# *Chromatic and Space Charge Effects in Nonlinear Integrable Optics*

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# *Outline*

- *Crash Survey of Integrable Optics*
- *Dispersion & Chromaticity*
- *Space Charge & Invariants*
- *Future work*

# *Crash Survey of Integrable Optics*

# The properties of linear strong focusing

- *Strong focusing is robust because it is integrable*
  - *Two transverse Courant-Snyder invariants*

$$I = \beta p^2 + 2\alpha qp + \gamma q^2$$

    - *orbits are integrable — regular, bounded, periodic motion*
    - *KAM theorem notably does not apply to linear systems*
  - *KAM Th$^m$ does not apply to linear systems*
    - *single tune makes whole system unstable to resonant perturbations*
    - *higher-order effects such as chromaticity restore some stability*
  - *Linearity leaves system susceptible to parametric resonances*
    - *core-halo*
    - *resistive wall instability*
    - *beam break-up*
    - *…*

# *Additional stability from nonlinear integrable optics*

- *Key ideas:*
  - *A system with large tune spread…*
    - *fast Landau damping*
    - *suppresses parametric resonances*
    - *promises beam transport with lower losses*
  - *… but integrable dynamics*
    - *KAM Thm provides stability*
    - *on-momentum orbits are bounded and regular*
    - *perturbations lead to resonant lines…*
    - *…but orbits must diffuse out of dynamic aperture*
  - *so we expect stable beam dynamics in space charge*

# Conditions for Integrability

- *Bertrand-Darboux equation*

$$xy\left(\partial_x^2 - \partial_y^2\right)U + \left(y^2 - x^2 + c^2\right)\partial_{xy}U + 3\left(y\partial_x - x\partial_y\right)U = 0$$

  - *Hamiltonians with 2nd invariants quadratic in momentum satisfy:*
    - *differential equation is **linear***
    - *any superposition of potentials that satisfy this differential equation will have a 2nd invariant and be integrable*
  - *Other auxiliary conditions for accelerators:*
    - *matched beta functions in the drifts with these nonlinear elements*
    - *equal vertical and horizontal linear tunes*

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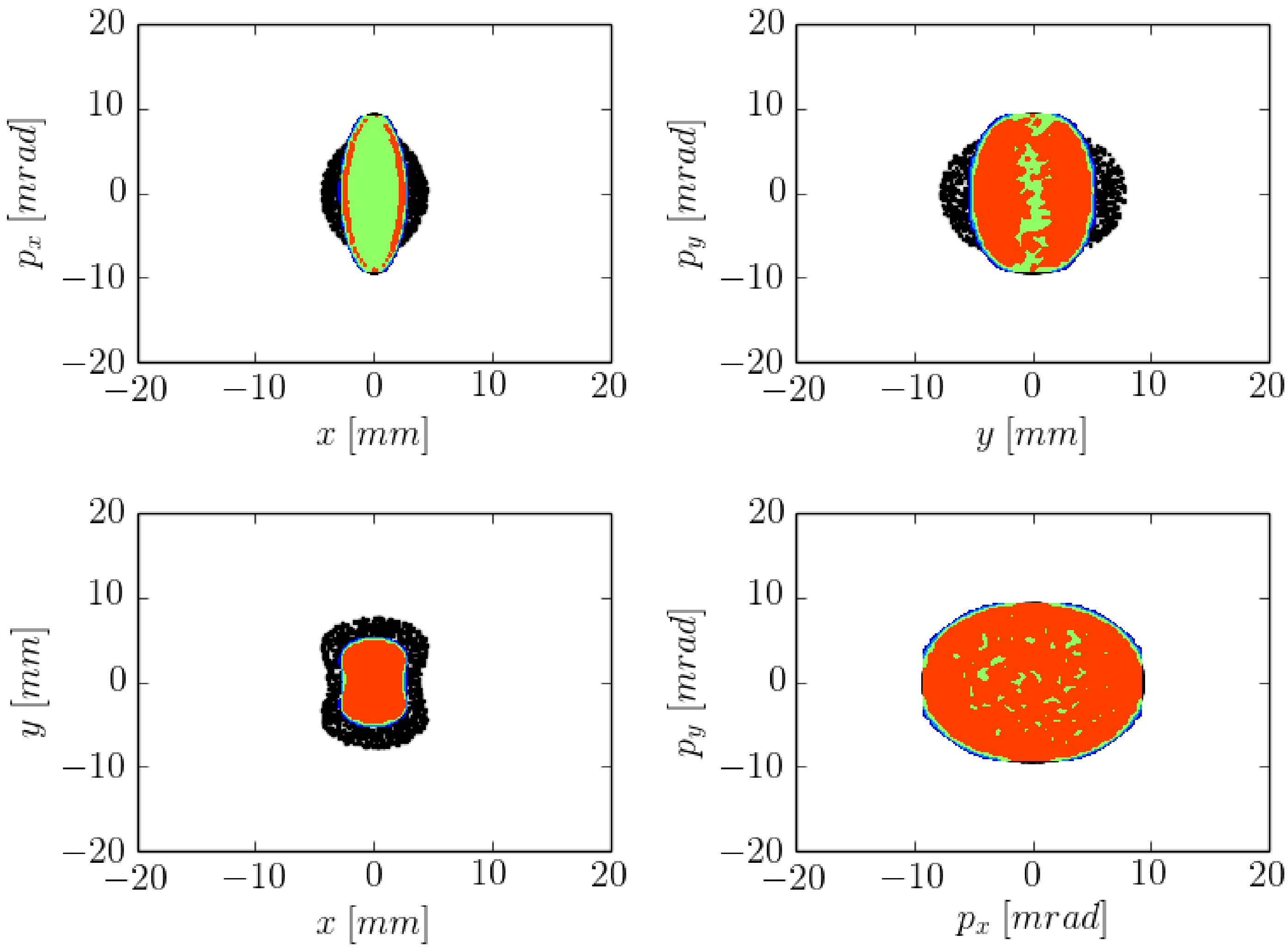
# *Nonlinearities suppress parametric resonances*

