Chromatic and Space Charge Effects in Nonlinear Integrable Optics*

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Abstract

The IOTA test accelerator is under construction at FNAL to study a novel method of advancing the intensity frontier in storage rings: nonlinear integrable optics. For particles at the design momentum, the lattice has two invariants and the dynamics is integrable. In the ideal single-particle two-dimensional case, this yields bounded, regular orbits with extremely large tune spreads. Off-momentum effects such as dispersion and chromaticity, and collective effects such as direct space charge, break the integrability. We discuss the origin of this broken integrability for both single- and many-particle effects, and present simulation results for the IOTA lattice used as a high intensity proton storage ring.

INTRODUCTION

Future generations of intense, multi-megawatt accelerators have applications for discovery science as drivers for spallation sources, neutrino physics, and the next generation of high energy colliders. Such intense beams are prone to collective instabilities including, but not limited to: space charge driven beam halo, resistive wall instability, head-tail instability, and the various beam break-up instabilities. The physical origin of these instabilities is the constant transverse tunes in linear strong-focusing lattices. In the SNS accumulator ring, for example, it was found [1] that these instabilities did not appear for the natural chromaticity of the lattice, which was very large. It is then natural to conclude that the large tune spreads associated with these chromaticities are desirable for mitigating such instabilities.

The trouble with this is that the large chromaticities in the linear lattices will span an entire integer or more of tune space, which will cross many single-particle resonances. A more robust method is required to obtain large tune spreads without losing dynamic aperture due to single-particle dynamics. Enter the nonlinear integrable optics designed by Danilov and Nagaitsev [2], which introduce very large tune spreads while keeping the orbits regular. This work has already shown promise in preventing space charge driven beam halo [3] In this proceeding, we discuss how these invariants change in two real-machine situations: energy spread and space charge.

In the next section, we discuss how energy spread in coasting beams breaks the single-particle integrability, and how we may design the lattice to restore that integrability. How direct space charge changes a matched distribution of the invariants is explored in the following section. We conclude with preliminary simulations of using the integrable optics to prevent resistive wall instability in an intense proton ring.

OFF-MOMENTUM EFFECTS: CHROMATICITY & DISPERSION

The work in [2] considers purely two-dimensional particle dynamics – transverse oscillations with no energy spread. In real intense accelerators energy spread and the associated chromaticity, as well as the dispersion in the lattice, will modify the integrable Hamiltonian. Before asking the nonlinear integrable lattices to mitigate intensity-driven effects, it is important to restore the integrability which makes it so robust.

As we show in [4], a Lie operator treatment of a ring designed for integrable optics that includes off-momentum effects and nonlinear elements such as sextupoles and octupoles yields a correction to the Hamiltonian in [2] due to dispersion in the elliptic magnet sections and the lattice chromaticity. The integrable lattice factors into a product of maps:

$$\mathcal{M} = \mathcal{A}^{-1} e^{-t:\int ds \mathcal{U}(x-\delta\eta(s),y):} e^{-:h:} e^{-t:\int ds \mathcal{U}(x-\delta\eta(s),y):} \mathcal{A} \tag{1}$$

where $\mathcal{A}$ is the normalizing map, and h is the Hamiltonian that generates the single turn map for the integrable optics lattice when the nonlinear elliptic potential strength is zero. Thus, h includes drifts, dipoles, and quadrupoles, as well as chromaticity-correcting families of nonlinear magnets. The details of this calculation may be found in [4] and are too lengthy to include here.

