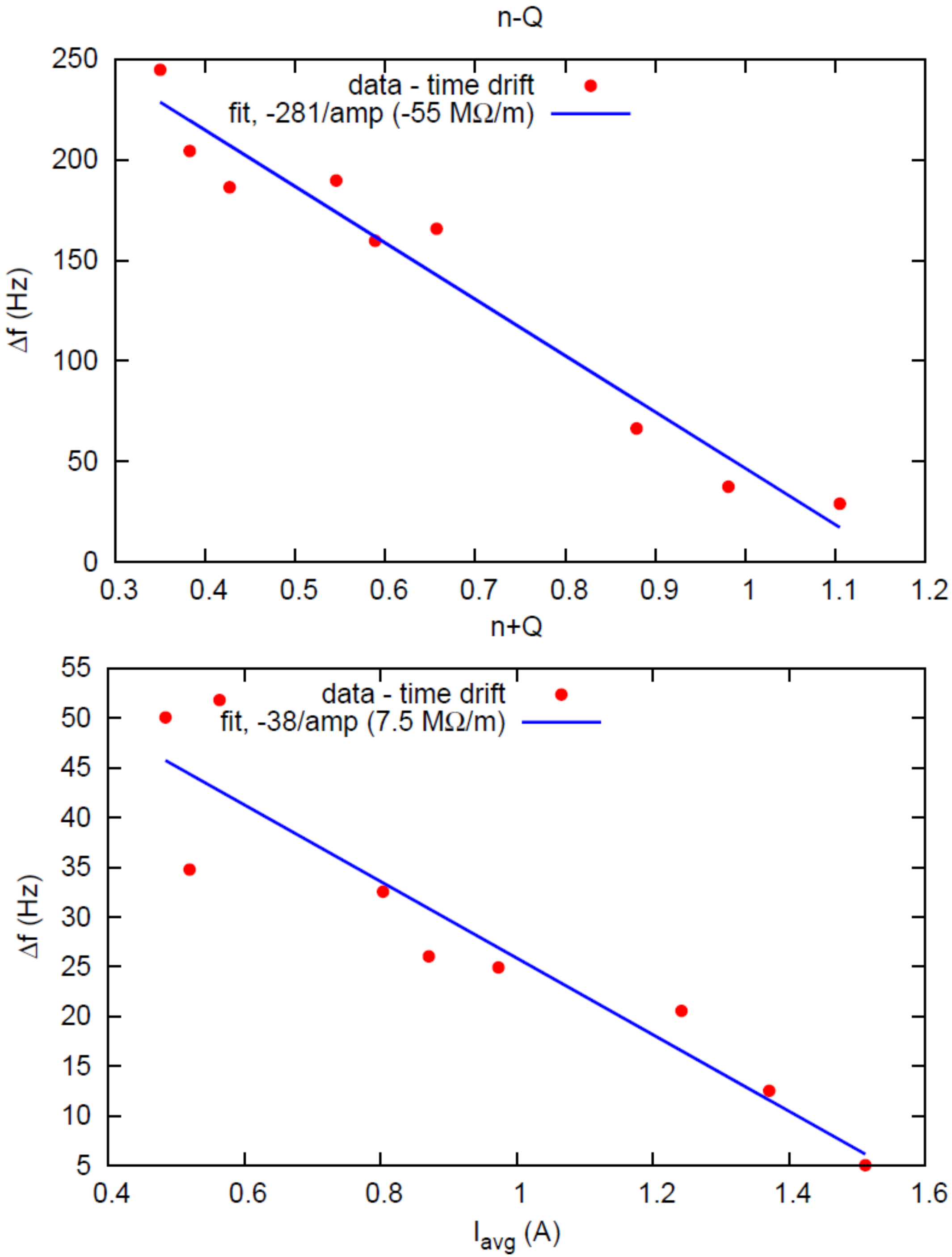
$$S21 = \frac{Vreceived}{Vdrive}$$



- Protons at γ=25.5 in RHIC
- Average bunch currents between 0.3 and 1.5 amps

$$I_{avg} = \frac{\int_{bunch} I^2(t)dt}{\int_{bunch} I(t)dt}$$





- Peaks were fitted with a parabolic cap to obtain the center frequency.
- Notice difference in scales
- Q increases with I_{avg} for n-Q sideband
- Q decreases with I_{avg} for n+Q sideband
- This behavior is consistent with a coasting beam dispersion relation when Z_{sc} and Z_{wall} are included.

