Cyclotron (and Isochronous FFAG) Space Charge Limit

Rick Baartman, TRIUMF

November 10, 2014

Summary

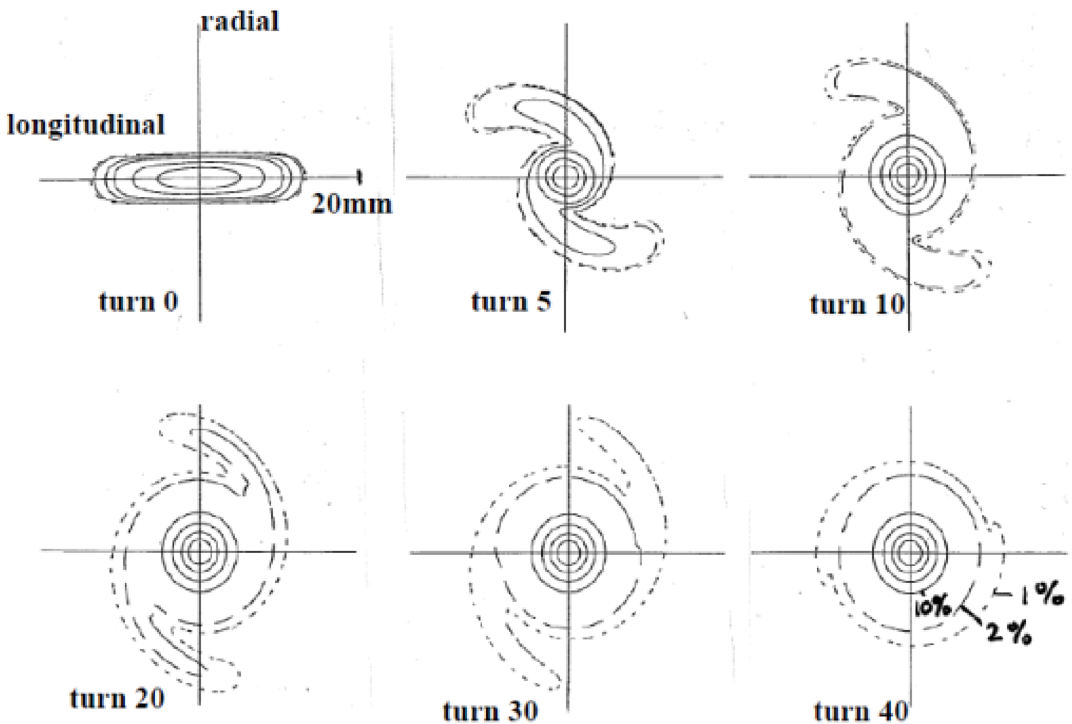
Space charge in separated turn cyclotrons causes bunches to reshape.

Resulting bunches experience “vortex effect” and tend to become circular in median plane.

We determine:

- the “tune” of the rotating bunches,
- the size of the bunches vs. charge,
- the bunch charge limit,
- the effects of acceleration.

Longitudinal Space Charge in Injector II Cyclotron



Simulation of a 1mA beam, circulating in Injector II at 3 MeV for 40 turns without acceleration.

The core stabilizes faster than the halos (calculations by Stefan Adam)

W.Joho 2013

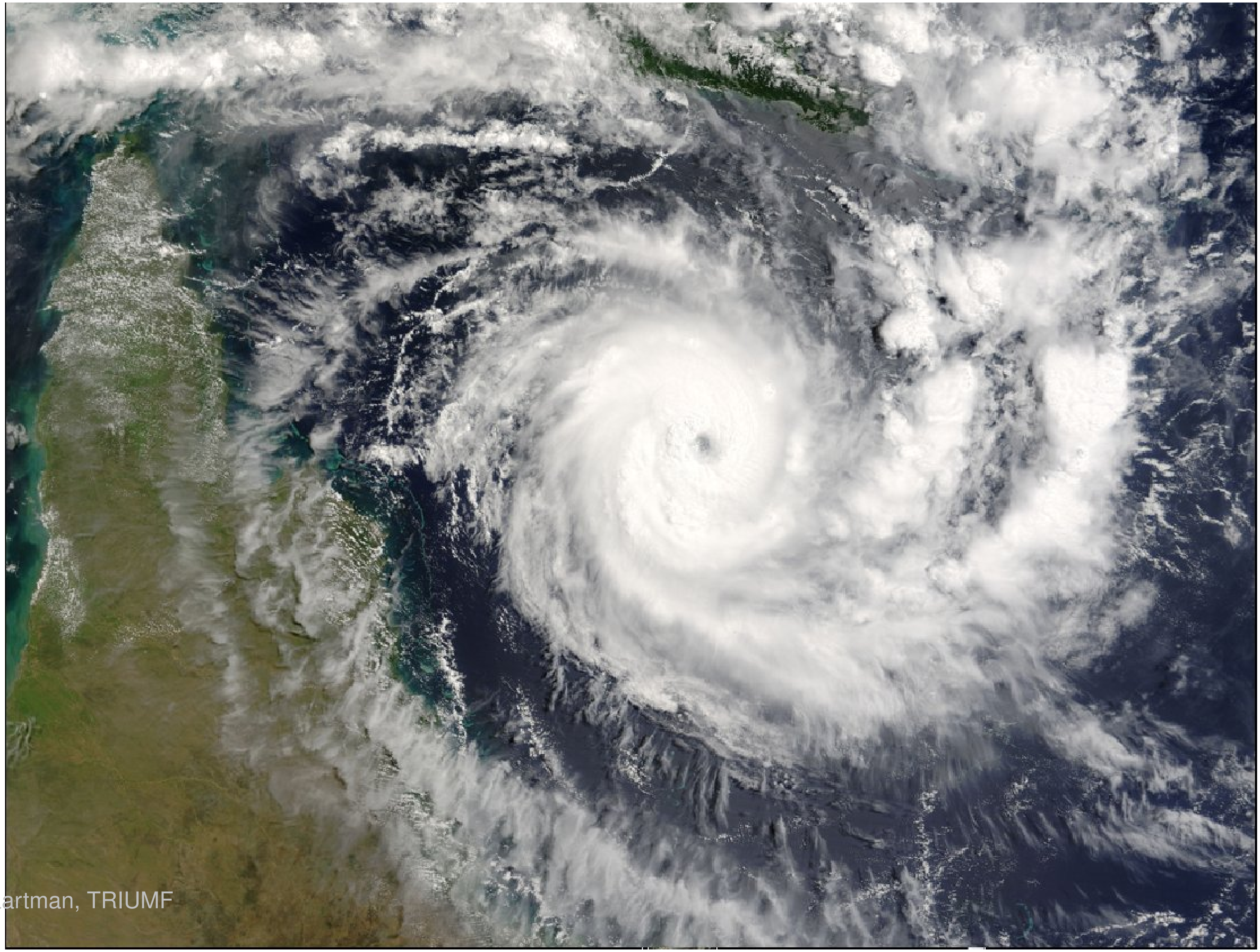
Vortex Effect

At least some of this is easily understood. Leading particles are “pushed” by space charge, but cannot advance because of isochronism and instead gain energy and so go sideways to higher radius. Trailing particles do the reverse. Particles at the outsides move back and those at the inside move forward.

Another way to understand is via the Coriolis effect. A typhoon results from pressure gradients in a rotating frame: The low pressure area cannot simply collapse because the earth’s rotation causes the particles to move sideways instead of along the pressure gradient.

But interesting questions raised:

1. How fast does it twist? (What is its “tune”?)
2. Does this effect stabilize at any bunch charge, or is there a limit?



M.M. Gordon (1969):

Proceedings of the Fifth International Cyclotron Conference

The longitudinal space charge effect and energy resolution*

M. M. Gordon
Michigan State University, Michigan, U.S.A.

Consider the non-relativistic motion of a charge q in the isochronous magnetic field B as viewed in a reference frame rotating with constant angular velocity $\underline{\omega}$ whose direction is perpendicular to B and whose magnitude is the isochronous angular frequency. If there are no electric fields present, then on a time-average basis the charge q will be at rest. Considering the electric field vector $\underline{F}$ as a perturbation, the steady-state (non-oscillatory) velocity vector $\underline{v}$ is then given by:

$$m_o\, \underline{\omega} \times \underline{v} = q\underline{F} \qquad \text{(F = Electric Field)} \qquad (1)$$

that is, the Coriolis force has the effect of reversing the magnetic field direction as seen by q in the rotating frame. Since this steady-state velocity is perpendicular to both $\underline{\omega}$ and $\underline{F}$, it is directed along the equipotential curves associated with $\underline{F}$. In the rotating frame the rf electric field has a time-average component in the azimuthal direction given by: $V_1/(2\pi r)$, and Eqn (1) shows that this field causes the particles to move radially outward with the velocity: $dr/dt = q V_1/2\pi m_o \omega r$, so that $q V_1$ is the average energy gain per turn. If the rf electric field is absent, the $\underline{F}$ is produced entirely by the space charge and Eqn (1) indicates that the charges will then circulate clockwise about the point of maximum potential, thereby establishing a 'vortex' in the space charge cloud as viewed in the rotating frame. This vortex motion is depicted in Fig. 1. Since both electric fields are actually present, the resultant steady-state motion is a superposition of these two phenomena.

The longitudinal space charge effect has been extensively investigated for other types of accelerators.[8] Synchrotron oscillations as viewed in the charges will then circulate clockwise about the point of maximum potential, thereby establishing a 'vortex' in the space charge cloud as viewed in the rotating frame. This vortex motion is depicted in Fig. 1. Since both electric fields are actually present, the resultant steady-state motion is a superposition of these two phenomena.