The resulting Hamiltonian for the total single turn map is given, to lowest order, by:

$$\overline{\mathcal{H}} = \frac{\mu_0}{2}\Big\{[1-C_x(\delta)]\left(\overline{p}_x^2+\overline{x}^2\right)+[1-C_y(\delta)]\left(\overline{p}_y^2+\overline{y}^2\right)+ \frac{t}{1-\delta}\int_0^{\ell_{\text{drift}}} \mathcal{U}\left(\overline{x}-\eta(s')\,\delta),\overline{y}\right) ds'\Big\} + \dots \tag{2}$$

where ... are higher order terms, including any nonlinear terms left over after adjusting the chromaticity. Here we have assumed a coasting beam with no synchrotron oscillations, thus δ is a constant. $\eta(s)$ is the dispersion function through the drift where the elliptic magnetic element will be placed, and $\mathcal{U}$ is the nonlinear elliptic potential from [2]. This means that the vertical and horizontal chromaticities, $C_y(\delta)$ and $C_x(\delta)$ respectively, are general functions of δ. We also concluded that conventional chromaticity correction schemes – using sextupoles to correct

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linear chromaticity with $(2n+1)\pi$ phase advances between the sextupoles to minimize dynamic aperture loss, for example – still apply. The goal is to bring this lowest-order Hamiltonian into a form which has two invariants and thus preserves the two-dimensional integrability

The Hamiltonian in [2] derives from a self-consistent solution for free-space magnetic fields and the Bertrand-Darboux partial differential equation for Hamiltonians with a second invariant which is quadratic in the momentum. One of the fundamental assumptions going into the derivation of the Bertrand-Darboux equation is that the coefficients of the vertical and horizontal momenta must be equal. In doing see, we obtain a modification of the Hamiltonian in eqn. (22) of [2]:

$$H = (1 - C(\delta)) \left[\frac{1}{2} \left(p_x^2 + p_y^2 + x^2 + y^2 \right) + \frac{1}{1 - C(\delta)} V(x, y) \right] \quad (3)$$

This leads us to four design principles for building a lattice ready for the nonlinear integrable optics to obtain an integrable Hamiltonian, even in the presence of energy spread:

1. Vertical and horizontal linear tunes must be equal
2. Vertical and horizontal beta functions inside the drift where the nonlinear magnet is to be placed must be equal
3. Dispersion must vanish inside this drift
4. Vertical and horizontal chromaticities must be equal

Non-dispersive sections of rings are fairly standard, and make the integral over the dispersion function into simply a multiplication by ℓ_{drift}. The conclusion of equal chromaticities is based on the following line of reasoning.

Chromaticity correction in conventional strong-focusing linear lattices is a balancing act between having enough tune spread to Landau damp instabilities, while keeping the tune spread small enough to avoid crossing nonlinear resonances. Because the nonlinear integrable lattices already have large tune spreads, which will already cross nonlinear resonances, it is most important to keep the integrability of the unperturbed Hamiltonian so that the KAM theorem applies near these resonances. It is thus sufficient to restore the conditions required for the Bertrand-Darboux equation, specifically that the transverse momenta have the same coefficient. We can therefore adjust C_x and C_y until they are equal. This is actually beneficial to the dynamic aperture, as this reduces the strength required of the chromaticity-correcting nonlinear elements. It also gives us the freedom to choose whether we correct one or both chromaticities, depending on which choice will lead to the best dynamic aperture.

SPACE CHARGE & THE INVARIANTS

We now return to study two-dimensional effects – how space charge changes the distribution of a longitudinally cold beam. We consider here the distribution of the Hamiltonian, H, and second invariant, I, described in [2]. We began with a beam with an exponential distribution in H, viz.

$$f = \frac{N}{\varepsilon} e^{-H/\varepsilon} \quad (4)$$

where N is the number of particles per unit length in the beam and ε is the transverse emittance. This distribution reduces to a Gaussian distribution for a linear lattice, with ε being simply the RMS emittance, equal in the vertical and horizontal. Because the elliptic potential creates strong transverse coupling, such a separation into "vertical" and "horizontal" emittance is not possible for these lattices.