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More About BTFs

Consider a horizontal kicker and pickup.

Transverse voltage $V_x(t) = V_0 \cos(\omega t) e^{\alpha t} = \int ds \left(E + v \times B\right)_x$

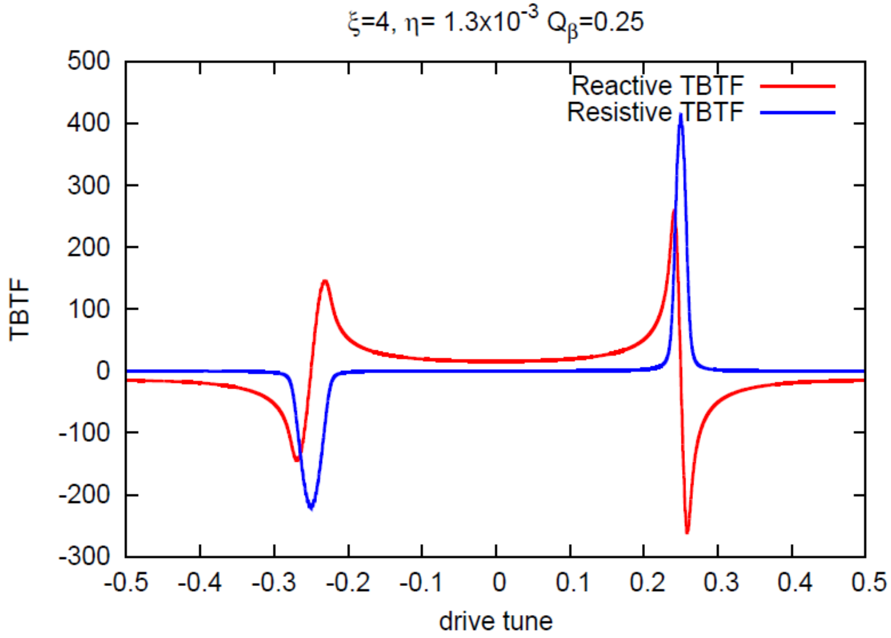
Growing kicks are used for finite resolution bandwidth or to allow the study of growing modes, α<0 often ill-defined.

There is a response at the pickup,

$$x(t)I(t) = V_0 X(\omega)\cos(\omega t)e^{\alpha t} - V_0 R(\omega)\sin(\omega t)e^{\alpha t} + aliases$$

X(ω) is the reactive part of the BTF and R(ω) is the resistive part.

Kick used in plot has 360° of phase advance across the bunch, $\alpha > \omega_s$



Simulation outline

Only consider broad band transverse impedance here

$$\frac{d^2 x_j}{d\theta^2} + Q^2(\varepsilon_j)x_j = \kappa \sum_{k=1}^{N} \left\{(x_j - x_k)Z_{sc} + x_k Z_{wall}\right\}\lambda(\tau_j - \tau_k) + F_e(\theta, \tau_j)$$

$$\frac{d\varepsilon_j}{d\theta} = \frac{qV(\tau_j)}{2\pi}, \qquad \frac{d\tau_j}{d\theta} = \frac{T_{rev}}{2\pi}\eta \frac{\varepsilon_j}{\beta^2 E_T}$$

Update 10 times per turn with $1 < Q_{sim} < 2$ spot checked with 20

Multiharmonic RF but no longitudinal kick from Z_{wall} or Z_{sc}

There is a code and user manual TRANFT which is close.