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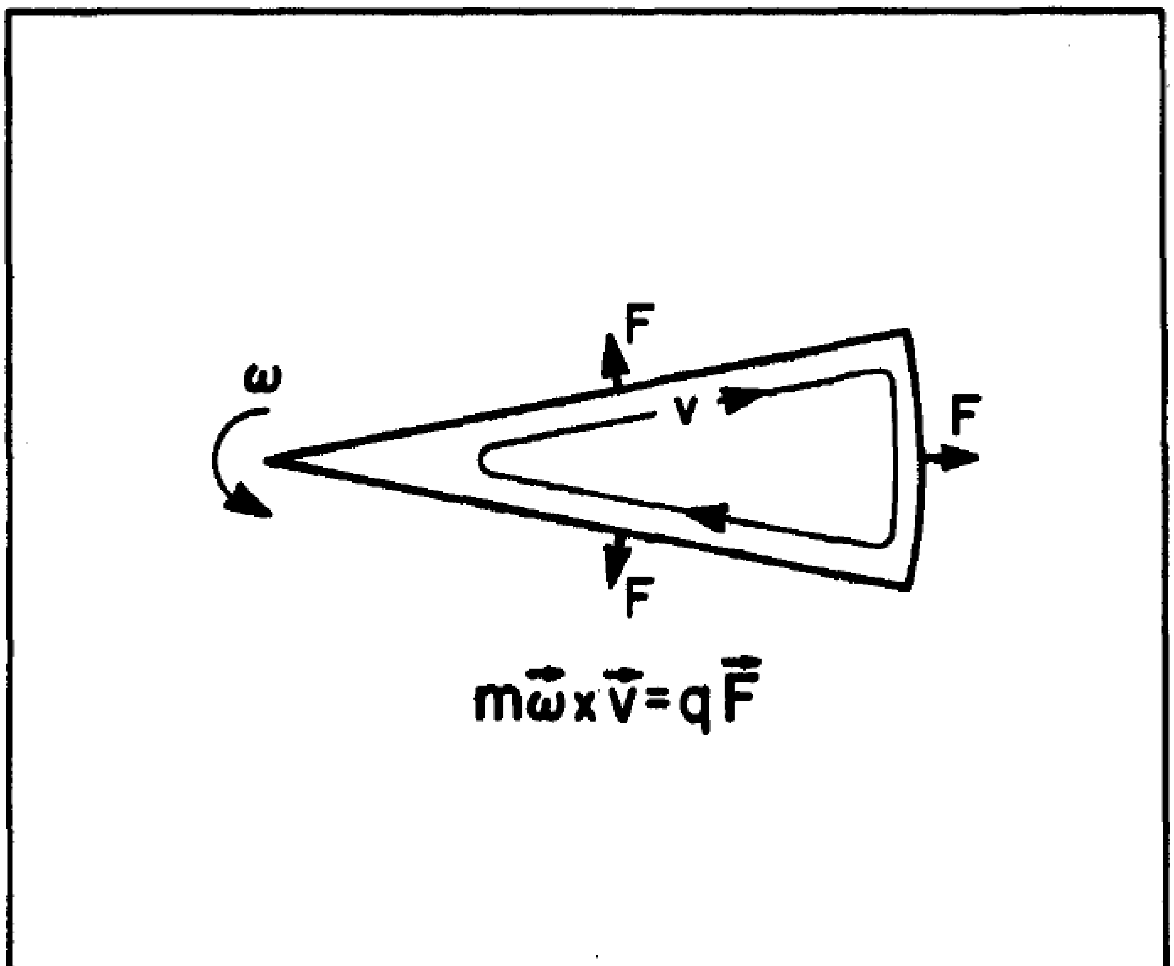
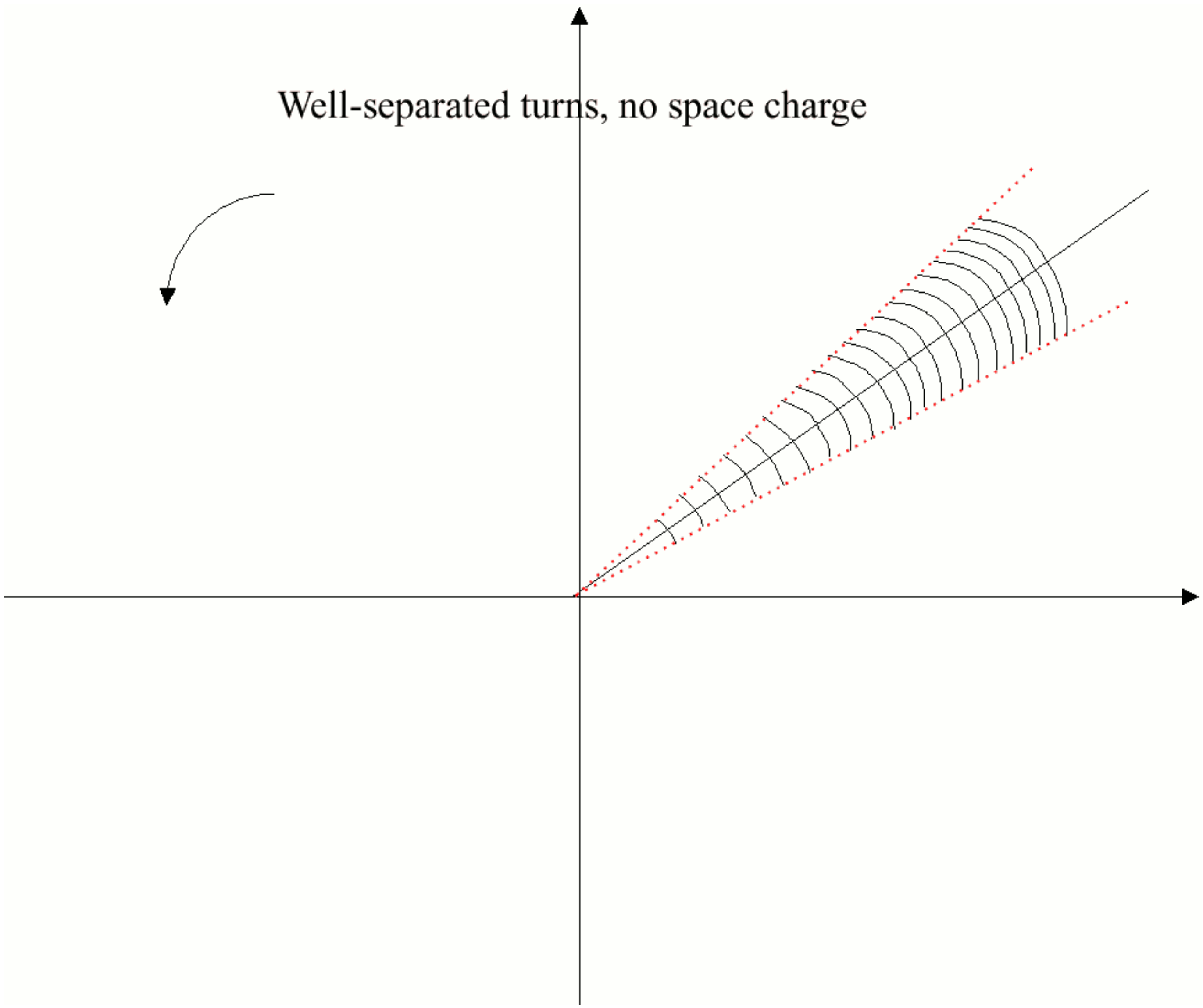



Well-separated turns, no space charge

2.1. *Local vortices*

In an isochronous cyclotron with separated turns the charge density and resultant electrostatic potential have local maxima at the centre of each ion bunch or turn. As a result, the central region of each turn will execute a local vortex motion in which the ions remain within the same turn; the ions outside this region, however, will partake in the overall vortex motion of the total charge cloud and it is only this motion which tends to destroy the turn separation. Since the length $r\Delta\theta$ of the turn is generally much greater than the radius gain per turn, the local vortices are so small and feeble that their presence can be neglected entirely. This rule may not apply when the cyclotron operates under pulsed conditions; in the MSU cyclotron, for example, where nine out of ten ion pulses can be completely rejected, the radial separation between ion bunches is then always greater than the length $r\Delta\theta$, so that the local vortices are quite significant in this case. However, this special situation will not be treated here. It should be noted that in those cases where the vortex motion seriously changes the charge distribution, an iterated calculation may then be necessary to achieve adequate self-consistency.

Equipotential Contours $\neq$ Density Contours

Motion in a bunch is along equipotential lines. This seems ideal until one remembers that the equipotential contours are not the same as the density contours.

E.g. with a beam shaped as elliptic cylinder, the potential is

$$V \propto \frac{x^2}{a(a+b)} + \frac{y^2}{b(a+b)}, \tag{1}$$

but the boundary is different:

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} \tag{2}$$

The natural nonlinearity of space charge forces a non-circular bunch to twist (spaghetti around a fork –W. Joho). If it were linear, it would act as a propeller.

What is the result?

In particular, what is the stationary state?

This was answered in a brilliant work by Wiel Kleeven in his thesis (1988): The stationary state is circular bunches in the (R, θ)-plane.

Simulations (E.g. Adelmann) and measurements (also at PSI: Dölling) confirm it.

Kleeven's full relativistically-correct theory is tricky, tedious, but first:

There is a very elegant model due to Ricaud and Bertrand (2001): Spherical bunches are a good approximation in cases where radial and vertical tunes are comparable. But main feature is that the electric field is simply derived from Coulomb's law:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^3}\,\vec{r} \equiv k\,\vec{r} \tag{3}$$

SPECIFIC CYCLOTRON CORRELATIONS UNDER SPACE CHARGE EFFECTS IN THE CASE OF A SPHERICAL BEAM

P.Bertrand, Ch. Ricaud, GANIL, Caen, France

Abstract

High intensity primary ion beams at GANIL are necessary to induce high radioactive production rates in the frame of the SPIRAL project. In this paper, we show that an intense beam can be tuned at injection in a cyclotron so as to result in a spatially spherical beam in the machine, with a reduced halo formation.

