THE INFLUENCE OF THE MAGNETIC FIELD INHOMOGENEITY ON THE SPONTANEOUS RADIATION AND THE GAIN IN THE PLANE WIGGLER

Koryun B. Oganesyan*, A.I.Alikhanyan National Science Lab, Yerevan, Armenia

Abstract
We calculate the spectral distribution of spontaneous emission and the gain of electronsmoving in plane wiggler with inhomogeneous magnetic field. We show that electrons do complicated motion consisting of slow(strophotron) and fast(undulator) parts. We average the equations of motion over fast undulator part and obtain equations for connected motion. It is shown, that the account of inhomogenity of the magnetic field leads to appearence of additional peaks in the spectral distribution of spontaneous radiation and the gain.

INTRODUCTION

Free-Electron Lasers are powerful, tunable, coherent sources of radiation, which are used in scientific research, plasma heating, condensed matter physics, atomic, molecular and optical physics, biophysics, biochemistry, biomedicine etc. FELs today produce radiation ranging from millimeter waves through to ultraviolet, including parts of the spectrum in which no other intense, tunable sources are available [1], [2]. This field of modern science is interesting from the point of view of fundamental research and very promising for further applications.

Usually FEL [3], [4] use the kinetic energy of relativistic electrons moving through a spatially modulated magnetic field(wiggler) to produce coherent radiation. The frequency of radiation is determined by the energy of electrons, the spatial period of magnetic field and the magnetic field strength of the wiggler. This permits tuning a FEL in a wide range unlike atomic or molecular lasers. In usual FEL magnetic field of wiggler is supposed to be uniform. But really the magnetic field is inhomogeneous in transverse direction(see for example [5]). It is important to take into account this inhomogeneity. This account leads to complex motion of electrons: fast undulator oscillations along the wiggler axis and slow strophotron motion [6], [7], [8], [9], [10], [11] in transverse direction .

In the Sec II we describe the equation of motion of electrons moving along the axis of the wiggler with transversal inhomogeneous magnetic field. In the Sec. III and IV we calculate the spectral distribution of spontaneous emission and the gain correspondingly.

* bsk@yerphi.am

EQUATIONS OF MOTION

The vector potential of undulator's magnetic field has a form [12]

$$\overrightarrow{A_W} = -\frac{H_0}{q_0}\cosh q_0 x \sin q_0 z \;\; \hat{j} \tag{1}$$

where H_0 is the strength of magnetic field and $q_0 = 2\pi/\lambda_0$, λ_0 - period of wiggler, $\hat{j}$ unit vector in y direction.
Corresponding magnetic field is

$$\begin{aligned}\vec{H} &= rot\vec{A} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) \\ &+ \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = -\hat{i}\frac{\partial A_y}{\partial z} + \hat{k}\frac{\partial A_y}{\partial x} \\ &= \hat{i}H_0 \cosh q_0 x \cos q_0 z - \hat{k}H_0 \sinh q_0 x \sin q_0 z \end{aligned} \tag{2}$$

So

$$\begin{aligned} H_x = H_0 \cosh q_0 x \cos q_0 z; H_y = 0; \\ H_z = -H_0 \sinh q_0 x \sin q_0 z. \end{aligned} \tag{3}$$

These fields satisfy Maxwell's equations $div\vec{H} = 0$ and $\Delta\vec{H} = 0$.

Equations of motion in the fields (3) have a form

$$\begin{aligned}\frac{dp_x}{dt} &= e[\vec{v}\vec{H}]_x = e(v_y H_z - v_z H_y) = e v_y H_z \\ \frac{dp_y}{dt} &= e[\vec{v}\vec{H}]_y = e(v_z H_x - v_x H_z) = e v_z H_x - e v_x H_z \\ \frac{dp_z}{dt} &= e[\vec{v}\vec{H}]_z = e(v_x H_y - v_y H_x) = -e v_y H_x \end{aligned} \tag{4}$$

and change of energy

$$\frac{d\varepsilon}{dt} = 0, \qquad \varepsilon = const \tag{5}$$

Further we consider paraxial approximation when

$$q_0 x < 1 \tag{6}$$

Taking into account (6) the magnetic field (2) becomes

$$\begin{aligned} H_x = H_0\left(1 + \frac{q_0^2 x^2}{2}\right)\cos q_0 z; H_y = 0; \\ H_z = -H_0 q_0 x \sin q_0 z \end{aligned} \tag{7}$$