$$xI = D = M\left[F - WD\right]$$

Principle involved:

$$D = \left(1 + MW\right)^{-1} MF$$

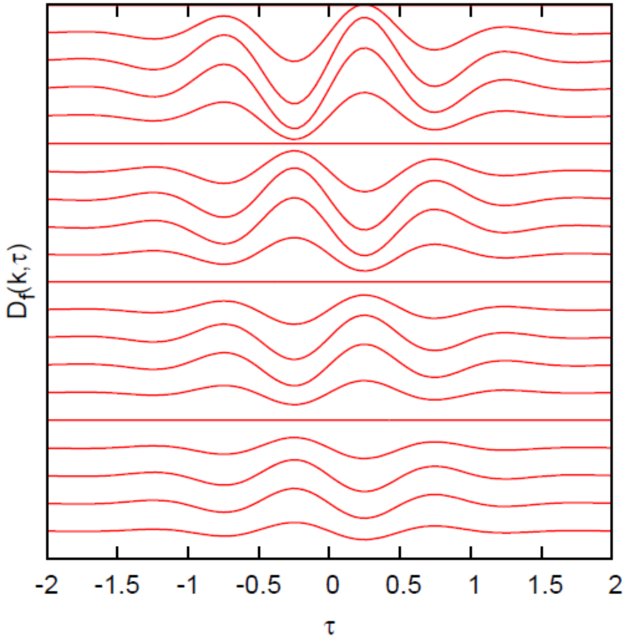
$$BTF(\omega) = \frac{\tilde{D}(\omega)}{\tilde{F}(\omega)}$$

Calculating BTFs using simulations (NAPAC13 with Vahid Ranjbar)

Starter problem: Consider a kicker giving a kick f(τ) as the bunch passes on turn 0. Let $D_f(k, \tau)$ be the beam response (dipole moment) to this kick on turns k=0,1,…

Now take the forcing function $F(n,\tau) = \cos(2\pi n Q) f(\tau) e^{gn}$

Assuming linear response



$$D(n,\tau) = \sum_{k=-\infty}^{\infty} D_f(n-k,\tau)\cos(2\pi k Q)e^{gk}$$

$$= \sum_{m=0}^{\infty} D_f(m,\tau)\cos[2\pi Q(n-m)]e^{g(n-m)}$$

$$= \mathrm{Re}\left\{ e^{2\pi j Q n + gn} \sum_{m=0}^{\infty} D_f(m,\tau) e^{-2\pi j Q m - gm} \right\}$$

Hence, only one simulation is needed for all Q.

Calculating BTFs using simulations

In a real BTF with $|\omega_1| < \omega_0/2 = \pi/T_{\mathrm{rev}}$

$$F(k,\tau) = A\cos[(n\omega_0 + \omega_1)(kT_{rev} + \tau)]$$
$$= A\cos[n\omega_0\,\tau + \omega_1 kT_{rev} + \omega_1\tau], \qquad |\,\omega_1\,\tau\,| \le \pi/2h$$
$$\approx A\cos[n\omega_0\,\tau + \omega_1 kT_{rev}]$$
$$= A\cos(\omega_1\,kT_{rev})\cos(n\omega_0\tau) - A\sin(\omega_1 kT_{rev})\sin(n\omega_0\tau)$$

Hence we need the impulse response to a sine kick and a cosine kick to get the response for all frequencies.

Also, BTF is one complex number and not a function of τ.

Take the Fourier component of the beam response at the drive frequency.