1 INTRODUCTION

The question on high intensity beams in cyclotrons is of great interest (Stammbach [1]).Various new applications require a fine beam tuning and a good comprehension of the space charge effects in order to limit the halo formation, and to avoid beam losses and activation in the machine. First, we establish the exact matched solution in the academic case where the electric space charge force is linear. Then we present a self-consistent approach allowing us to take into account the non-linear effects. Finally, we present simulation results obtained in the case of our compact injector C01.

2 LINEAR ANALYSIS

We consider a reference particle (q,m) rotating without acceleration on a circle according to the equations :

$$x_0(t) = r_0 \cos(\omega t)$$
$$y_0(t) = r_0 \sin(\omega t)$$

$$m\ddot{x} = qk(x - x_0) + qb_z\dot{y}$$
$$m\ddot{y} = qk(y - y_0) - qb_z\dot{x}$$

This gives in the complex plane, using $z = x + iy$:

$$\ddot{z} - i\omega\dot{z} - \lambda z = -\lambda z_0 \quad ; \quad \lambda = qk/m$$
$$z = A\,e^{r_1 t} + B\,e^{r_2 t} + r_0\,e^{i\omega t}$$
$$r^2 - i\omega r - \lambda = 0 \quad ; \quad r = r_1, r_2$$

The solution is stable for r_1,r_2 purely imaginary, which leads to the following condition on the intensity :

$$0 \le u = \frac{I}{I_{\max}} = \frac{4\lambda}{\omega^2} < 1 \;;$$

$$I_{\max} = \frac{2\pi^2}{\mu_0 c^2}\frac{f_{hf}^2\,|b_z|}{h}\Delta r^3$$

We can use now the matrix form :

$$\begin{bmatrix}\Delta x\\ \Delta y\\ \Delta\dot{x}\\ \Delta\dot{y}\end{bmatrix}(t) = \begin{bmatrix}x - x_0\\ y - y_0\\ \dot{x} - \dot{x}_0\\ \dot{y} - \dot{y}_0\end{bmatrix}$$

$$\begin{bmatrix}\Delta x_0 \\ \vdots\end{bmatrix} \quad \begin{bmatrix}\Delta x_0 \\ \vdots\end{bmatrix}$$

Stay in the lab frame, assume flat field B. Then the magnetic and electric forces on particle of charge q, mass m give:

$$\begin{aligned} m\ddot{x} &= +qB\dot{y} + qk(x - x_0) \\ m\ddot{y} &= -qB\dot{x} + qk(y - y_0) \end{aligned}$$

where $(x_0, y_0) = R(\cos\omega t, \sin\omega t)$ is the equilibrium orbit and $\omega = qB/m$.

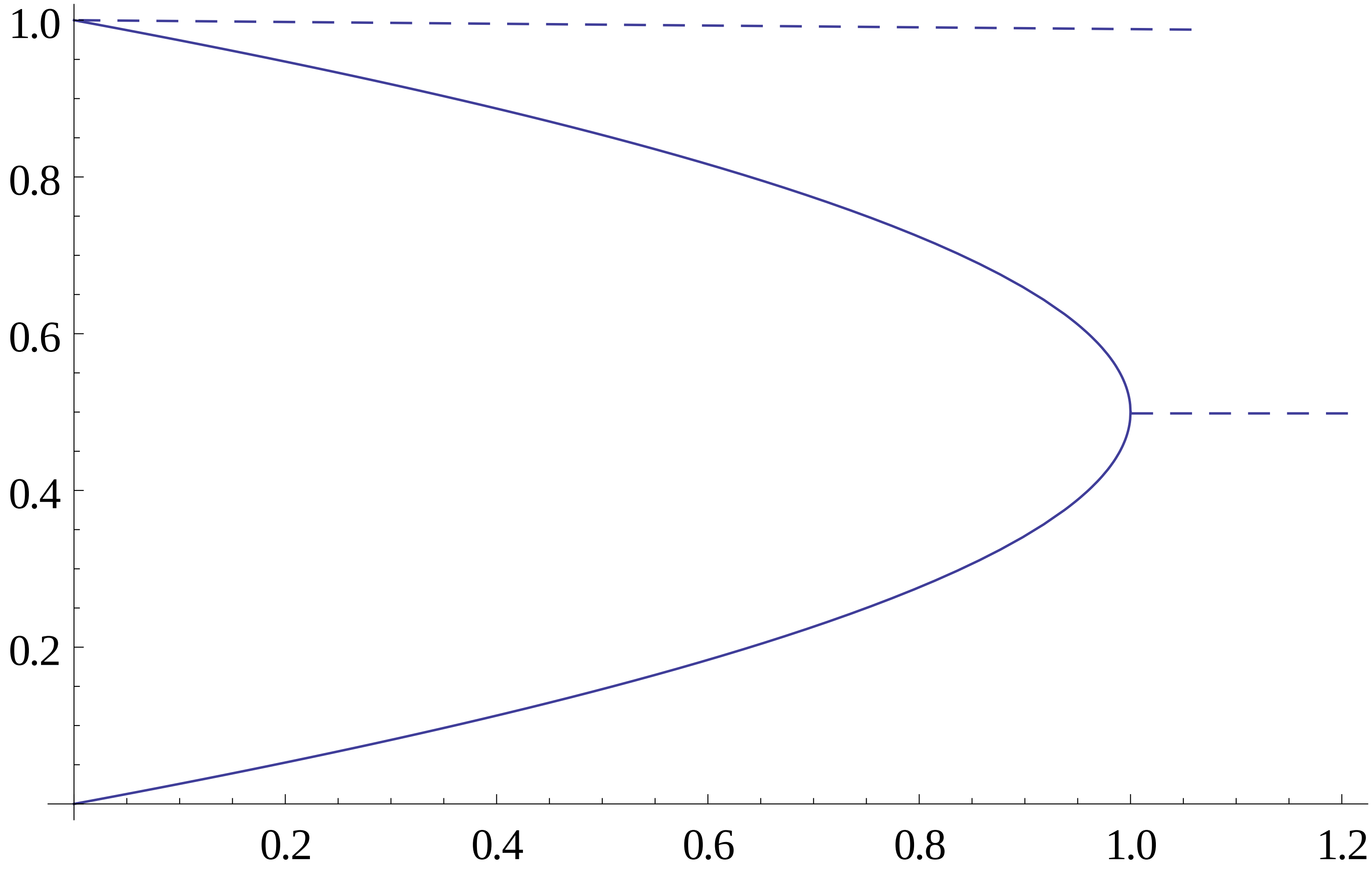
Solve using complex $z = x + iy$, let $z = R\exp(i\omega t) + C\exp(pt)$, find

$$p^2 - i\omega p - \frac{qk}{m} = 0 \rightarrow p = \frac{i\omega}{2} \pm \sqrt{-\frac{\omega^2}{4} + \frac{qk}{m}} \tag{4}$$

Divide p by $i\omega$ to get the tunes of the modes:

$$\nu_{r\pm} = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{Q}{Q_{\max}}}\right) \text{ where } Q_{\max} = \pi\epsilon_0\left(\frac{m}{q}\right)\omega^2 r^3 \tag{5}$$

Mode tunes $\nu_{r\pm}$ vs. $Q/Q_{\max}$



For $Q > Q_{\rm max}$, p has a real part allowing exponential growing solutions. As we approach this limit from below, the acceptance approaches zero; at the limit beam must have zero emittance in horizontal plane.

For $Q \ll Q_{\rm max}$, we find the tune shift.

$$\Delta\nu_r = \frac{Q}{4Q_{\rm max}} = -\frac{NR^2 r_p}{\beta^2 r^3} \tag{6}$$

(To connect with Laslett tune shift, note "bunching factor" $B_f \propto r/R$. $\Delta\nu_r$ is actually 1/2 the "Laslett space charge tune shift", as though the full shift is shared between radial and longitudinal.)

Here's a simpler formula for maximum charge: Notice $\omega = c/R_\infty$, $mc^2/q \equiv V_m$ (938 MV for protons), $\epsilon_0 = (cZ_0)^{-1}$, where $Z_0 = 377\,\Omega$:

$$Q_{\rm max} = \pi \left(\frac{V_m}{cZ_0}\right) \frac{r^3}{R_\infty^2} \tag{7}$$

Example: The PSI Injector II: $V_m = 938$ MV, $R_\infty = 9.54$ m, try $r = 6.5$ mm. This yields $Q_{\max} = 78$ pC; multiply by rf frequency of 50 MHz, we get