Then we can write equations of motion

ISBN 978-3-95450-126-7

$$\begin{aligned}\frac{dp_x}{dt} &= -eH_0 q_0 x v_y \sin q_0 z\\ \frac{dp_y}{dt} &= eH_0\left[v_z\left(1+\frac{q_0^2x^2}{2}\right)\cos q_0 z + q_0 v_x x \sin q_0 z\right]\\ \frac{dp_z}{dt} &= -eH_0 v_y\left(1+\frac{q_0^2x^2}{2}\right)\cos q_0 z \qquad (8)\end{aligned}$$

or taking into account (5) ($p_{x,y,z} = v_{x,y,z}\varepsilon$)

$$\begin{aligned}\ddot{x} &= -\frac{eH_0q_0}{\varepsilon}x\dot{y}\sin q_0 z\\ \ddot{y} &= \frac{eH_0}{\varepsilon}\left[\dot{z}\left(1+\frac{q_0^2x^2}{2}\right)\cos q_0 z + q_0 x\dot{x}\sin q_0 z\right]\\ \ddot{z} &= -\frac{eH_0}{\varepsilon}\left(1+\frac{q_0^2x^2}{2}\right)\dot{y}\cos q_0 z \qquad (9)\end{aligned}$$

One can see, that

$$\left(\frac{q_0x^2}{2}\sin q_0 z\right)' = q_0x\dot{x}\sin q_0 z + \frac{q_0^2x^2}{2}\dot{z}\cos q_0 z \qquad (10)$$

and

$$\int \dot{z}\cos q_0 z\ dt = \int \cos q_0 z\ dz = \frac{\sin q_0 z}{q_0} \qquad (11)$$

Using relations (10) and (11), we can integrate the second equation of (9) and obtain

$$\dot{y} = \frac{eH_0}{\varepsilon q_0}\left(1+\frac{q_0^2x^2}{2}\right)\sin q_0 z \qquad (12)$$

Plugging $\dot{y}$(12) into first and third equations of (9), we obtain(taking into account (6))

$$\begin{aligned}\ddot{x} &= -\left(\frac{eH_0}{\varepsilon}\right)^2 x\sin^2 q_0 z\\ \ddot{z} &= -\frac{1}{2q_0}\left(\frac{eH_0}{\varepsilon}\right)^2 \sin 2q_0 z(1+q_0^2x^2) \qquad (13)\end{aligned}$$

After averaging first equation of Eq.(13) on period $2\pi/q_0$ and taking into account that $\overline{\sin^2 q_0 z} = 1/2$ we have

$$\ddot{x} + \Omega^2\ x = 0 \qquad (14)$$

which has a solution

$$x = a_0\cos(\Omega t + \theta_0) \qquad (15)$$

where

$$\begin{aligned}\Omega &= \frac{eH_0}{\sqrt{2}\varepsilon}; \qquad a_0 = \sqrt{x_0^2 + \frac{\alpha^2}{\Omega^2}};\\ \cos\theta_0 &= \frac{x_0}{a_0}; \quad \sin\theta_0 = -\frac{\alpha/\Omega}{a_0};\\ \theta_0 &= -\arctan\frac{\alpha}{x_0\Omega} \qquad (16)\end{aligned}$$

Averaging of the second equation of Eq.(13) gives

$$\ddot{z^{(0)}} = 0; \quad \dot{z^{(0)}} = v; \quad z^{(0)} = vt. \qquad (17)$$

The solution of the second equation of Eq.(13) using Eq.(15) and Eq.(17) has a form

$$\begin{aligned}\delta z &= -\frac{\Omega^2}{2q_0^2}t + \frac{\Omega^2}{4q_0^3}\sin 2q_0 t\\ &+ \frac{a_0^2\Omega^2}{16q_0}\sin\{2(q_0+\Omega)t + 2\theta_0\}\\ &+ \frac{a_0^2\Omega^2}{16q_0}\sin\{2(q_0-\Omega)t - 2\theta_0\} \qquad (18)\end{aligned}$$