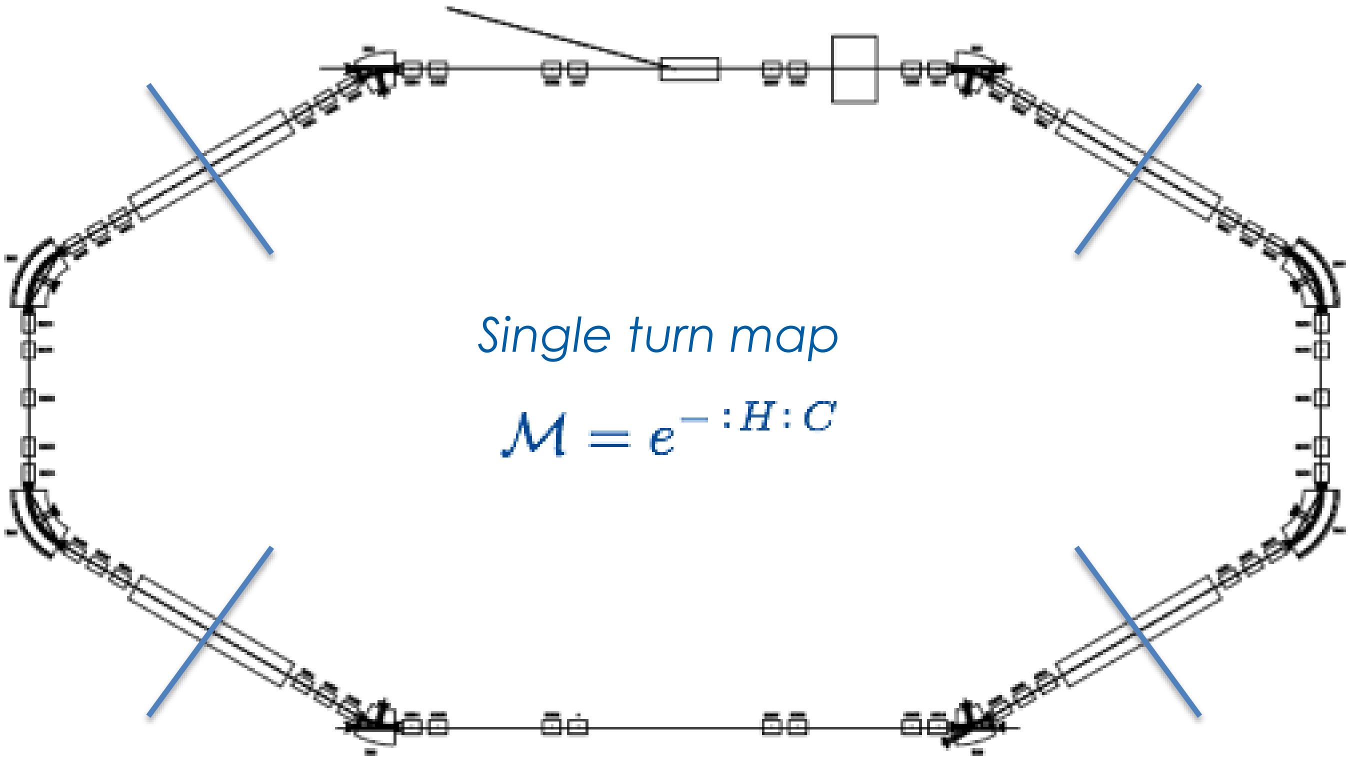


## *Dispersion & Chromaticity*

# Dispersion & Chromaticity I

- *Off-momentum particles couple motion to energy*
  - *Linear lattice chromaticity:*
    - *energy-dependent tune could cross nonlinear resonance*
    - *no loss of integrability (assuming linear RF bucket/coasting beam)*
  - *Linear lattice dispersion:*
    - *large dispersion can cause large beam size*
  - *Potential problems for elliptic potential*
    - *unequal tunes violates the Bertrand-Darboux equation*
    - *dispersion violates the equal beta function requirement*
  - *Conclusions:*
    - *defocussing quadratic perturbation due to differing chromaticities*
    - *already have large tune spreads — no need to remove all the chromaticity*