The addition of space charge adds a self-consistent term to the Hamiltonian, so that the total Hamiltonian is given by

$$\mathcal{H} = H_0(J) + \mathcal{V}[f] \quad (5)$$

where $\mathcal{V}$ is the space charge potential. This has the effect of changing the invariants, and so the initial distribution will evolve. Here, $\mathcal{V}[f]$ is a functional of the phase space distribution, which can be thought of as the Green's function for the potential as a function of the charge distribution, as expressed in action-angle variables.

We use as a figure of merit the value of $H_0(p, q)$ as a function of time. The initial distribution is a delta function, $\delta(H_0 - \varepsilon)$. This function is known from [2] eqn. (22), and is analogous to the Courant-Snyder invariant for linear lattices. Thus, this bunch is analogous to a Kapchinskij-Vladimirskij distribution with emittance $\varepsilon = 15\,\text{mm} - \text{mrad}$. We therefore expect that the spread in H_0 should be a good indication of the beam evolution. In Fig. 1, we see that the spread in H_0 grows dramatically on two time scales. There is one time scale that is a handful of turns that marks a dramatic increase in H_0, and then a slower diffusive-like process that takes over after this.

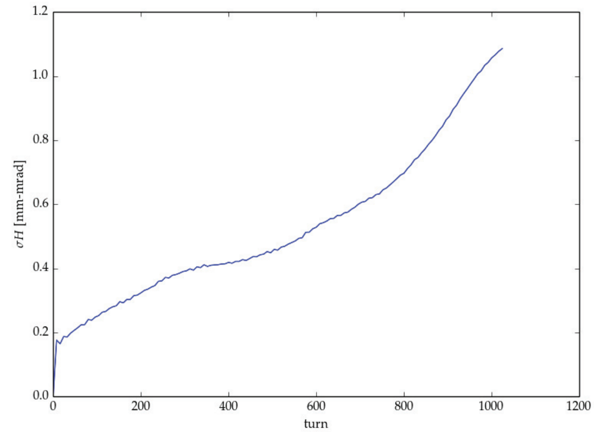

Figure 1: The r.m.s. value of H_0 as a function of turn.

Based on this, we would expect the bunch distribution to be expanding quite rapidly. However, this is not the case, as we see in the phase space projections from turn 712 (Fig. 2), well after the diffusion has taken hold. The

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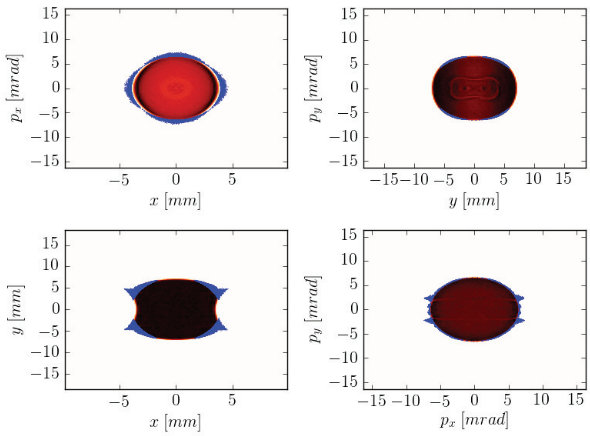

Figure 2: The transverse phase space projections after 712 turns.

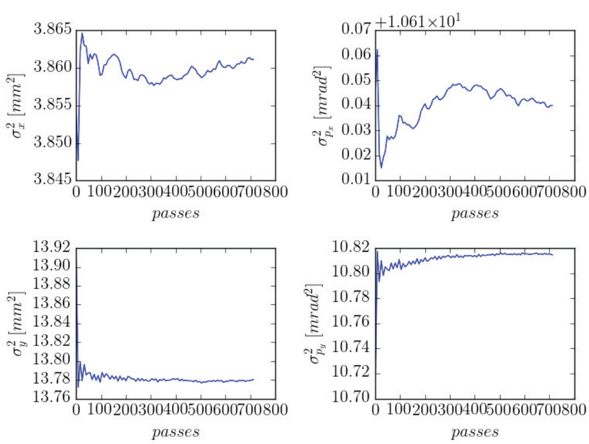

Figure 3: RMS beam size and momentum spread versus time.