So, for z we have

$$\begin{aligned}z &= t\left(1 - \frac{1}{2\gamma^2} - \frac{\Omega^2}{2q_0^2}\right) + \frac{\Omega^2}{4q_0^3}\sin 2q_0 t\\ &+ \frac{a_0^2\Omega^2}{16q_0}\sin\{2(q_0+\Omega)t + 2\theta_0\}\\ &+ \frac{a_0^2\Omega^2}{16q_0}\sin\{2(q_0-\Omega)t - 2\theta_0\} \qquad (19)\end{aligned}$$

Here we take into account, that $1 - v = \frac{1}{2\gamma^2}$, where $\gamma = \frac{E}{mc^2}$ is a relativistic factor, m- mass of electron, c- velocity of light and E- energy of electrons.

Limitations used in above consideration are

$$a_0 q_0 < 1; \quad \frac{\Omega}{q_0} < 1; \quad a_0\Omega_0 < 1 \qquad (20)$$

In longitudinal direction(along axis z, wiggler's axis) electrons perform fast undulator oscillations, while in transverse direction they perform slow strophotron oscillations in one direction(x direction) and fast undulator oscillations in another direction(y direction).

SPONTANEOUS EMISSION

Now using the solutions for x Eq.(15), $\dot{y}$ Eq.(12) and z Eq.(19), we can find the spectral intensity of a spontaneous emission. The spectral intensity of emission in the z axis direction(wiggler's axis) is determined by the formula [13]

$$\frac{d\varepsilon}{d\omega do} = \frac{e^2\omega^2}{4\pi^2}\left|\int_0^T dt[\vec{n}\times\vec{v}]e^{i\omega(t-z)}\right|^2 \qquad (21)$$

where do is an infinitely small solid angle in the z direction and T is electron traveling time through wiggler.

Using formulae [14]

$$e^{-iA\sin x} = \sum_{-\infty}^{\infty} J_n(A)e^{-inx} \qquad (22)$$

with Bessel functions $J_n(A)$ we find

$$\frac{d\varepsilon}{d\omega do} = \frac{e^2\omega^2\Omega^2T^2}{8\pi^2q_0^2}$$

ISBN 978-3-95450-126-7

$$\times \sum_{n,m,k=-\infty}^{+\infty} \left[(I_{n+1,k,m} - I_{n,k,m})^2 \frac{\sin^2 u_y}{u_y^2} + \frac{a_0^2 q_0^2}{2} (I_{n,k,m+1} - I_{n,k,m})^2 \frac{\sin^2 u_x}{u_x^2} \right] \tag{23}$$

where

$$\begin{aligned} u_y &= \frac{T}{2}\left[\omega\left(\frac{1}{2\gamma^2} + \frac{\Omega^2}{2q_0^2}\right) - (2n+1)q_0 - 2m\Omega\right] \\ u_x &= \frac{T}{2}\left[\omega\left(\frac{1}{2\gamma^2} + \frac{\Omega^2}{2q_0^2}\right) - 2nq_0 - (2m+1)\Omega\right] \\ I_{nkm} &= J_{n-k}(Z_1) J_{\frac{k+m}{2}}(Z_2) J_{\frac{k-m}{2}}(Z_2); \\ Z_1 &= \frac{\omega\Omega^2}{4q_0^3}; \quad Z_2 = \frac{\omega a_0^2 \Omega^2}{16 q_0} \end{aligned} \tag{24}$$

Equation Eq. (23) describes the spectrum of emission consisting of a superposition of the spectral lines located at the combination frequencies of odd harmonics $(2n+1)\omega_{res,und}$ and even harmonics $2m\Omega_{res,st}$, and even harmonics $2n\omega_{res,und}$ and odd harmonics $(2m+1)\Omega_{res,st}$, $(m, n = 0, 1, 2, ...)$ where

$$\omega_{res,und} = \frac{2\gamma^2 q_0}{1+\gamma^2 \frac{\Omega^2}{q_0^2}}, \qquad \omega_{res,str} = \frac{2\gamma^2\Omega}{1+\gamma^2\frac{\Omega^2}{q_0^2}} \tag{25}$$

are resonance frequencies in undulator and strophotron correspondingly.