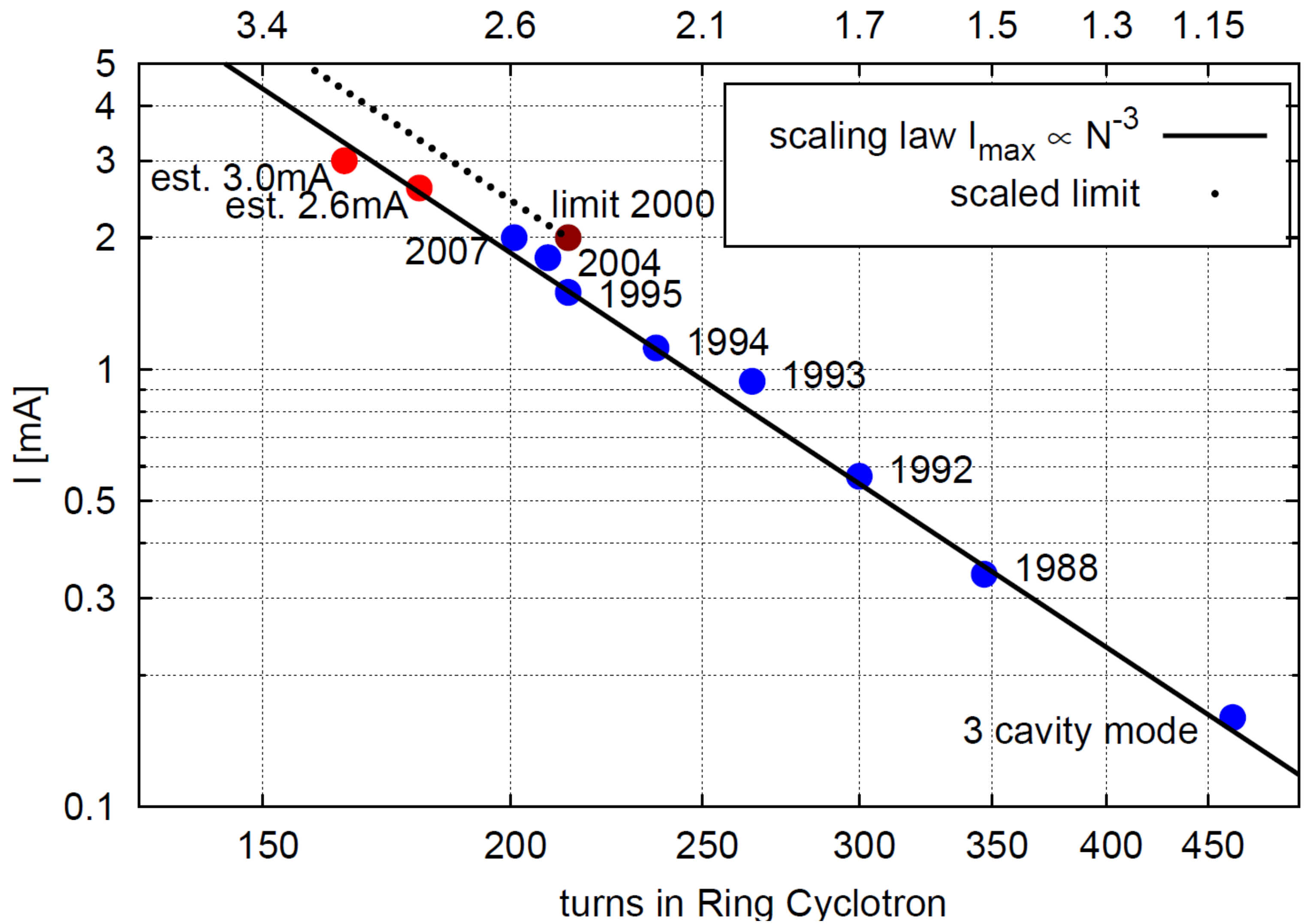
$$I_{\max} = 4\,\text{mA} \tag{8}$$

Don't quote this, though; it's not the whole story (yet). Field is still flat $\nu_x = 1$, it is non-relativistic, bunches are spheres.

However, we have established that: $Q_{\max} \propto r^3$. Since the rf voltage needed for clean extraction is $V_{\rm rf} \propto r$, we have

$$I_{\max} \propto V_{\rm rf}^3 \propto \text{turns}^{-3}, \tag{9}$$

in agreement with PSI's oft-quoted "scaling law".


3.4
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est. 3.0mA
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2004
1995
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1993
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1988
3 cavity mode
150
200
250
300
350
400
450
turns in Ring Cyclotron

Let r expand: Quartic Equation

Bertrand and Ricaud go on to self-consistent case of a spherical bunch with finite emittance. They derive the following quartic equation.

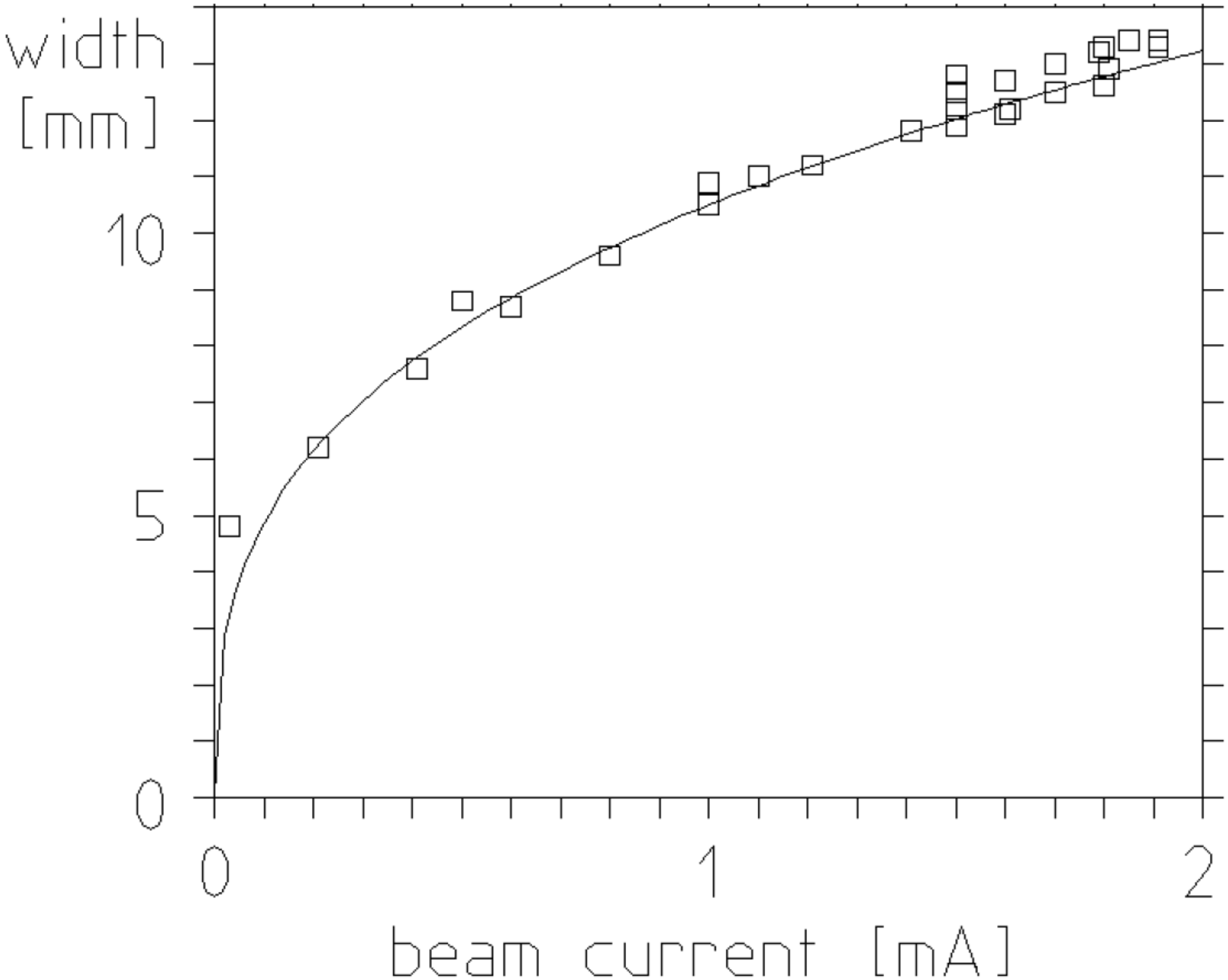
$$r^4 - C\,r - r_0^4 = 0 \tag{10}$$

where $C = \frac{qQ}{\pi\epsilon_0 m\omega^2}$, $r_0 = \sqrt{2R\epsilon_x}$.

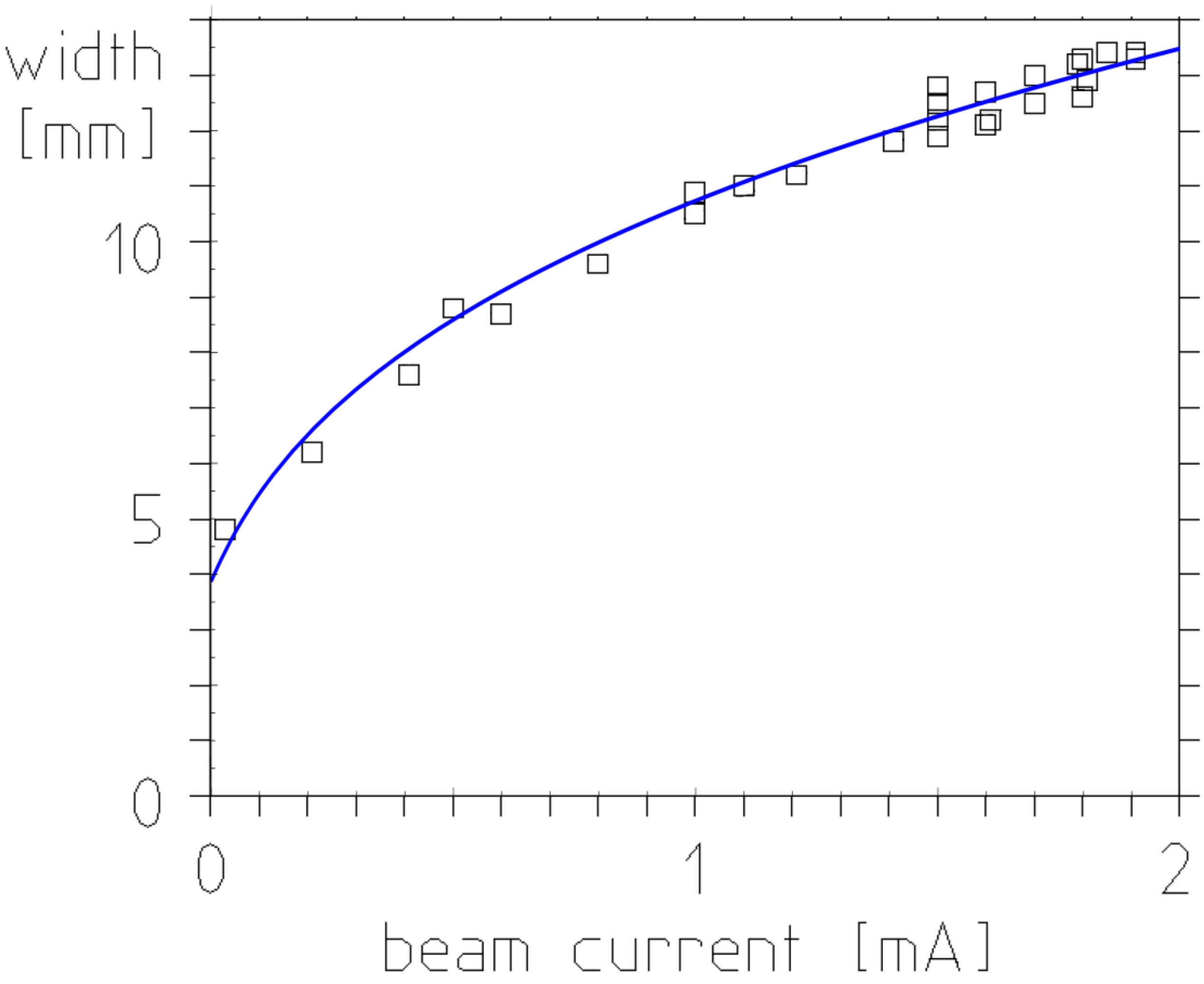
At zero charge, one would expect $r_0 = \sqrt{R\epsilon_x}$ since the Courant-Snyder beta-function $\beta_x = R$ for this flat magnet. However, there is a factor of 2 arising from the fact that circular bunches can only be stationary if the longitudinal and radial emittances are equal. Since the beam must be dispersion-matched, each emittance contributes to the beam size.