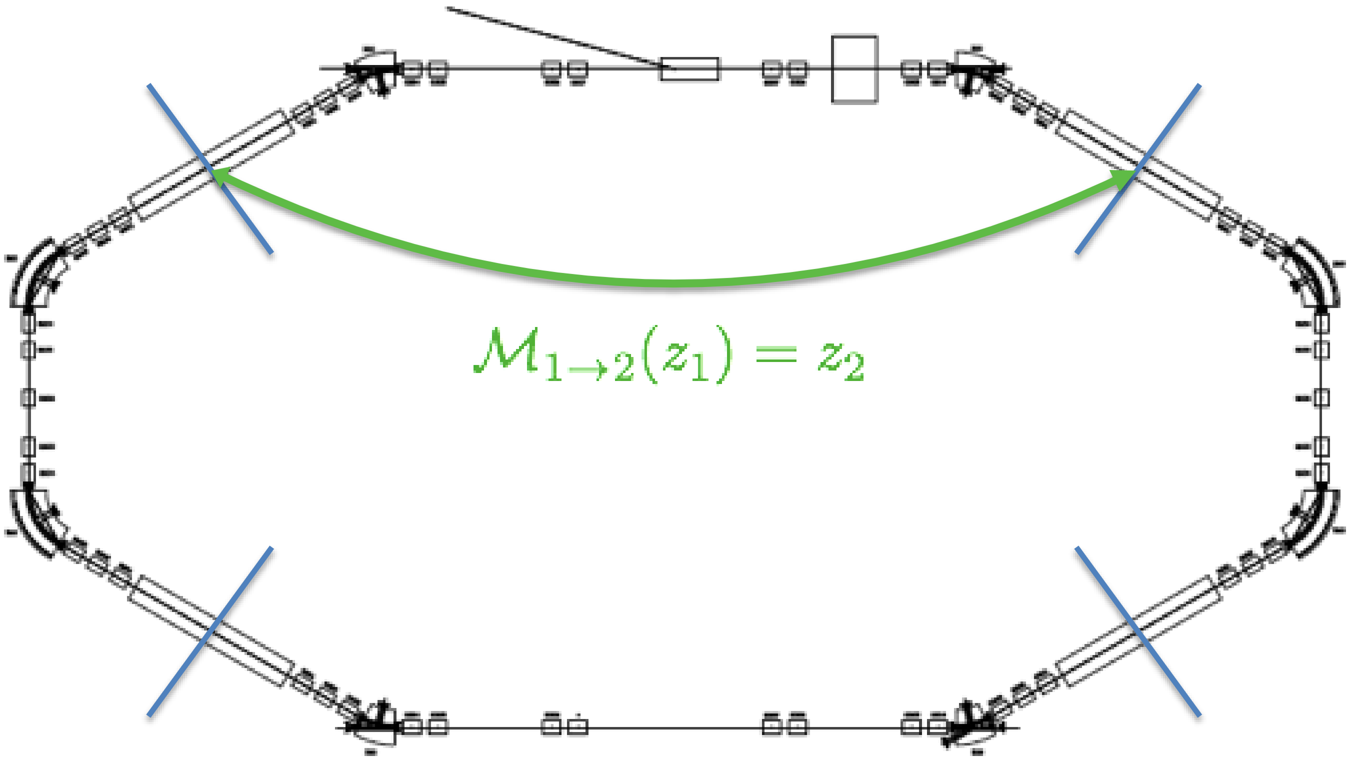
# *Single-turn Map*




$$\mathcal{M} = e^{-:H:C}$$

Figure from S. Nagaitsev, "IOTA Physics Goals" (2012)

# *Single-turn Map*



# Dispersion & Chromaticity II

- Computed for the continuously varying magnet
  - Details in extra slides…
  - Compute single-turn map as

$$\mathcal{M}_{\mathrm{IOTA}} = \mathcal{A}^{-1} \exp\left\{ -:\frac{t}{1-\delta}\int_{\ell/2}^{\ell} ds\, \mathcal{U}(\overline{x}-\eta(s)\delta, \overline{y}): \right\} e^{-:\overline{H}:} \exp\left\{ -:\frac{t}{1-\delta}\int_{0}^{\ell/2} ds\, \mathcal{U}(\overline{x}-\eta(s)\delta, \overline{y}): \right\} \mathcal{A}$$

  - and the related Hamiltonian

$$\overline{\mathcal{H}} = \frac{\mu_0}{2}\left\{ [1-C_x(\delta)]\left(\overline{p}_x^2+\overline{x}^2\right) + [1-C_y(\delta)]\left(\overline{p}_y^2+\overline{y}^2\right) + \frac{t}{1-\delta}\int_0^{\ell_{\mathrm{drift}}} \mathcal{U}\left(\overline{x}-\eta(s')\ \delta), \overline{y}\right) ds' \right\} + \mathrm{h.o.t.}$$

  - For the Bertrand-Darboux potential, we require:
    - Very particular form for U
    - equal vertical and horizontal linear tunes

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# Dispersion & Chromaticity III

- *New Set of Design Rules:*
  - *Twiss parameters*
    - *require equal beta functions to get desired cancellation*
    - *effective double-focusing lens for on-momentum linear map*
  - *Chromaticity*
    - *transverse tunes must be equal*
    - *familiar chromaticity correction schemes sufficient*
    - *correct to make $C_x = C_y$*
  - *Dispersion*
    - *dispersion modifies the integrable potential*
    - *drift section for elliptic magnets must be dispersion-free*