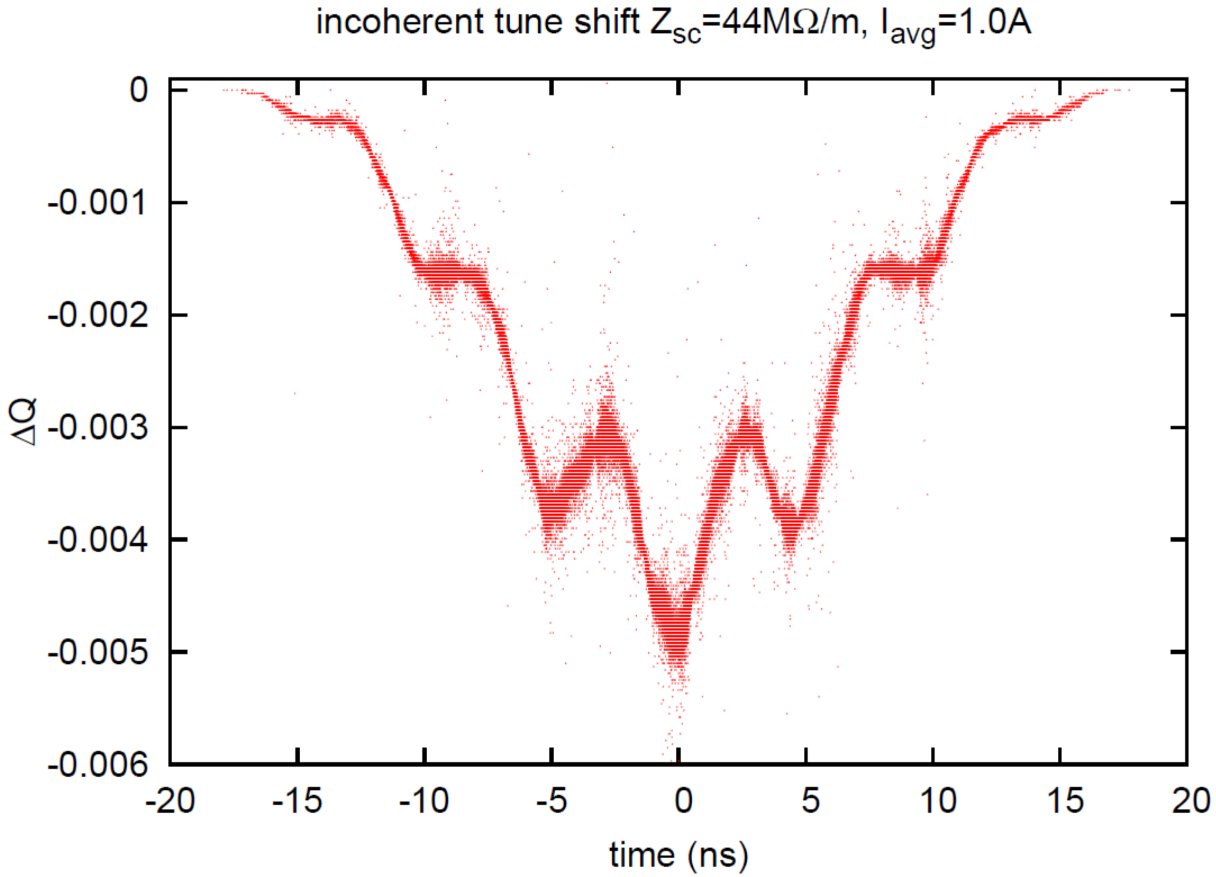
$$BTF(\omega_1) = \sum_{m=0}^{\infty} e^{-(j\omega_1+\varepsilon)mT_{rev}} \int_{-\tau_b}^{\tau_b} e^{-jn\omega_0\tau}\left\{D_{\cos}(m,\tau) + jD_{\sin}(m,\tau)\right\}d\tau$$