For non-spherical bunches, neighbouring turn effects, etc. C may be different by a factor, so leaving this as a free parameter but using their quoted $\epsilon_x = 1\,(\pi)$mm-mrad, we find the blue curve:

PSI Injector II, their graph (Stammbach et al.2001)



PSI Injector II, quartic solution fit



Digression: RMS Envelope Equations

To get to the fully relativistic, non-spherical theory, we need some background on space charge and Second Moments.

In 1971, Frank Sacherer published one of the most important papers in accelerator physics:

RMS ENVELOPE EQUATIONS WITH SPACE CHARGE[1]

Frank J. Sacherer
CERN, Geneva, Switzerland

Summary

Envelope equations for a continuous beam with uniform charge density and elliptical cross-section were first derived by Kapchinsky and Vladimirsky[2] (K-V). In fact, the K-V equations are not restricted to uniformly charged beams, but are equally valid for any charge distribution with elliptical symmetry, provided the beam boundary and emittance are defined by rms (root-mean-square) values. This results because (i) the second moments of any particle distribution depend only on the linear part of the force (determined by least squares method), while (ii) this linear part of the force in turn depends only on the second moments of the distribution. This is also true in practice for three-dimensional bunched beams with ellipsoidal symmetry, and allows the formulation of envelope equations that include the effect of space charge on bunch length and energy spread.

The utility of this rms approach was first demonstrated by Lapostolle[3] for stationary distributions. Subsequently, Gluckstern[4] proved that the rms version of the K-V equations remain valid for all continuous beams with axial symmetry. In this report these results are extended to continuous beams with elliptical symmetry as well as to bunched beams with ellipsoidal form, and also to one-dimensional motion.

Moment equations

Consider an ensemble of particles that obey the single-particle equations

$$\dot{x} = p \qquad\qquad \dot{p} = F(x,t) \ , \tag{1}$$

where $F(x,t)$ includes both the external force and the self-force, $F = F_e + F_s$. Averaging (1) over an arbitrary particle distribution $f(x,p,t)$, we obtain

moment equations, namely the equation for each moment involves the higher moments in an endless hierarchy. However, if the self-force is derived from the free-space Poisson equation, $\overline{xF_s}$ depends mainly on the second moments and very little, if at all, on the higher moments. This will be demonstrated in the following sections. The remaining term $\overline{pF_s}$ is associated with emittance growth; we will avoid considering it by assuming that the rms emittance

$$E = \sqrt{\overline{x^2}\ \overline{p^2} - \overline{xp}^2} \tag{5}$$

is either constant, or that its time dependence is known *a priori*. Then $\overline{p^2}$ is given in terms of $\overline{x^2}$, $\overline{xp}$, and $E(t)$ by (5), and the first two equations of (4) form a closed set. They can be combined to give the K-V type equation:

$$\ddot{\tilde{x}} + K(t)\tilde{x} - \frac{E^2}{\tilde{x}^3} - \frac{\overline{xF_s}}{\tilde{x}} = 0 \ , \tag{6}$$

where $\tilde{x}$ is the rms value, $\tilde{x} = \sqrt{\overline{x^2}}$.

The space-charge term in this equation has an interesting interpretation. If we define the linear part of the force $F_s(x,t)$ as $\varepsilon(t)x$, where $\varepsilon(t)$ is determined by minimizing the difference

$$D = \int \left[\varepsilon(t)x - F_s(x,t)\right]^2 n(x,t)\ dx \tag{7}$$

for a fixed t, where $n(x,t) = \int f(x,p,t)\ dp$, then

$$\varepsilon(t)x = \frac{\overline{xF_s}}{\tilde{x}^2}\ x \ . \tag{8}$$

In other words, the rms envelope equation depends only on the linear part of the forces, determined by least squares method.

It is convenient to put equation (4) into matrix form. The assumption of constant rms emittance is

Not as well known is that there is a much fuller version of this paper:

EE

EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

96

CERN/SI/Int. DL/70-12
18.11.1970

[redacted]

P00031499

RMS ENVELOPE EQUATIONS WITH SPACE CHARGE

Frank J. Sacherer

ABSTRACT

The envelope equations for a continuous beam with circular symmetry but otherwise arbitrary charge distribution have been derived by Lapostolle and Gluckstern. Their results are extended in this report to continuous beams with elliptical symmetry and to bunched beams with ellipsoidal form.

Main result is to generalize the Kapchinsky-Vladimirsky envelope equations to non-uniform distributions and arbitrarily-coupled optics.

> square) values. This results because (i) the second moments of any particle distribution depend only on the linear part of the force (determined by least squares method), while (ii) this linear part of the force in turn depends only on the second moments of the distribution. This is also true in practice for three-dimensional bunched beams with ellipsoidal symmetry, and allows the formulation of envelope equations that include the effect of space charge on bunch length and energy spread.

The "second moments" are nothing but the σ-matrix of TRANSPORT notation. E.g. $\sigma_{11} = \overline{x^2}$, $\sigma_{12} = \overline{xP_x}$, $\sigma_{13} = \overline{xy}$, etc.

Beam evolution for space charge and other non-analytic elements

When the transfer matrix does not have a closed-form expression, we do not have a transfer matrix $\mathbf{M}$, we can use the infinitesimal transfer matrix $\mathbf{F}$ instead. In the $\boldsymbol{\sigma}$-matrix equation $\boldsymbol{\sigma}_\mathbf{f} = \mathbf{M}\boldsymbol{\sigma}_\mathbf{i}\mathbf{M}^\mathbf{T}$, the transfer matrix of an infinitesimal length ds is $\mathbf{M} = \mathbf{I} + \mathbf{F}ds$, we find directly the equations of motion of the second moments.

$$\boldsymbol{\sigma}' = \mathbf{F}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{F}^T \tag{11}$$

If all elements are simple in the sense that the transfer matrices $\mathbf{M}$ are known, then they are simply multiplied together to find the matrix of the whole beamline or synchrotron, and the final beam is found from the initial. If not, as with space charge, 11 is solved with a Runge-Kutta integrator.

Sacherer 1971 Envelope Equation

For the envelope equation for a 3D bunch of charge, Sacherer did all the "heavy lifting": deriving the linear part of the space charge force from any 3D distribution of charge, as a function of its second moments:

$$g_x\left(\frac{b}{a}, \frac{c}{a}\right) = \frac{3}{2}\int_0^\infty \frac{ds}{(1+s)^{3/2}\left(\frac{b^2}{a^2}+s\right)^{1/2}\left(\frac{c^2}{a^2}+s\right)^{1/2}} \quad . \qquad (34)$$

The integral in (34) can be expressed in terms of elliptic integrals of the second kind, but direct numerical evaluation with the Gaussian integration method is easier and also quick and accurate. The complete envelope equation for $\tilde{x}$ is

$$\ddot{\tilde{x}} + K_x(t)\tilde{x} - \frac{E_x^{\,2}}{\tilde{x}^3} - \frac{e^2N\lambda_3}{m\tilde{x}^2}\, g_x\left(\frac{\tilde{y}}{\tilde{x}}, \frac{\tilde{z}}{\tilde{x}}\right) = 0 \;, \qquad (35)$$

Integral is now known as a "Carlson symmetric form".

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Carlson symmetric form

From Wikipedia, the free encyclopedia

In mathematics, the **Carlson symmetric forms of elliptic integrals** are a small canonical set of elliptic integrals to which all oth reduced. They are a modern alternative to the Legendre forms. The Legendre forms may be expressed in terms of the Carlson forr versa.

The Carlson elliptic integrals are:

$$R_F(x,y,z) = \tfrac{1}{2}\int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)}}$$

$$R_J(x,y,z,p) = \tfrac{3}{2}\int_0^\infty \frac{dt}{(t+p)\sqrt{(t+x)(t+y)(t+z)}}$$

$$R_C(x,y) = R_F(x,y,y) = \tfrac{1}{2}\int_0^\infty \frac{dt}{(t+y)\sqrt{(t+x)}}$$

$$R_D(x,y,z) = R_J(x,y,z,z) = \tfrac{3}{2}\int_0^\infty \frac{dt}{(t+z)\sqrt{(t+x)(t+y)(t+z)}}$$

Since R_C and R_D are special cases of R_F and R_J, all elliptic integrals can ultimately be evaluated in terms of just R_F and R_J.