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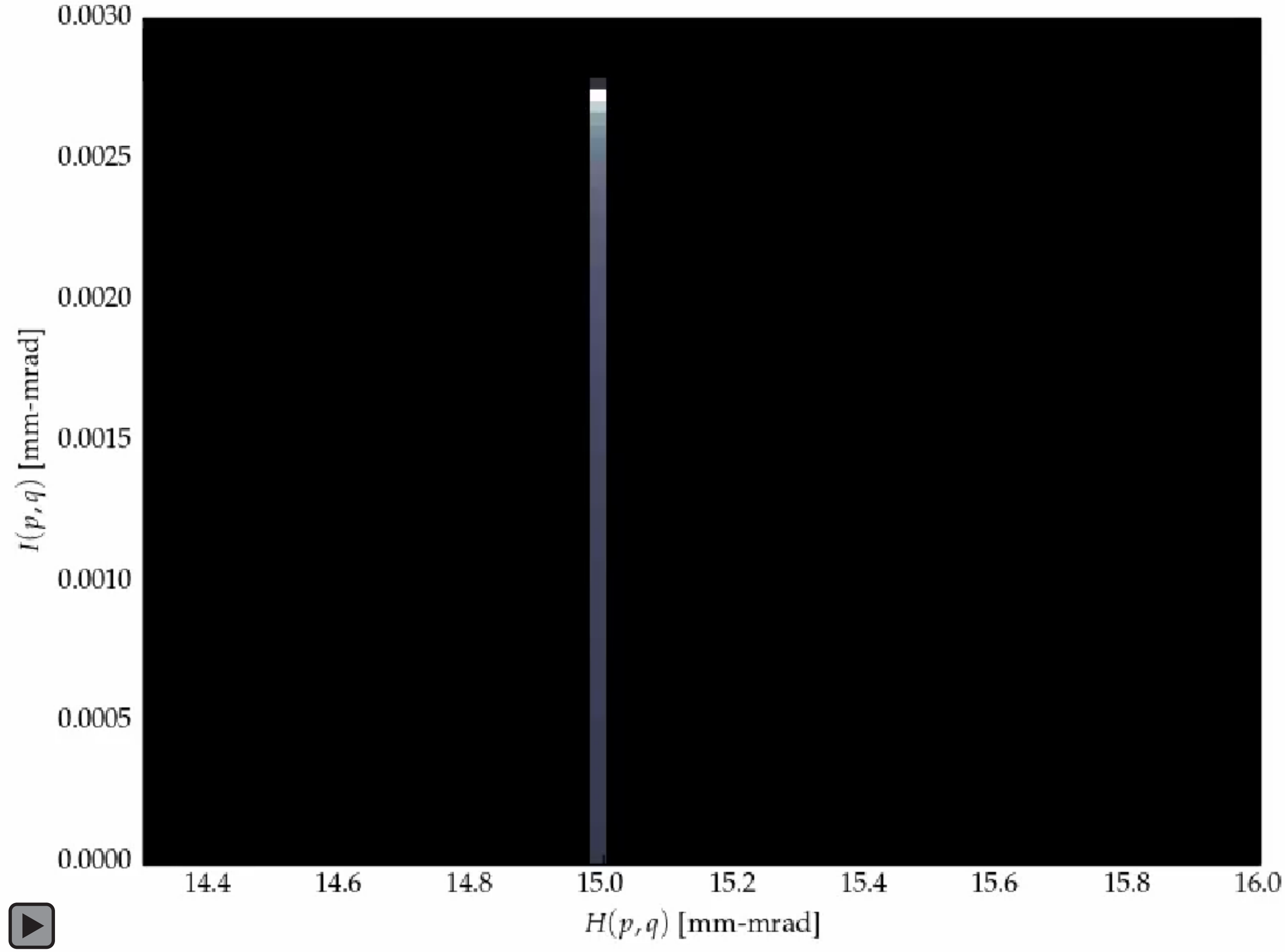
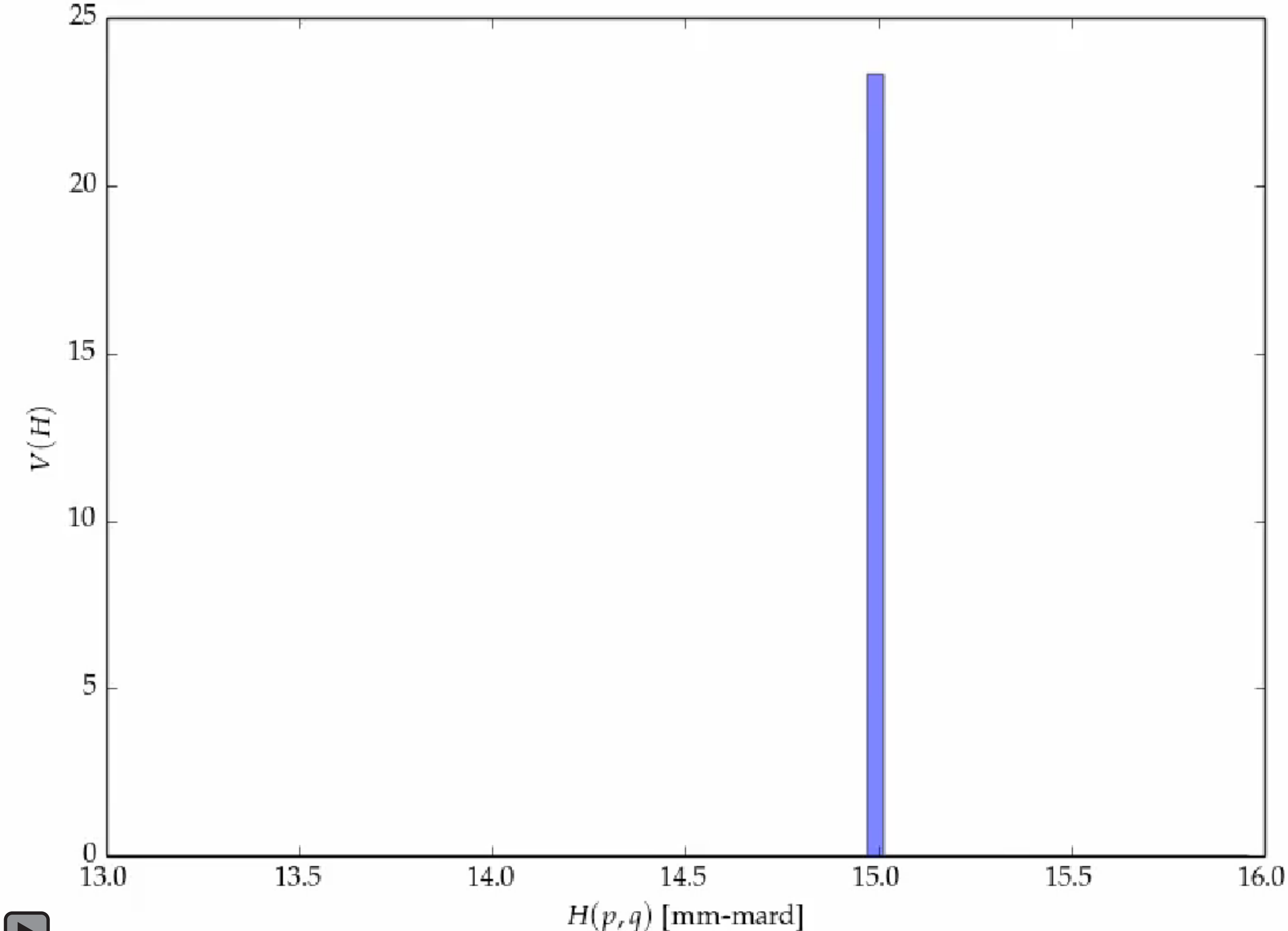
# *Space Charge & Invariants*