THE GAIN

The gain can be found from the above derived spectral intensity of a spontaneous emission (Eq. (23)) using Madey's theorem [15].

But in a derivation of these general relations between spontaneous and stimulated processes, some assumptions are used. They require a check in any new case. Therefore we prefer a direct derivation of the gain from equations of motion.

Now let electromagnetic wave propagates along z (wiggler axis) with vector potential

$$\overrightarrow{A_L} = -\frac{E_0}{\omega} \sin\omega(t-z) \left(\hat{\imath}\cos\alpha + \hat{\jmath}\sin\alpha\right) \tag{26}$$

where ω is the frequency of electromagnetic wave and E_0 its electrical field strength, α is angle between x direction and vector of electric field strength of the propagating electromagnetic wave, $\hat{\imath}$ and $\hat{\jmath}$ are the unit vectors in x and in y directions correspondingly.

The equations of electron motion in the wiggler(1) and electromagnetic(26) fields are

$$\begin{aligned} \frac{dp_x}{dt} &= -eH_0 q_0 x v_y \sin q_0 z \\ &+ eE_0 \cos\alpha (1 - v_z) \cos\omega(t-z) \\ \frac{dp_y}{dt} &= eH_0 \left[v_z \left(1 + \frac{q_0^2 x^2}{2}\right) \cos q_0 z + q_0 v_x x \sin q_0 z \right] \\ &+ eE_0 \sin\alpha (1 + v_z) \cos\omega(t-z) \\ \frac{dp_z}{dt} &= -eH_0 v_y \left(1 + \frac{q_0^2 x^2}{2}\right) \cos q_0 z \\ &+ eE_0 \cos\alpha \; v_x \cos\omega(t-z) \end{aligned} \tag{27}$$

and the rate of energy change is [13]

$$\frac{d\varepsilon}{dt} = e\vec{v}\vec{E} = eE_0(v_x\cos\alpha + v_y\sin\alpha)\cos\omega(t-z) \tag{28}$$

We are interesting in linear gain to find which is sufficient to obtain the first-order corrections $x^{(1)}(t)$, $y^{(1)}(t)$ and $z^{(1)}(t)$ to $x^{(0)}(t)$ Eq.(15), $y^{(0)}(t)$ Eq.(12) and $z^{(0)}(t)$ Eq.(19). These first-order corrections obey the equations(obtained from Eqs. (27))

$$\begin{aligned} \frac{dp_x^{(1)}}{dt} &= -\varepsilon_0 \Omega^2 x^{(1)} \\ &+ eE_0 \cos\alpha (1 - v_z^{(0)}) \cos\omega(t - z^{(0)}) \\ \frac{dp_y^{(1)}}{dt} &= eE_0 \sin\alpha (1 + v_z^{(0)}) \cos\omega(t - z^{(0)}) \\ \frac{dp_z^{(1)}}{dt} &= eE_0 \cos\alpha \; v_x^{(0)} \cos\omega(t - z^{(0)}) \end{aligned} \tag{29}$$

The linear(field-independent) gain is determined by the second order($\propto E_0^2$),

$$\begin{aligned} \frac{d\varepsilon}{dt} &= eE_0 \left(v_x^{(1)}\cos\alpha + v_y^{(1)}\sin\alpha\right)\cos\omega(t - z^{(0)}) \\ &+ eE_0\omega \left(v_x^{(0)}\cos\alpha + v_y^{(0)}\sin\alpha\right) z^{(1)} \\ &\times \sin\omega(t - z^{(0)}) \end{aligned} \tag{30}$$

which is found from Eq.(28)

Now finding $x^{(1)}(t)$, $y^{(1)}(t)$ and $z^{(1)}(t)$ from Eqs.(29) and using $x^{(0)}(t)$ Eq.(15), $y^{(0)}(t)$ Eq.(12) and $z^{(0)}(t)$ Eq.(19) we obtain expression of electron emitted energy ($\Delta\varepsilon$) during time T and gain ($G = \frac{8\pi N_e}{E_0^2}\Delta\varepsilon$, N_e is electron beam concentration).