initial distribution is much the same as this one, despite the RMS value of H_0 being 5 % of the initial H_0. Longer time simulations do not show a substantial departure from this distribution. Indeed, we find that variations in the RMS beam size vary at the 0.1 % level, in Fig. 3. This is contrary to our intuition based on linear optics – the beam radius is given by $\sigma_r \propto \sqrt{\beta\varepsilon}$, and therefore an increase in ε should correspond to a similar increase in the beam size.

We speculate that the origin of this can be derived from a Fokker-Planck equation for diffusion in action space due to the space charge perturbation [5]. Taking the phase space distribution $f = f(J)$ as a pure function of the action, the stochasticity due to chaotic trajectories and phase mixing leads to a Fokker-Planck equation

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial J}\left\{\frac{1}{4}(\Delta t)^2\left(\frac{\partial}{\partial J}\left\langle\left(\frac{\partial\mathcal{V}[f]}{\partial\psi}\right)^2\right\rangle_\psi\right)\frac{\partial f}{\partial J}\right\} \tag{6}$$

The total Hamiltonian $\mathcal{H}$ is broken into three parts: the single particle integrable Hamiltonian, the component of space charge that remains integrable, and the component of space charge that induces diffusion. Specifically:

$$\mathcal{H} = H_0(J) + \mathcal{V}_0[f](J) + \sum_{n\neq 0}\mathcal{V}_n[f](J)e^{in\psi} \tag{7}$$

When we generate a distribution, we are matching it to H_0 – tracking that distribution over many thousands of turns shows no change in the phase space distribution when current is zero. The integrable space charge component, $\mathcal{V}_0[f](J)$, represents a potential well distortion of sorts, and is the source of the rapid early filamentation that occurs over only a handful of turns. This is the origin of the abrupt growth in measured H_0 we see in a handful of turns in Fig. 1 The remaining terms in the space charge, varying with ψ, cause the diffusion we see in H_0 in Fig. 1.

A possible limit for space charge in the nonlinear integrable lattices may be the existence of a stable stationary solution to the Fokker-Planck equation for the phase space distribution. Thus, the figures of merit for a nonlinear integrable lattice to determine the importance of space charge could be related to the functional form of the Fokker-Planck equation, and the existence of stationary solutions. This would go a long way to explaining the stability of these beams in real space, despite their apparently poor behavior in terms of the single-particle invariant distributions.

FUTURE WORK

Nonlinear integrable optics remains a very naïve field – it is not clear how many concepts from conventional linear lattices survive, and how they might be modified. We suggest that there are two lines of inquiry worth pursuing: how the chromatic corrections affect the dynamic aperture, and how space charge affects the distribution.

Chromatic correction schemes are designed to minimize the impact on on-energy dynamic aperture. The π phase advances meticulously cancel terms that are not proportional to the energy offset, and the chromatic correction scheme is built around the terms, say, linear in δ for a sextupole correction. There then remains $\mathcal{O}(\delta^2)$ terms which reduce the dynamic aperture for off-energy particles. In the linear lattice case, this is avoided by avoiding the resonant lines in tune space these remaining terms generate. The nonlinear integrable optics has an enormous tune spread – this is the origin of its great robustness against parametric resonances. It is necessary to gain theoretical guidelines to how this affects the dynamic aperture. There are three specific questions in this line:

1. Where are the resonant lines for these terms in the nonlinear integrable optics?
2. What chromatic correction schemes can be used to get the equal chromaticities while minimizing the impact on dynamic aperture?
3. What is the diffusion time for particles on those resonant lines?