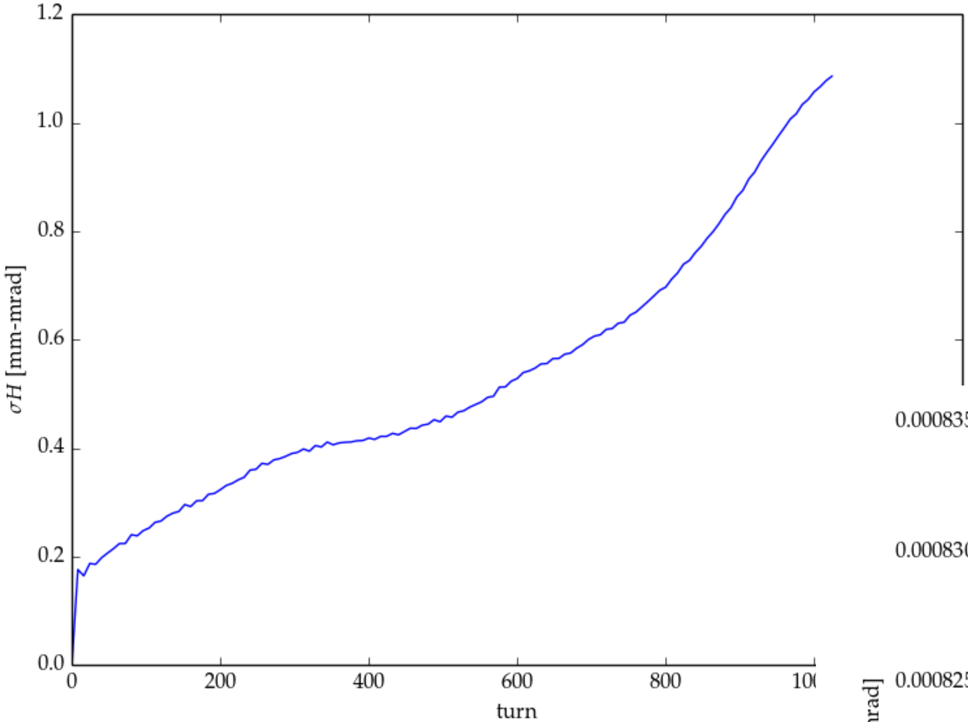
# Presence of Space Charge Changes Distribution