All these calculations are simple, although rather cumbersome. Here we present only the found result:

$$\begin{aligned} G &= \frac{e^2\omega^2\Omega^2 T^2}{8\pi^2 q_0^2} \\ &\times \sum_{n,m,k=-\infty}^{+\infty} \left[(I_{n+1,k,m} - I_{n,k,m})^2 \sin^2\alpha \right. \\ &\times \frac{d}{du_y}\frac{\sin^2 u_y}{u_y^2} + \frac{a_0^2 q_0^2}{2}(I_{n,k,m+1} - I_{n,k,m})^2 \\ &\times \left. \cos^2\alpha \, \frac{d}{du_x}\frac{\sin^2 u_x}{u_x^2} \right] \end{aligned} \tag{31}$$

From Eq.(31) we conclude that maximum gain is achieved when vector of electromagnetic wave E_0 is directed in y direction($\alpha = \pi/2$).

ISBN 978-3-95450-126-7

We calculate the spectral distribution of spontaneous emission and the gain of electrons moving in plane wiggler with inhomogeneous magnetic field. It is shown, that electrons do complex motion consisting of slow(strophotron) and fast(undulator) parts. We average the equations of motion over fast undulator part and obtain equations for connected motion. It is shown, that the account of inhomogeneity of the magnetic field leads to appearance of additional peaks in the spectral distribution of spontaneous radiation and the gain. Having much peaks and using the well-known mode-locking one can obtain ultrashort impulses.

REFERENCES

[1] A. Doria, V.B. Asgekar, D. Esposito et al., "Long wavelength compact-FEL with controlled energy-phase correlation," Nucl. Instrum. Methods Phys. Res., Sect. A**475**, 296-302, (2001).

[2] W.B. Colson, "Short wavelength free electron lasers in 2000," Nucl. Instrum. Methods Phys. Res., Sect. A **475**, 397-4000, (2001).

[3] C.A. Brau, Free-Electron Lasers, Academic, Boston, 1990.

[4] M.V. Fedorov, Atomic and Free Electrons in a Strong Light Field, World Scientific, Singapore, 1997.

[5] K.B. Oganesyan, M.L. Petrossyan, "The magnetic fields of hellical wiggler," Yerevan 1981, YERPHI-475(18) - 81

[6] M.V.Fedorov,E.A.Nersesov,K.B.Oganesyan, "Potential laser realization on relativistic strophotron-type free-electrons," Sov. JTP 56, 2402-2404 (1986).

[7] Fedorov M.V., Oganesyan K.B. "Classical theory of emission at high harmonics in the relativistic strophotron FEL," IEEE Journal of Quant. Electr. **21**, 1059-1068 (1985).

[8] D.F. Zaretsky, E.A. Nersesov, K.B. Oganesyan, M.V. Fedorov, "A laser utilizing free-electrons moving in transverse-gradient fields," Sov. Quant. Electronics, **13**, pp.685-692, (1986).

[9] K.B. Oganesyan, M.V. Fedorov, "Non-linear amplification theory in free-electron laser of the relativistic strophotron type," Sov. ZHTF, **57**, 2105-2114, (1987).

[10] M.V. Fedorov, K.B. Oganesyan, A.M. Prokhorov, "Free-electron laser based on the effect of channeling in an intense standing light-wave," Appl. Phys. Lett., **53**, 353-354 (1988).

[11] K.B. Oganesyan, A.M. Prokhorov, M.V. Fedorov, "Transverse channeling and free-electron laser on intense standing wave," Sov. ZHETF, **53**, 80-86, (1988); Sov. Phys. JETP **68**, 1342 (1988).

[12] E. Jerby, "The axial-velocity distribution function and the longitudinal susceptibility of an e-beam in a plannar wiggler FEL," NIM A**27**, 457-466 (1988).

[13] L.D. Landau, E. M. Lifshits, The classical theory of fields. New York, Academic, 1975.

[14] I.S. Gradstein and I.M. Ryzhik, Tables of integrals, Series and Products, New York, Academic, 1966.

[15] J.M.J. Madey, "Relationship between mean radiated energy, mean squared radiated energy and spontaneous power spectrum in a free-electron laser," Nuovo Cimento, **50b**, 64-88, 1979.

ISBN 978-3-95450-126-7