THEORY OF ACCELERATED ORBITS AND SPACE CHARGE EFFECTS IN AN AVF CYCLOTRON

PROEFSCHRIFT

ter verkrijging van de graad van doctor aan de Technische Universiteit Eindhoven, op gezag van de Rector Magnificus, prof. dr. F.N. Hooge, voor een commissie aangewezen door het College van Dekanen in het openbaar te verdedigen op vrijdag 19 augustus 1988 te 16.00 uur

door

WILLEM JAN GERARD MARIE KLEEVEN

geboren te Horst

The first (and only?) person to apply the Sacherer 1971 space charge second moments technique to cyclotrons is Wiel Kleeven (thesis, 1988).

The vertical motion (here called z) separates and has exactly same envelope equation as Sacherer's. But the median plane (x, P_x, s, P_s) has coupling due to dispersion. The equations of motion of second moments follow:

$$\frac{d}{d\tau}\langle \hat{x}^2 \rangle = 2\langle \hat{x}\hat{p}_x\rangle + \upsilon_x \langle \hat{x}\hat{s}\rangle$$
$$\frac{d}{d\tau}\langle \hat{x}\hat{p}_x \rangle = \langle \hat{p}_x^2\rangle - \frac{1}{4}(\upsilon_x^2 - 4a)\langle \hat{x}^2\rangle + \frac{1}{2}\upsilon_x(\langle \hat{x}\hat{p}_s\rangle + \langle \hat{s}\hat{p}_x\rangle) + d\langle \hat{x}\hat{s}\rangle$$
$$\frac{d}{d\tau}\langle \hat{p}_x^2 \rangle = \upsilon_x \langle \hat{p}_x\hat{p}_s\rangle - \frac{1}{2}(\upsilon_x^2 - 4a)\langle \hat{x}\hat{p}_x\rangle + 2d\langle \hat{s}\hat{p}_x\rangle$$

a, b, d are the elliptic integrals

$$\frac{d}{d\tau}\langle \hat{s}^2 \rangle = 2\langle \hat{s}\hat{p}_s\rangle - \upsilon_x \langle \hat{x}\hat{s}\rangle$$
$$\frac{d}{d\tau}\langle \hat{s}\hat{p}_s \rangle = \langle \hat{p}_s^2\rangle - \frac{1}{4}(\upsilon_x^2 - 4b)\langle \hat{s}^2\rangle - \frac{1}{2}\upsilon_x(\langle \hat{x}\hat{p}_s\rangle + \langle \hat{s}\hat{p}_x\rangle) + d\langle \hat{x}\hat{s}\rangle$$
$$\frac{d}{d\tau}\langle \hat{p}_s^2 \rangle = -\upsilon_x \langle \hat{p}_x\hat{p}_s\rangle - \frac{1}{2}(\upsilon_x^2 - 4b)\langle \hat{s}\hat{p}_s\rangle + 2d\langle \hat{x}\hat{p}_s\rangle$$

$$\frac{d}{d\tau}\langle \hat{x}\hat{s} \rangle = \frac{1}{2}\upsilon_x(\langle \hat{s}^2\rangle - \langle \hat{x}^2\rangle) + \langle \hat{x}\hat{p}_s\rangle + \langle \hat{s}\hat{p}_x\rangle \qquad (4.77)$$
$$\frac{d}{d\tau}\langle \hat{p}_x\hat{p}_s \rangle = \frac{1}{2}\upsilon_x(\langle \hat{p}_s^2\rangle - \langle \hat{p}_x^2\rangle) - \frac{1}{4}(\upsilon_x^2 - 4a)\langle \hat{x}\hat{p}_s\rangle + - \frac{1}{4}(\upsilon_x^2 - 4b)\langle \hat{s}\hat{p}_x\rangle + d(\langle \hat{x}\hat{p}_x\rangle + \langle \hat{s}\hat{p}_s\rangle)$$
$$\frac{d}{d\tau}\langle \hat{x}\hat{p}_s \rangle = \langle \hat{p}_x\hat{p}_s\rangle + \frac{1}{2}\upsilon_x(\langle \hat{s}\hat{p}_s\rangle - \langle \hat{x}\hat{p}_x\rangle) - \frac{1}{4}(\upsilon_x^2 - 4b)\langle \hat{x}\hat{s}\rangle + d\langle \hat{x}^2\rangle$$
$$\frac{d}{d\tau}\langle \hat{s}\hat{p}_x \rangle = \langle \hat{p}_x\hat{p}_s\rangle + \frac{1}{2}\upsilon_x(\langle \hat{s}\hat{p}_s\rangle - \langle \hat{x}\hat{p}_x\rangle) - \frac{1}{4}(\upsilon_x^2 - 4a)\langle \hat{x}\hat{s}\rangle + d\langle \hat{s}^2\rangle$$

Stationary case

To find stationary case, simply set LHS to zero and solve. Result is:

$$\langle \hat{s}^2 \rangle = \langle \hat{x}^2 \rangle, \qquad \langle \hat{s}\hat{p}_s \rangle = \langle \hat{x}\hat{p}_x \rangle, \qquad \langle \hat{p}_s^2 \rangle = \langle \hat{p}_x^2 \rangle$$

$$\langle \hat{x}\hat{s} \rangle = \langle \hat{p}_x \hat{p}_s \rangle = 0, \qquad \langle \hat{s}\hat{p}_x \rangle = -\langle \hat{x}\hat{p}_s \rangle = \bar{L}/2N \tag{4.84}$$

which is a circular beam. (QED)

However: the variable s is actually the coordinate in the direction of motion, multiplied by γ. Thus, to be precise, the distribution is not circular but shortened in azimuthal direction by factor γ.

Envelope equations for Cyclotron

Kleeven combines the equations of motion of second moments:

$$\frac{d^2\hat{x}_m}{d\tau^2} + \frac{\nu_x^2}{4}\hat{x}_m - \frac{(\hat{\epsilon}^2 - 8\,\bar{L}^2/N^2)}{32\,\hat{x}_m^3} - \frac{I}{I_o}\frac{1}{\hat{x}_m^2}\,g\left(1, \frac{\hat{z}_m}{\hat{x}_m}\right) = 0.$$

$$\frac{d^2\hat{z}_m}{d\tau^2} + \nu_z^2\,\hat{z}_m - \frac{\hat{\epsilon}_z^2}{16\,\hat{z}_m^3} - \frac{I}{I_o}\frac{1}{\hat{z}_m^2}\,g\left(\frac{\hat{x}_m}{\hat{z}_m}, \frac{\hat{x}_m}{\hat{z}_m}\right) = 0 \qquad (4.85)$$

The $8\overline{L}^2/N^2$ is an angular momentum term.

Bertrand-Ricaud-Kleeven Quartic Equation

To find the stationary beam size, set the derivative to zero. We recover exactly the same equation as Bertrand-Ricaud, except that the constant is multiplied by $\frac{g(1,r_z/r)}{\gamma^2\nu_x^2}$; takes into account: relativity (higher energy raises space charge limit), radial focusing (more focusing raises space charge limit), non-sphericity (stretching bunch in B direction raises space charge limit).

$$r^4 - C_x\, r - r_0^4 = 0 \qquad (12)$$

where

$$C_x = \frac{Q}{\pi\epsilon_0}\,\frac{q}{m(\gamma\nu_x\omega)^2}\,g_x, \qquad r_0 = \sqrt{\frac{2R\epsilon_x}{\nu_x}}. \qquad (13)$$

$(g_x \equiv g(1, \frac{\tilde{z}}{r}))$

To find $Q_{\rm max}$ for a given r, simply set emittance to zero:

$$C_{x\rm max} = r^3_{\rm max}. \qquad (14)$$

General bunch shapes

The quartic equation is still correct for non-hard-edge and even for non-ellipsoids. This was proved by Sacherer, 1971.

But then r is $\sqrt{5}$ times the rms size, and the emittance is 5 times the rms emittance. See Sacherer 1971.

If the bunch shape is far from stationary, then it will change with time and so the rms emittance also will change with time. In that case, the envelope equations, while still correct, are not very useful.

Intensity Limit

The formula for maximum charge is now

$$Q_{\max} = \frac{\pi}{g_x}\left(\frac{V_m}{cZ_0}\right)\frac{r^3}{R_\infty^2}\nu_x^2\gamma^2 \tag{15}$$

The maximum allowed size r is some factor, say ξ smaller than the radius gain per turn at extraction. Thus,

$$\xi r_{\mathrm{f}} = \frac{R_\infty}{\beta_{\mathrm{f}}\gamma_{\mathrm{f}}\nu_{x,\mathrm{f}}^2}\frac{V_{\mathrm{rf,f}}}{V_m} \tag{16}$$

where $V_{\mathrm{rf,f}}$ is the rf voltage per turn on the final orbit and $V_m = mc^2/q$.

Let h be the number of bunches per turn, convert the charge per bunch to current $I = \frac{hQc}{2\pi R_\infty}$, we find a simple expression for maximum current

$$I_{\rm max} = \frac{h}{2g_x \xi^3 \beta^3 \gamma \nu_x^4} \, \frac{V_{\rm rf}^3}{V_m^2 Z_0} \qquad (17)$$

where g_x, β, γ, ν_x, and $V_{\rm rf}$ have of course their extraction values.

Note: for aspect ratios in the range $1/2 \le \zeta/r \le 2$, the approximation $g_x \approx 1 - \frac{3}{5}\log(\zeta/r)$ works well.

Examples:

Let us take spherical bunches, $\xi = 2.7$; this means the allowed turn width is $2.7\sqrt{5} = 6$ times the rms size.

- PSI Ring; $590\,$MeV; $h = 6$; $V_{\text{rf}} = 3\,$MV $\rightarrow I_{\max} = 2.2\,$mA

- PSI Inj.2; $72\,$MeV; $h = 10$; $V_{\text{rf}} = 0.75\,$MV $\rightarrow I_{\max} = 2.1\,$mA

Reminder: Joho “sector model” fails for PSI Injector 2.