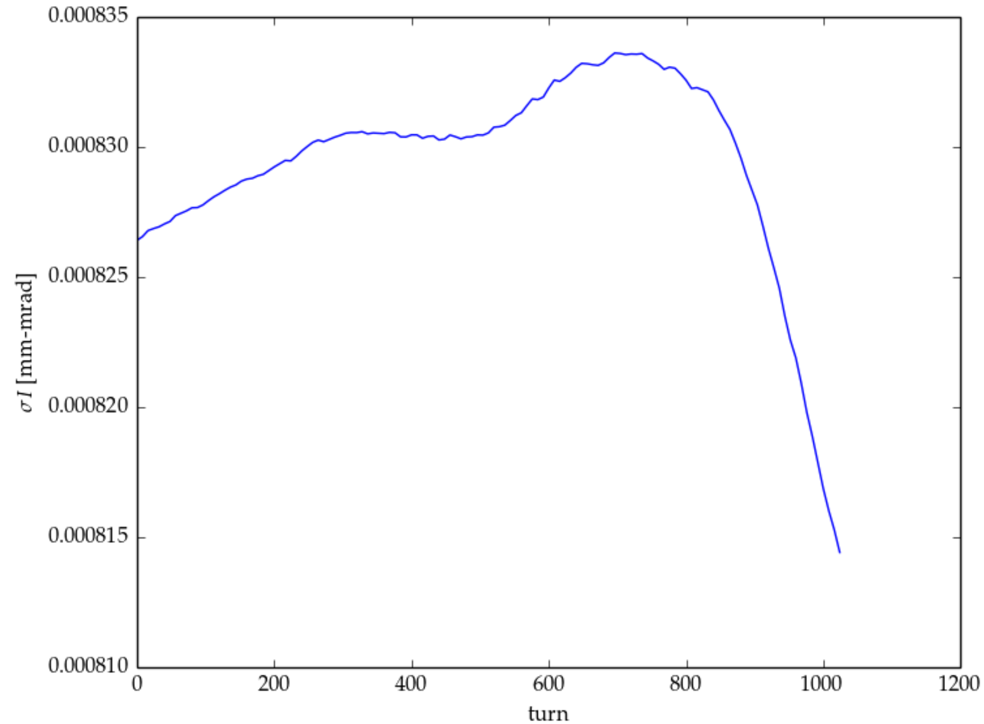


# Presence of Space Charge Changes Distribution



# Presence of Space Charge Changes Distribution

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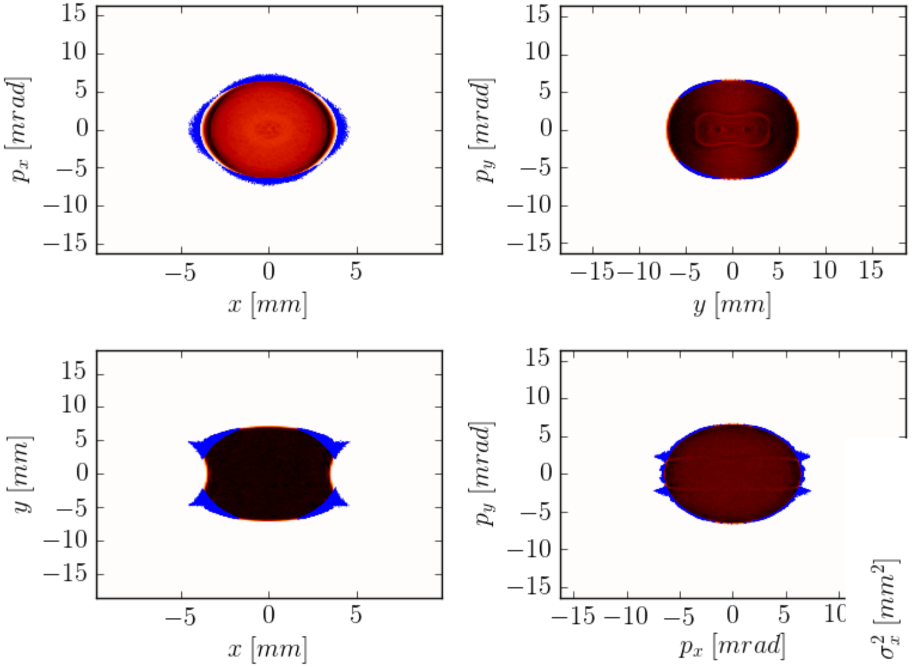
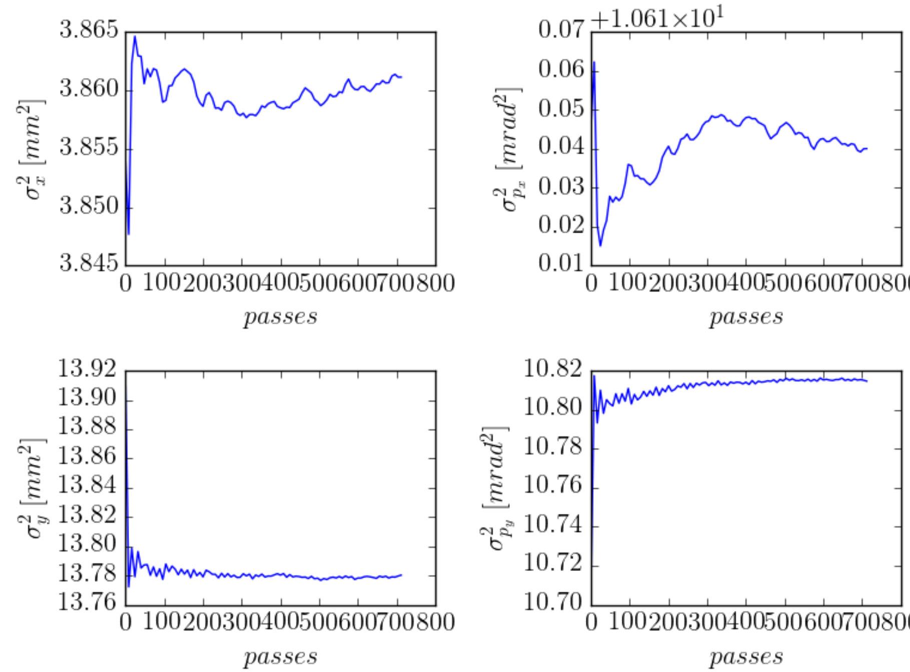
# *But the transverse beam distribution is static…*




*After 700+ turns, transverse phase space remains static*




*Transverse beam size has initial growth, followed by very small variations*

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# What's going on?

- *Hamiltonian now contains self-consistent space charge*
  - *Hamiltonian given by*

$$H = H_0(J) + \int dJ' d\psi' G(J, \psi; J', \psi') f(J')$$

  - *[G] ∝ [current]*
    - *intensity-dependent effects induce diffusion*
    - *distribution diffuses to fill "potential"*
    - *achieves steady state through space charge induced stochasticity*
  - *Speculation, requires better evidence*
    - *diffusion rate ∝ current*
    - *actual calculation (unlikely)*

[1] *see, e.g., Lichtenberg & Lieberman, §5.4*

# What's going on?

- *Diffusion in Action Space*
  - *Fokker-Planck Equation for perturbed integrable systems*[1]

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial J}\left[\frac{1}{4}(\Delta t)^2 \left(\frac{\partial}{\partial J}\left\langle\left(\frac{\partial H}{\partial \psi}\right)^2\right\rangle_\psi\right)\frac{\partial f}{\partial J}\right]$$

  - *Modified Hamiltonian with space charge*

$$H = \underbrace{H_0(J) + \int dJ' g_0(J,J') f(J')}_{\text{coherent tune shift}} + \underbrace{\sum_{n\neq 0} g_n(J,J') e^{in\psi} f(J')}_{\text{diffusive dynamics}}$$

  - *Particles drift in effective potential and diffuse*
    - *some steady state reached*
    - *some particles diffuse out of the potential well*
    - *complicated by resonance islands, correlations, nonlinear Fokker-Planck…*

[1]*see, e.g., Lichtenberg & Lieberman, §5.4*

# *Future Work*

- *What do the chromaticity corrections do to the invariants?*
  - *What do sextupoles, etc., do the the dynamic aperture?*
  - *How to minimize the impact on the beam?*
  - *What is the diffusion time for particles on resonance?*

- *What does space charge **do**?*
  - *How does space charge affect the invariants?*
  - *What can be done to compensate space charge?*
  - *Is there a collective invariant that remains?*

## ***Thank you for your attention***

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# Digression on Lie Operators

- *Lie operators from Poisson brackets*

$$\dot{z} = -\{H, z\} \mapsto \dot{z} = -:H:z \qquad z(t) = e^{-:H:t} z(0)$$

  - *Advantages*
    - *can multiply maps, cannot multiply Hamiltonians*
    - *maps make coördinate transformations into similarity transformations*
  - *Disadvantages*
    - *a lot of formalism to get to the physics*
    - *difficult to work with time-varying Hamiltonians*