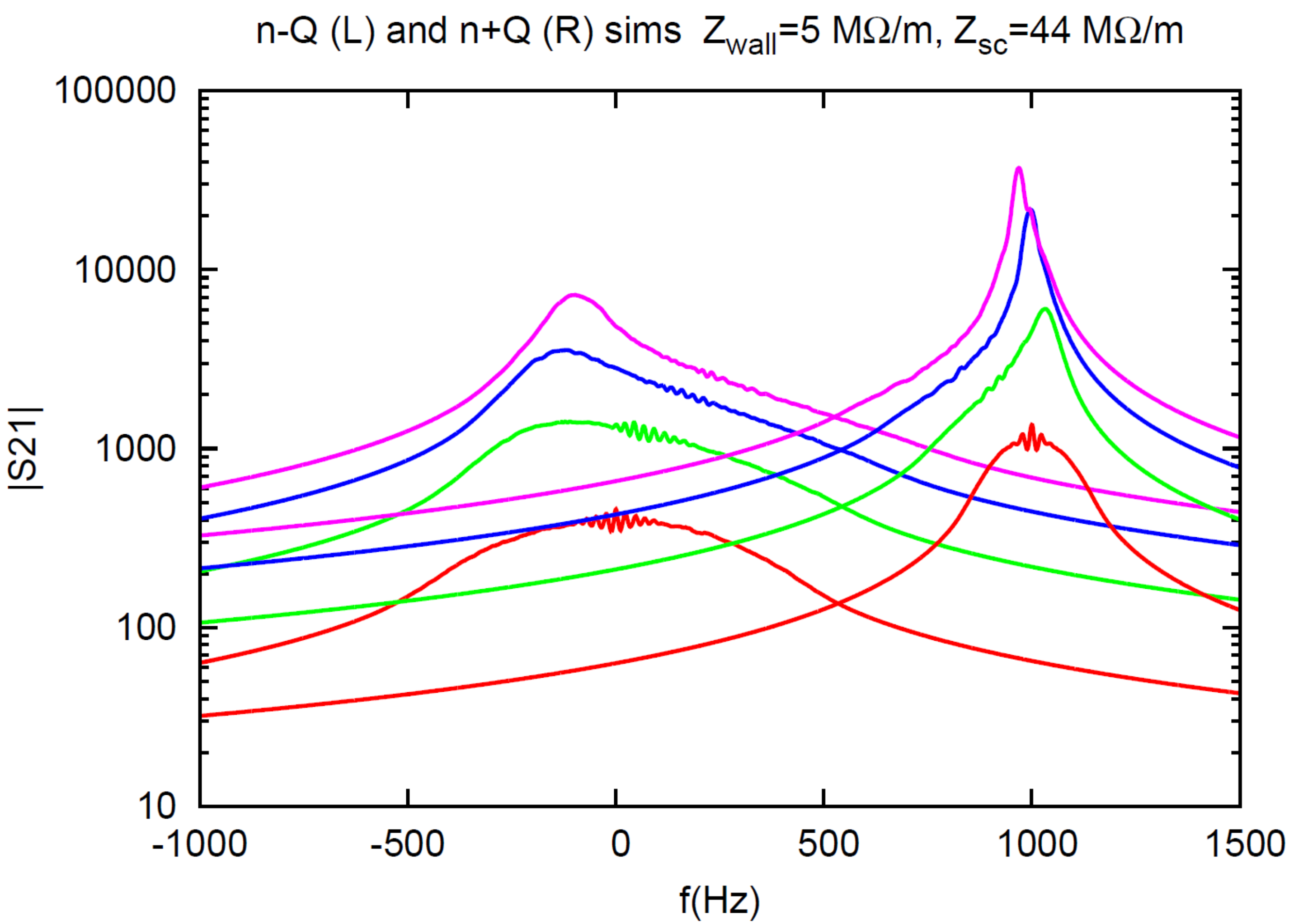
Measuring tunes in multiparticle simulations without FFTs

Identical updates 10 times per turn. For each particle minimize:

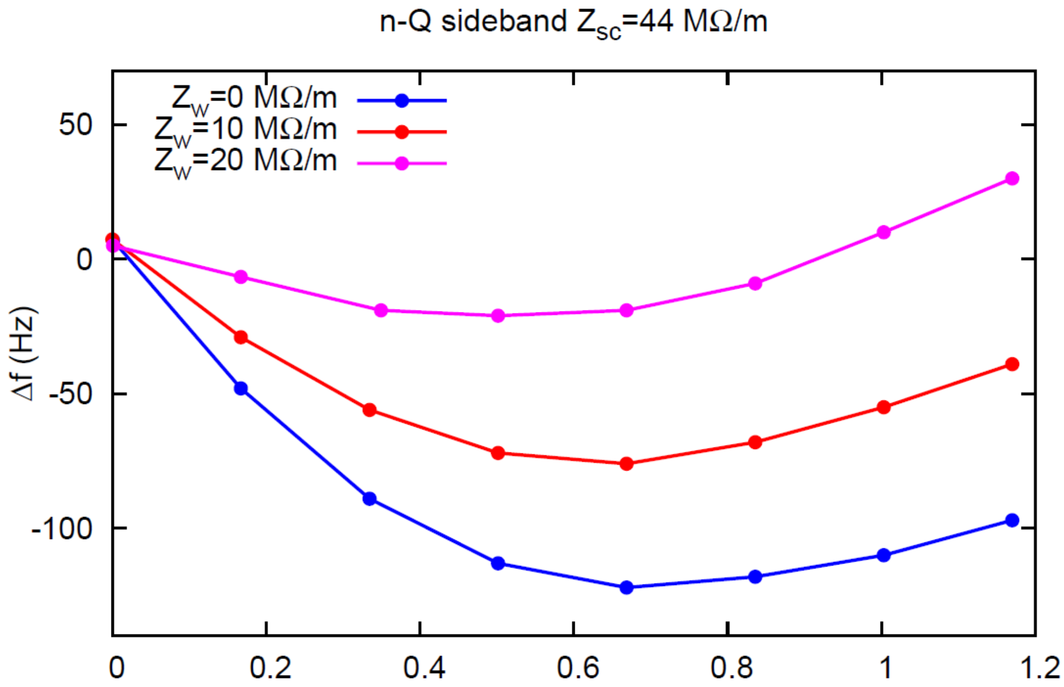
$$L=\sum_{k=1}^{N_{turn}-1}\left(x_{k+1}-a_{11}x_k-a_{12}p_k\right)^2+\left(p_{k+1}-a_{21}x_k-a_{22}p_k\right)^2 \qquad set \qquad \frac{\partial L}{\partial a_{ij}}=0$$

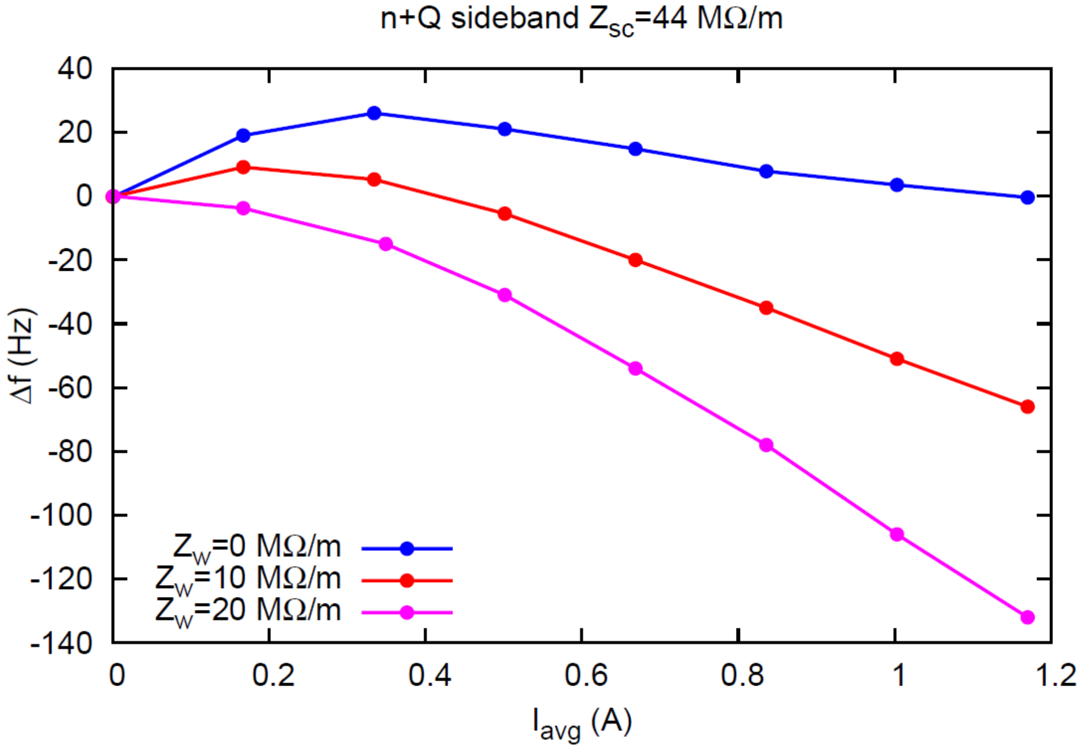
Solve for the a_{ij}s and get the particle tune from the matrix.