Current Limit in 72 MeV Injector II

$$I_L = \frac{\mu_n}{f_n} \frac{(U_f - U_i)}{Z_I} \frac{\beta_f}{n^3} \frac{\Delta\Phi}{2\pi} = \text{current limit from long. space charge}$$

$U_f = 72\,\text{MV}$, $\beta_f = 0.37$

$n = 85$ $(\Delta E_n \approx 0.75\,\text{MeV})$, $\Delta\Phi \approx 6^0$

$f_n \approx 1/4$ (rough estimate only !)

$\mu_n \approx 1/3$ for centered beam

$I_L \approx 0.3\,\text{mA}$ (present record = 2.7 mA !)

=> sector model fails due to phase mixing

(space charge forces produce spherical bunch, "spaghetti effect")

=> this lowers energy spread from longitudinal space charge

=> much higher current limit !

W.Joho 2013

It's only Approximate...

These can be expected to be only within a factor of ~ 2 of the real limit, because of the following considerations.

- It's too high, because this is the limit at unrealistic $\epsilon_x = 0$.
- It's too low, because it does not inlcude "tricks" like coherent oscillations (especially the PSI ring).
- It's too low, because it assumes spherical bunches. Can gain a factor $1/g_x$ by increasing the vertical beam size. For example if aspect ratio is $2:1$, gain a factor of 1.6.
- The ξ parameter needed depends crucially on the amount of "halo". For example, if radius gain per turn at extraction is 5 times rms rather than 6 times, $I_{\max}$ increases by a factor $(6/5)^3 = 1.73$.

Scaling

Thus we have the "cubic scaling law" with rf voltage, but further:

- For given energy per nucleon, heavier particles hinder rather than help.
- Large radial tune hinders rather than helps: it increases the space charge limit for a given beam size, but it reduces radius gain per turn and latter effect dominates.
- More bunches per turn (higher harmonic number) always helps. But may cause difficulty at low energy.
- Higher magnetic field neither helps nor hinders, provided h is unchanged, thus higher rf frequency for higher B. Also, more difficult to get the needed V_{rf} into the smaller machine.

Paradox

Surprising features of the stationary distribution shape:

- Constant size means rf phase length of bunch decreases during acceleration.

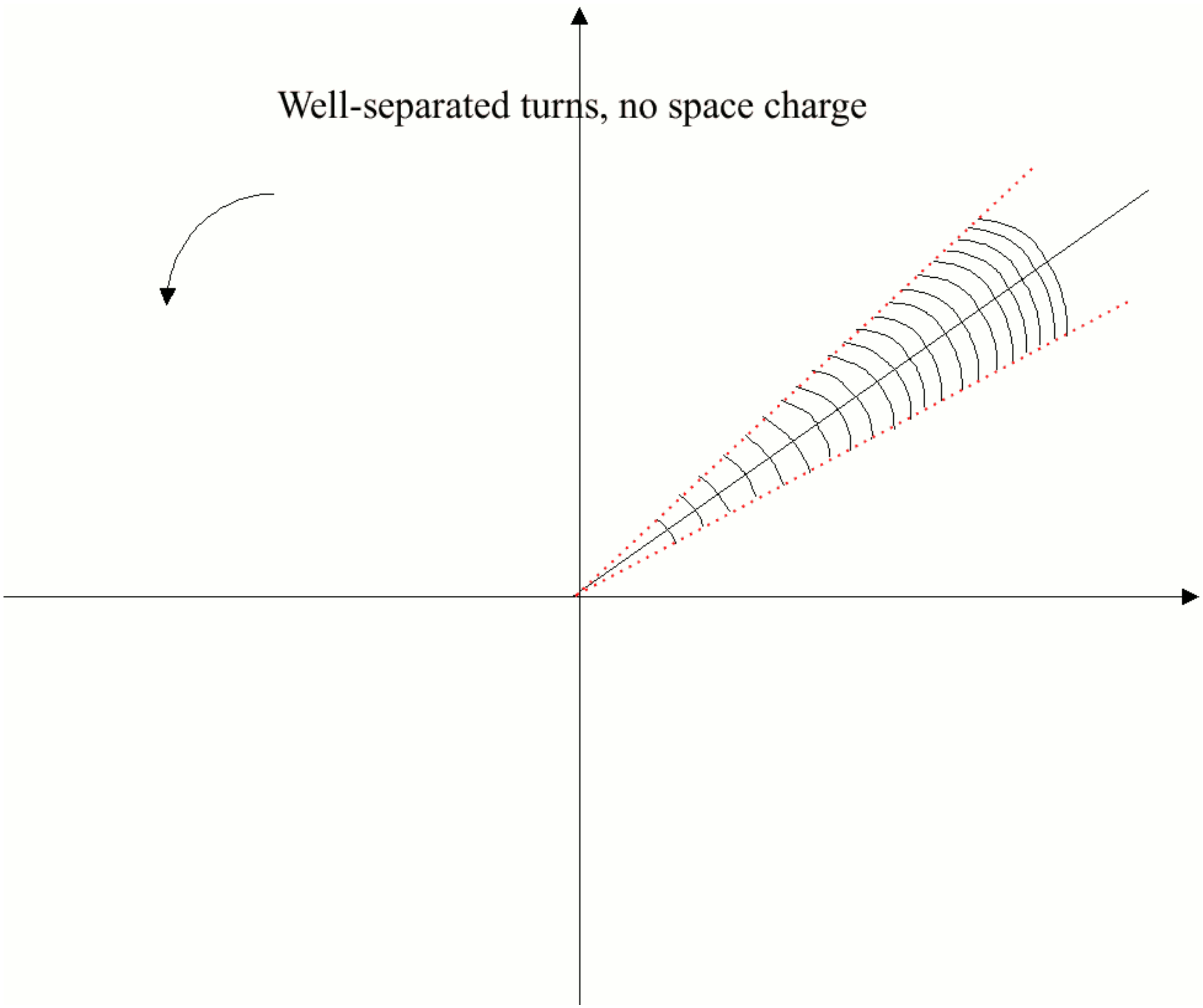
- Round beam stationary shape is independent of intensity.

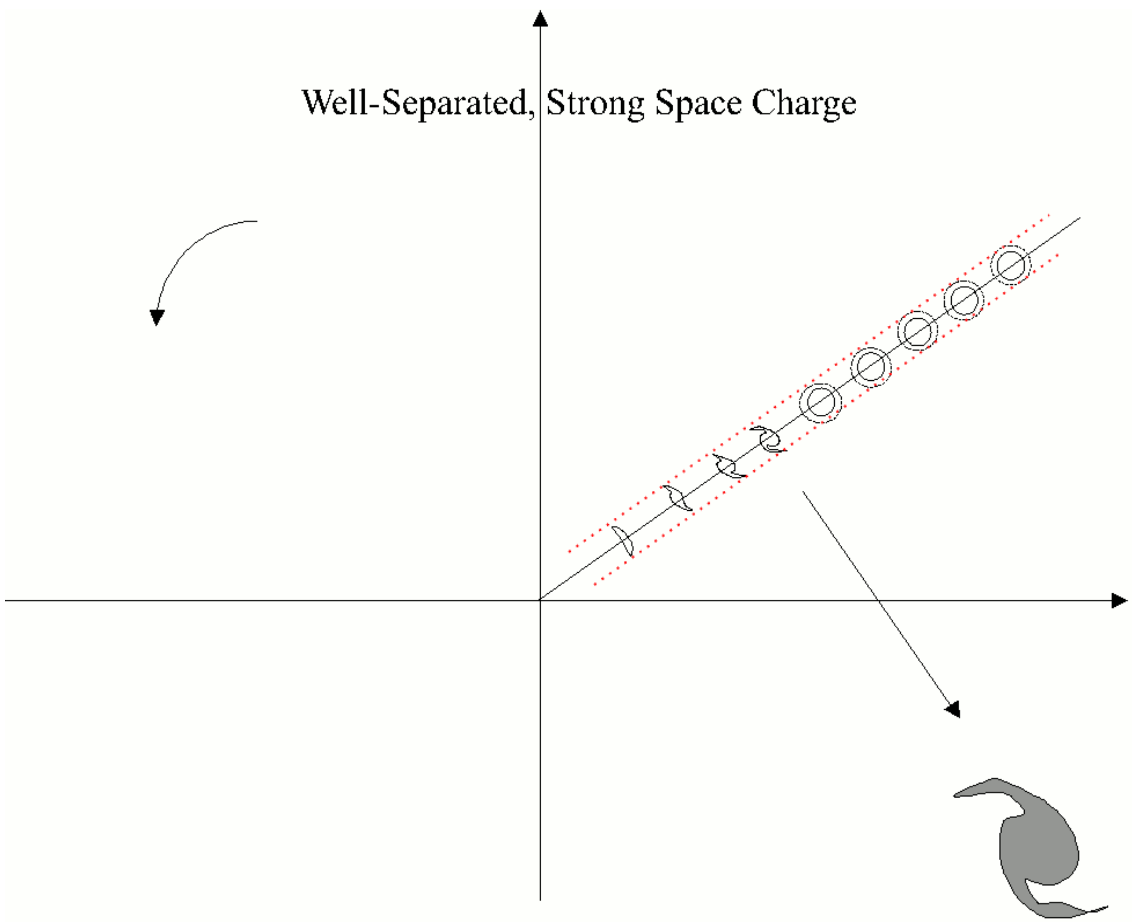
How can this be, since without space charge bunch length increases with acceleration maintaining same phase length?