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This requires understanding x and y in terms of two invariants I_1 and I_2, and their associated angle variables, to understand resonance lines and the relevant parameters for these computations. Analytical or semi-analytical results are absolutely necessary for achieving guidance here.

We have established thus far that the nonlinear decoherence prevents the onset of beam halo, and we have studied how the single-particle invariants evolve under space charge. We have seen that space charge prompts diffusion-like behavior in the H and I invariants, which is to be expected. What is curious is that the actual transverse profile does not change dramatically under space charge, even as the spread in the H and I quantities increases dramatically. Understanding this behavior is critical for moving to increasing intensity. This preliminary work suggests three additional questions in the realm of space charge:

1. How are the relevant parameters for characterizing space charge strength?
2. Is there a collective invariant consistent with space charge but different from the single-particle invariant?
3. What can be done to compensate space charge?

The suggestion here is that there are a new set of invariants (H', I') which are better-preserved in the presence of space charge. This is one way to explain how the distribution in the single-particle invariants grows so rapidly, but the actual beam envelope remains relatively stable. Until the effects of space charge are fully understood, we cannot make intelligent decisions about compensation, beam transport limits, *etc.*

The current set of results are promising for implementing a working accelerator that uses the nonlinear integrable optics as a method of transporting intense beams with low loss. We have given a theoretical conclusion for handling the chromaticity. Simulations indicate high-power beams may be transported without any loss due to space charge. But our theoretical understanding of the nonlinear integrable optics in the presence of space charge remains limited. In linear lattices, envelope models yield useful parameters such as the perveance and the Laslett tune shift [6]. No such clear-cut parameters exist to characterize the effects of space charge on the nonlinear integrable optics. It is necessary to determine the relevant physics behind the space charge dynamics in these beams, and in doing so determine new figures of merit. Only then will we be able to understand the real intensity-driven limits to beam transport in the nonlinear integrable optics.

CONCLUSIONS

We have thus discussed two new aspects of the nonlinear integrable lattices of Danilov and Nagaitsev. Through a Lie operator formalism, we were able to obtain two new design requirements for the lattices: (1) the dispersion through the drifts where the elliptic elements will be inserted must be zero, or as small as possible and (2) the vertical and horizontal chromaticities must be equal. By following these two rules, we are able to maintain single-particle integrability even in the presence of energy spread and chromatic effects. We have also studied how space charge affects the beam distribution, and have found some surprising results. While studying the distribution of the single-particle invariant ε, we find a large growth in the RMS spread. Our intuition from linear lattices is that this would lead to a monotonic increase in the RMS beam size. However, our study of the RMS size in the transverse plane show that the size of the beam fluctuates but is not increasing after some initial filamentation. We speculate that there are two processes at work here: a rapid filamentation due to the integrable component of the space charge potential distorting the potential well, and diffusion due to the non-integrable components of the space charge. We believe that a detailed study of this may lead to useful figures of merit characterizing the stability of beams in the presence of intense space charge effects in the nonlinear integrable lattices.

REFERENCES

[1] V.V. Danilov, *et al.*, "Accumulation of high intensity beam and first observations of instabilities in the SNS accumulator ring", HB2014, TUAX01, p. 59 (2006).

[2] V.V. Danilov and S. Nagaitsev, "Nonlinear accelerator lattices with one and two analytic invariants", Phys. Rev. ST-Acc. Beams **13**, 084002 (2010).

[3] S. D. Webb, *et al.*, "Effects of nonlinear decoherence on halo formation", (2012), submitted for publication, arXiv:1205:7083.

[4] S.D. Webb, *et al.*, "Chromatic and Dispersive Effects in Nonlinear Integrable Optics", (2014), submitted for publication.

[5] A. J. Lichtenberg and M. A. Lieberman, *Regular and Chaotic Dynamics*, 2^{nd} ed., Springer-Verlag (1992).

[6] M. Reiser, *Theory and Design of Charged Particle Beams*, Wiley-VCH (2004).

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