- BTFs with I_{avg} = 1.0, 0.66, 0.33, 0.1 Amps
- No shoulder on the n+Q sideband.





- Neither sideband shows linear behavior for any set impedance.
- RHIC beam probably had intensity dependent emittance, varying Z_{sc}.
- RHIC rings probably have different Z_{wall}.

Conclusions

- Space charge forces are still tricky.
- The effect of space charge nonlinearity on long term behavior is not settled. This is a fundamental question we need to answer.
- Getting solutions with linear transverse forces is fairly straightforward.
- There are qualitative differences between the data and the simulations (shoulder absent in sims).
- Simulations with PIC codes require a serious commitment.
- If form of nonlinearity is unimportant a fast algorithms exists for

$$F_j = C_{sc} \sum_m \sin[k(x_j - x_m)] \lambda(\tau_j - \tau_m)$$

- Perhaps PIC verification could be done for an ‘easy’ case

Growth rate scaling with N

$$\ddot{x}_j + \Omega_j^2 x_j = -\frac{2g\Omega_0}{N}\sum_{k=1}^{N}\dot{x}_k$$

$$\Omega_j = \Omega_0 + \omega_j, \qquad |\omega_j| << \Omega_0$$

$$x_j = a_j \exp(-\lambda t - i\Omega_0 t)$$

$$(\lambda - i\omega_j)a_j = \frac{g\Omega_0}{N}\sum_{k=1}^{N} a_k$$

$$1 = \frac{g\Omega_0}{N}\sum_{k=1}^{N}\frac{1}{\lambda - i\omega_k}$$

$$\int_{-\infty}^{\omega_k} f(\omega)d\omega = \frac{k-1/2}{N}$$

$$\lambda \approx i\omega_K, \quad \Delta\omega = \frac{1}{Nf(\omega_K)}$$

$$\begin{aligned}
&\sum_{m=1}^{N}\frac{1}{\lambda - i\omega_m} \\
= \quad & \sum_{|m-K|<M}\frac{1}{\lambda - i\omega_m} + \sum_{|m-K|\geq M}\frac{1}{\lambda - i\omega_m} \\
\approx \quad & \sum_{|m|<M}\frac{1}{\lambda - i\omega_K - im\Delta\omega} + \sum_{|m-K|>M}\frac{i}{\omega_m - \omega_K} \\
\approx \quad & \sum_{k=-\infty}^{\infty}\frac{1}{\lambda - i\omega_K - ik\Delta\omega} \\
+ \quad & iN\int_{-\infty}^{\infty}\frac{\omega - \omega_K}{0^+ + (\omega - \omega_K)^2}f(\omega)d\omega. \qquad (13)
\end{aligned}$$

$$\lim_{M\to\infty}\sum_{k=-M}^{M}\frac{1}{z - ik} = \pi\frac{\exp(2\pi z)+1}{\exp(2\pi z)-1},$$