- *Key Identities*
    - *BCH Identity*

$$e^{:C:} = e^{:A:}e^{:B:} \qquad C = A + B + \frac{1}{2}:A:B + \frac{1}{12}(:A:^2 B + :B:^2 A) + \ldots$$

    - *Similarity transformation*

$$e^{:A:}e^{:B:}e^{-:A:} = \exp\left(:e^{:A:}B:\right)$$

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# *When are sextupoles optically transparent?*

- *Lie operator approach*

$$\mathcal{M} = e^{-S_n :z^n:} e^{-:h_2:} e^{-S_n :z^n:}$$

$$e^{-:h_2:} = \mathcal{A}^{-1} \underbrace{\mathcal{R}(\theta)}_{\text{pure rotation}} \mathcal{A}$$

$$\mathcal{M} = e^{-:h_2:} \exp(-S_n :e^{:h_2:} z^n:) \exp(-S_n :z^n:)$$

$$\mathcal{M} = \mathcal{A}^{-1} \mathcal{R} \exp(-S_n :\mathcal{R}(\overline{z})^n:) \exp(-S_n :\overline{z}^n:)$$

$$\mathcal{R} \propto -1, \theta = (2n+1)\pi \implies \exp(-S_n :\mathcal{R}(\overline{z})^n:) \exp(-S_n :\overline{z}^n:) \mathcal{A} = 1$$

- *Off-momentum particles do not cancel exactly because θ is energy-dependent. This is the basis of chromaticity correction.*

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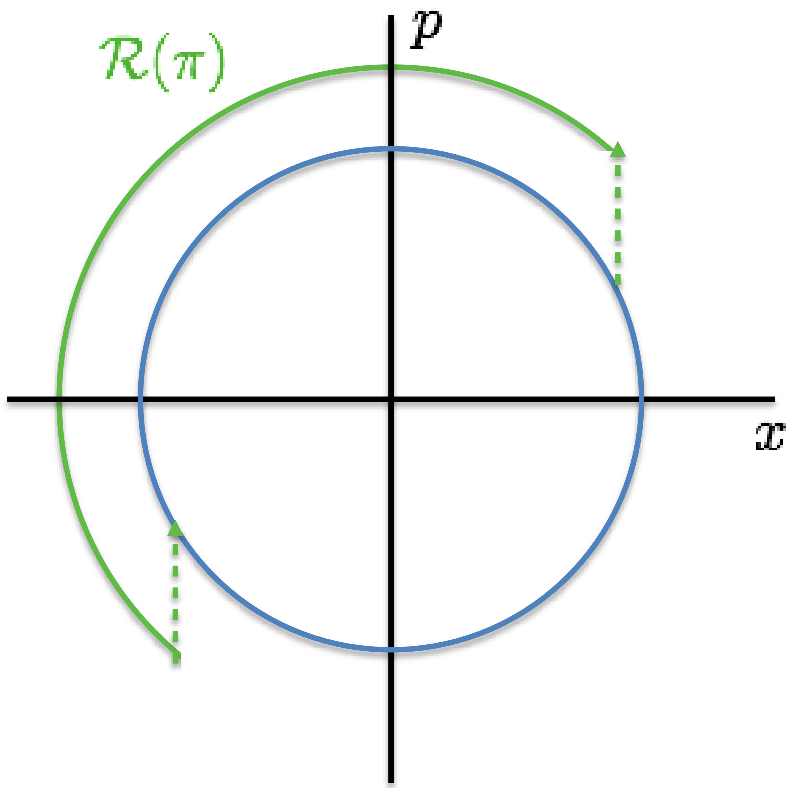
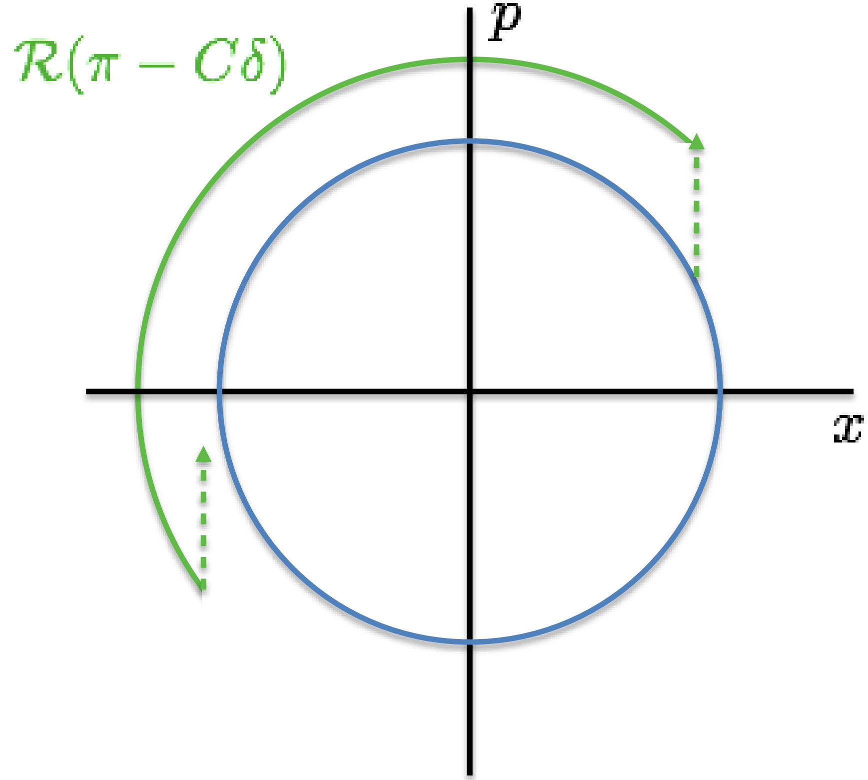
# When are sextupoles optically transparent?