Journées Accélérateurs de la SFP
5-7 Novembre 2001
Roscoff

Faisceaux sphériques dans les cyclotrons
(Théorie et simulation)

P. BERTRAND, C. RICAUD
GANIL FRANCE

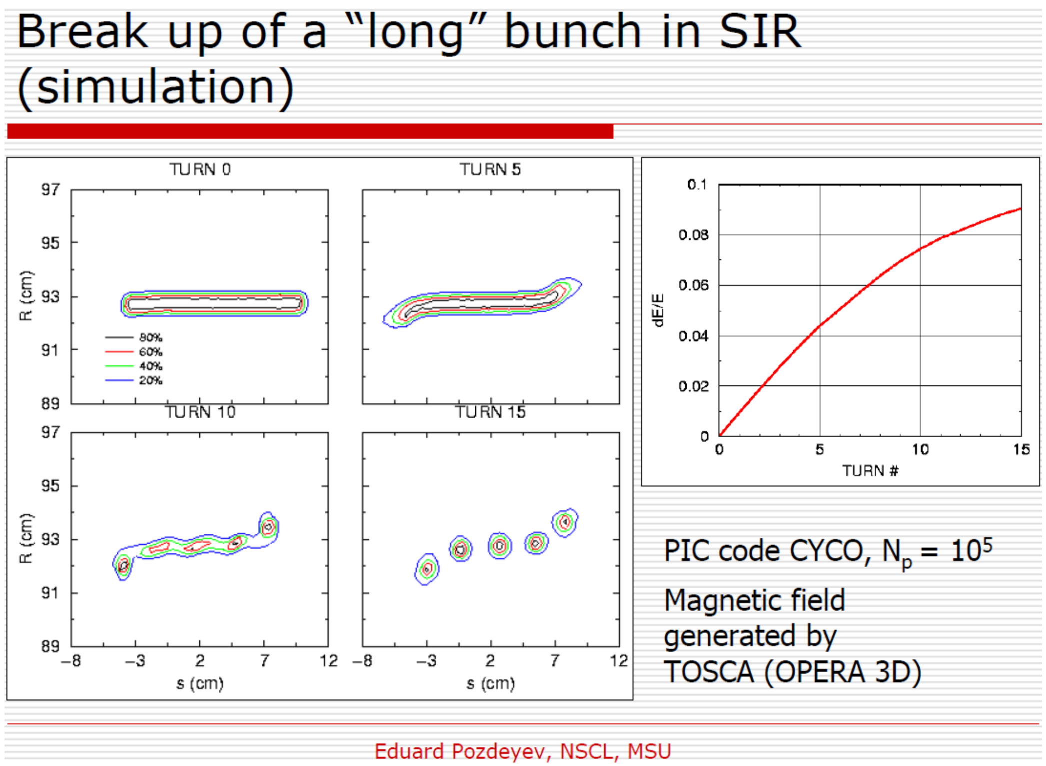

Well-separated turns, no space charge


Well-Separated, Strong Space Charge

Long Bunches

That was for short bunches. But what happens if bunches are much longer than wide?

They split into many droplets Eduard Pozdeyev (2003):



Condition for Pure Vortex State

There is a competition between vortex effect and constant phase length effect. The winner can be found by comparing:

$$\text{azimuthal change per turn from acceleration} = \delta R \Delta\theta \tag{18}$$

$$\text{radial change per turn from space charge} = 2\pi \Delta\nu_r R \Delta\theta \tag{19}$$

$R = \beta R_\infty$, $\Delta\theta$ is azimuthal extent. Thus if

$$2\pi \Delta\nu_r \gg \frac{\delta\beta}{\beta}, \tag{20}$$

then a launched circular dispersion-matched bunch will remain circular. A non-matched non-circular bunch will match itself after a number of turns $\gg 1/\Delta\nu_r$, and the generated halo will depend upon the initial mismatch.

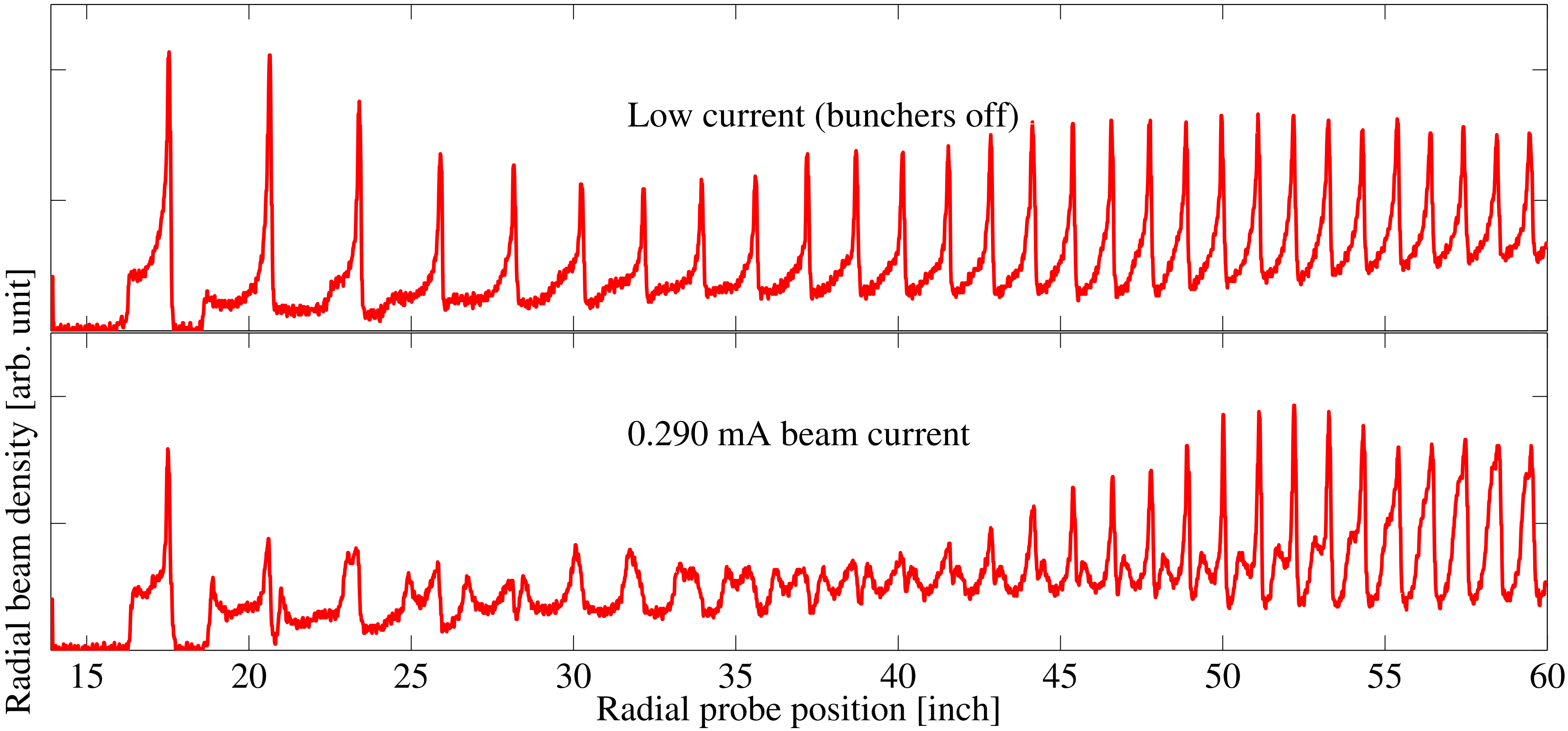
Importantly, remember that the cyclotron space charge tune shift at non-relativistic energy is independent of energy: $\Delta\nu_r = -\frac{qQ}{4\pi\epsilon_0 r^3 m\omega^2}$, r and ω constant, and that the space charge limit is where the tune is depressed by half. Thus an injector cyclotron operating near the space charge limit will have $\Delta\nu_r \sim \nu_x \sim 1$. Since usually in such a machine $\delta\beta \sim \beta$, it will start and remain in a vortex state. ($\delta\beta$ rapidly decreases.)

This is the case for PSI Inj.2, already at turn 1.

For TRIUMF, just after the injection gap, energy is $390\,$keV, and after one turn it is $750\,$keV, so $\frac{\delta\beta}{\beta} = 0.6$. The tune shift at $250\,\mu$A is $\sim 1/40$, so $2\pi\Delta\nu_r \sim 0.16$. So it is clearly in an in-between state: bunches stretch and also have some vortex character.

This is verified when we look at the turn structure.

TRIUMF cyclotron's first 30 turns



Extraction by Stripping does not care about ΔE of a Turn

TRIUMF's messy turns have little consequence. The reason is that TRIUMF extracts by stripping and separated turns are not needed. Irregular turns do not contribute to extracted emittance or energy spread because the distortions happen in such a way as to maintain the close correlation between energy and radius.

In general, this means that high output of compact cyclotrons, which is attained by very large phase acceptance, is not applicable to separated turn cyclotrons. In fact, Pozdeyev's experiments show that if you try, the bunches split into many droplets; separate turns become impossible.

Alternative route to high intensity:

Use high harmonic number, since $I_{\rm max} \propto h$.

Similar to condition 20, we can find a condition for space charge dominating over the effects of waveform non-linearity:

$$2\pi\Delta\nu_r \gg \frac{\text{turn separation}}{\text{bunch length}} \frac{(\Delta\phi)^2}{2} = \frac{\delta\beta}{\beta} \frac{h\Delta\phi}{2} \qquad (21)$$

($\Delta\phi = h\Delta\theta = h\Delta\theta$ is rf phase length.)

This criterion is satisfied for the PSI Inj.2; it explains why they do not need flattopping at their highest intensities.

But it indicates that increasing h will at some point decrease the space charge limit. (This holds some promise, though.)

Conclusions

- A new formula for maximum intensity for separated turn cyclotrons has been derived from envelope theory, and its scaling characteristics explored.
- The formula applies to cases where the injected bunch is sufficiently short that the vortex effect curls it up into a single droplet.
- A qualitative intensity threshold has been derived for the vortex effect to take place; below the threshold, bunches expand to maintain their phase length as β increases, but above it, the bunch maintains length and consequently decreases in phase length.
- The high output of compact cyclotrons, which is attained by very large phase acceptance, is not applicable to separated turn cyclotrons.