Growth rate scaling with N, II

Mixing and signal shielding are fully accounted for.

$$R(\omega_K) = \pi\Omega_0 f(\omega_K) \qquad X(\omega_K) = \Omega_0 \int_{-\infty}^{\infty} \frac{\omega - \omega_K}{0^+ + (\omega - \omega_K)^2} f(\omega) d\omega$$

$$\exp[2\pi N f(\omega_K)(\lambda - i\omega_K)] = \frac{1 + gR - igX}{1 - gR - igX} \approx 1 + 2gR \rightarrow \lambda \approx g\Omega_0 / N$$

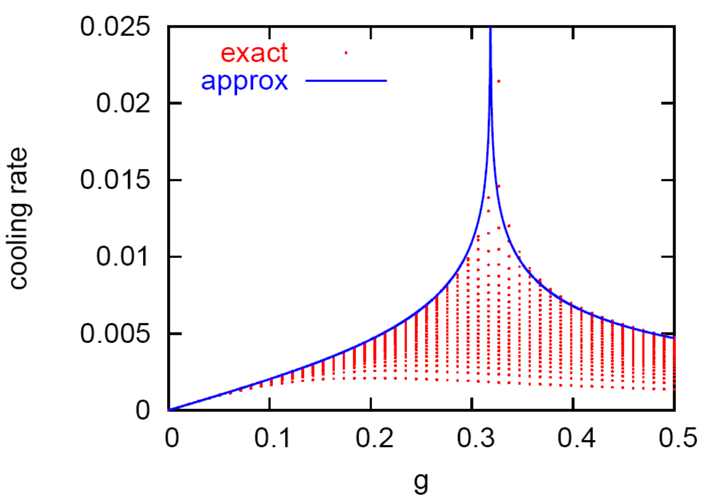


Figure 4: Comparison of actual values of $Re(\lambda)$ versus gain with those obtained from equation (14) with $X = 0$ for a rectangular frequency distribution with $N = 51$. The numerical solution had one eigenmode with a monotonically growing eigenvalue, which is not fully shown.

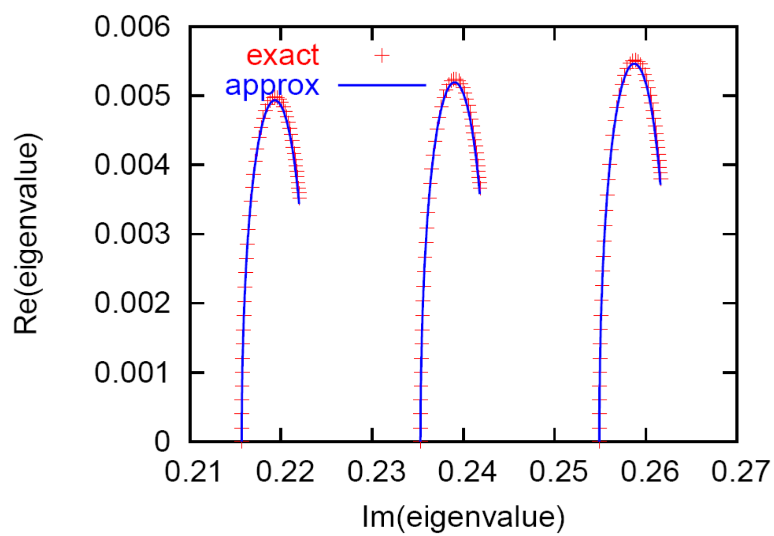


Figure 5: Evolution of λ as a function of gain for the exact, numerical solution and equation (14). The oscillator frequencies were uniformly spaced with $\omega_j = j/N$ and $N = 51$.