- *Pictorial approach (design momentum)*



# When are sextupoles optically transparent?

- *Pictorial approach (off-momentum)*



# *The horror…*

$$
\begin{aligned}
\mathcal{M} &= \left( \prod_{i=0}^{N/2} \exp\left\{ - : \frac{p^2}{2} + t_i \mathcal{V}_i(x,y) : \Delta s \right\} \right) e^{-:h_0:} \left( \prod_{i=N/2}^{N} \exp\left\{ - : \frac{p^2}{2} + t_i \mathcal{V}_i(x,y) : \Delta s \right\} \right) \\
&= \left( \prod_{i=0}^{N/2} \exp\left\{ -t_i : e^{-(i+1/2):p^2/2:\Delta s} \mathcal{V}_i(x,y) : \Delta s \right\} \right) \circ \\
&\quad \underbrace{e^{-:p^2/2:\ell/2} e^{-:h_0:} e^{-:p^2/2:\ell/2}}_{e^{-:h_2:}} \circ \\
&\quad \left( \prod_{i=N/2}^{N} \exp\left\{ -t_i : e^{(i+1/2):p^2/2:\Delta s} \mathcal{V}_i(x,y) : \Delta s \right\} \right)
\end{aligned}
$$

# … the horror

$$e^{-:h_2:} = \mathcal{A} e^{-:\overline{h_2}:} \mathcal{A}^{-1}$$

*Normalized coördinates*

$$\mathcal{A}^{-1} = \begin{pmatrix} 1/\sqrt{\beta_x} & 0 & 0 & 0 & 0 & -\eta/\sqrt{\beta_x} \\ \alpha_x/\sqrt{\beta_x} & \sqrt{\beta_x} & 0 & 0 & 0 & -\alpha_x\eta+\beta_x\eta'/\sqrt{\beta_x} \\ 0 & 0 & 1/\sqrt{\beta_y} & 0 & 0 & 0 \\ 0 & 0 & \alpha_x/\sqrt{\beta_y} & \sqrt{\beta_y} & 0 & 0 \\ \eta' & \eta & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

*Courant-Snyder Parameterization*

$$\mathcal{M} = \left( \prod_{i=0}^{N/2} \exp\left\{ -t_i : e^{-(i+1/2) : p^2/2 : \Delta s} \mathcal{V}_i(x,y) : \Delta s \right\} \right) \circ \left( \mathcal{A} e^{-:\overline{h_2}:} \mathcal{A}^{-1} \right) \circ \left( \prod_{i=N/2}^{N} \exp\left\{ -t_i : e^{(i+1/2) : p^2/2 : \Delta s} \mathcal{V}_i(x,y) : \Delta s \right\} \right)$$

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$$\mathcal{A}^{-1}\exp\left\{-t_i :e^{(i+1/2):p^2:2}\mathcal{V}_i(x,y): \Delta s\right\} =$$

$$\mathcal{A}^{-1}\exp\left\{-t_i :e^{(i+1/2):p^2:2}\mathcal{V}_i(x,y): \Delta s\right\}\mathcal{A}\mathcal{A}^{-1} =$$

$$\underbrace{\mathcal{A}^{-1}e^{-(i+1/2):p^2/2:\Delta s}}_{\mathcal{A}_i^{-1}}\exp\left\{-t_i :\mathcal{V}_i(x,y): \Delta s\right\}\underbrace{e^{(i+1/2):p^2/2:\Delta s}\mathcal{A}}_{\mathcal{A}_i}$$

*The Danilov-Nagaitsev potential normalizing trick as follows:*

$$A_0^{(i)} = \begin{pmatrix} 1/\sqrt{\beta_x} & 0 & 0 & 0 \\ \alpha_x/\sqrt{\beta_x} & \sqrt{\beta_x} & 0 & 0 \\ 0 & 0 & 1/\sqrt{\beta_y} & 0 \\ 0 & 0 & \alpha_x/\sqrt{\beta_y} & \sqrt{\beta_y} \end{pmatrix}$$

$$\mathcal{V}_i(x,y) = \mathcal{V}_i\left(A_0^{(i)}(x,y)\right)$$

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$$\mathcal{A}^{-1} e^{-(i+1/2)\,:p^2/2:\,\Delta s} \exp\left\{-t_i :\mathcal{V}_i(x,y): \Delta s\right\} e^{(i+1/2)\,:p^2/2:\,\Delta s} \mathcal{A} =$$

$$\exp\left\{-t :\mathcal{V}\left(\overline{x} - \delta \frac{\eta}{\sqrt{\beta_x}}, \overline{y}\right): \Delta s\right\}$$

*Final transfer map in normalized coordinates*

$$\left(\prod_{i=0}^{N/2} \exp\left\{-:\frac{p^2}{2} + t_i \mathcal{V}_i(x,y): \Delta s\right\}\right) e^{-\,:h_0:} \left(\prod_{i=N/2}^{N} \exp\left\{-:\frac{p^2}{2} + t_i \mathcal{V}_i(x,y): \Delta s\right\}\right) =$$

$$\mathcal{A}\ \exp\left\{\sum_i -(1-\delta) t :\mathcal{V}\left(\overline{x} - \delta \frac{\eta_i}{\sqrt{\beta_i}}, \overline{y}\right):\right\} e^{-\,:\overline{h_2}:} \exp\left\{\sum_i -(1-\delta) t :\mathcal{V}\left(\overline{x} - \delta \frac{\eta_i}{\sqrt{\beta_i}}, \overline{y}\right):\right\}\ \mathcal{A}^{-1}$$

$$\overline{h}_2 = \frac{\mu_0}{2}\left[(1 - C_x\delta)\left(\overline{p}_x^2 + \overline{x}^2\right) + (1 - C_y\delta)\left(\overline{p}_y^2 + \overline{y}^2\right)\right]$$