QUASIPERIODIC METHOD OF AVERAGING APPLIED TO PLANAR UNDULATOR MOTION EXCITED BY A FIXED TRAVELING WAVE *

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INTRODUCTION

We summarize our mathematical study in [1] on planar motion of energetic electrons moving through a planar dipole undulator, excited by a fixed planar polarized plane wave Maxwell field in the X-Ray FEL regime.

We study the associated 6D Lorentz system as the wavelength of the traveling wave varies. The 6D system is reduced, without approximation, to a 2D system. There are two small parameters in the problem, $1/\gamma_c$, where γ_c is a characteristic energy γ, and ε which is a measure of the energy spread. Using these parameters, the 2D system is then transformed into a system for a scaled energy deviation, χ, and a generalized ponderomotive phase, θ, both of which are slowly varying. When the two small parameters are related the system is in a form for an application of the Method of Averaging (MoA); a rigorous long time perturbation theory which leads to error bounds relating the exact and approximate solutions. As the wavelength varies the system passes through resonant and nonresonant zones and we develop nonresonant and near-to-resonant normal form approximations based on the MoA. For a special initial condition, on resonance, we obtain the well-known FEL pendulum system.

In [1] we prove nonresonant and near-to-resonant first-order averaging theorems, in a novel way, which give optimal error bounds for the approximations. The nonresonant case is an example of quasiperiodic averaging where the small divisor problem enters in the simplest possible way and the near-to-resonant case is an example of periodic averaging. To our knowledge the analysis has not been done with the generality in [1] nor has the standard FEL pendulum system been derived with error bounds. Our main emphasis here is to summarize the derivation of the normal form approximations, discuss their behavior and state in a rough way the results of the error analysis. The FEL pendulum appears on resonance in the near-to-resonant normal form and we discuss the near-to-resonant behavior with phase plane plots.

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GENERAL PLANAR UNDULATOR MODEL

In this section we state the basic problem and put the equations of motion in a standard form for the MoA.

Lorentz Force Equations

The 6D Lorentz equations of motion in SI units with z as the independent variable are

$$\frac{dx}{dz} = \frac{p_x}{p_z}\,, \quad \frac{dy}{dz} = \frac{p_y}{p_z}\,, \quad \frac{dt}{dz} = \frac{m\gamma}{p_z}\,, \tag{1}$$

$$\begin{aligned}\frac{dp_x}{dz} &= -\frac{e}{c}[cB_u\cosh(k_u y)\sin(k_u z) \\ &\quad -\frac{p_y}{p_z}cB_u\sinh(k_u y)\cos(k_u z) \\ &\qquad +E_r(\frac{m\gamma c}{p_z}-1)h(\breve{\alpha}(z,t))]\,,\end{aligned} \tag{2}$$

$$\frac{dp_y}{dz} = -\frac{e}{c}\frac{p_x}{p_z}cB_u\sinh(k_u y)\cos(k_u z)\,, \tag{3}$$

$$\begin{aligned}\frac{dp_z}{dz} &= -\frac{e}{c}[-\frac{p_x}{p_z}cB_u\cosh(k_u y)\sin(k_u z) \\ &\qquad +E_r\frac{p_x}{p_z}h(\breve{\alpha}(z,t))].\end{aligned} \tag{4}$$

Here (x, y, z) are Cartesian coordinates, z is the distance along the undulator, $t(z)$ is the arrival time at z, (p_x, p_y, p_z) are Cartesian momenta, $\gamma^2 = 1+\mathbf{p}\cdot\mathbf{p}/m^2c^2$, m is the electron mass, $-e$ is the electron charge and c is the vacuum speed of light.

The planar undulator model magnetic field which we use satisfies the Maxwell equations and is given by

$$\begin{aligned}\mathbf{B}_u &= -B_u \\ &\cdot[\cosh(k_u y)\sin(k_u z)\mathbf{e}_y + \sinh(k_u y)\cos(k_u z)\mathbf{e}_z]\,,\end{aligned} \tag{5}$$

where $B_u > 0$ is the undulator strength, $k_u > 0$ is the undulator wave number and $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ are the standard unit vectors.

The traveling wave radiation field we choose is also a Maxwell field and is given by

$$\begin{aligned}\mathbf{E}_r &= E_r h(\breve{\alpha})\mathbf{e}_x\,, \\ \mathbf{B}_r &= \frac{1}{c}(\mathbf{e}_z\times\mathbf{E}_r) = \frac{E_r}{c}h(\breve{\alpha})\mathbf{e}_y\,,\end{aligned} \tag{6}$$

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where $E_r > 0$ is a constant, h is a real valued function on $\mathbb{R}$, and

$$\breve{\alpha}(z,t) = k_r(z-ct) \; , \tag{7}$$

with $k_r > 0$. In this section we will keep h general, in the next section we treat the monochromatic case where

$$h(\breve{\alpha}) = \cos(\nu\breve{\alpha}) \; , \tag{8}$$

and where we derive a generalized pendulum system.

Standard Form for Method of Averaging

We confine ourselves to planar motion with no approximation since $y(0) = p_y(0) = 0$ implies that $y(z) = p_y(z) = 0$. Thus the six ODE's (1)-(4) reduce to four. The righthand sides of (1)-(4) are independent of x thus the x equation need not be considered. The quantity $p_x/mcK - \cos(k_u z) - (E_r/cB_u)(k_u/k_r)H(\alpha)$ is conserved where H is any antiderivative of h, i.e., $H' = h$ thus the p_x ODE can be eliminated. Only the two ODE's for t and p_z remain and everything can determined from them.

We now scale z by defining $\zeta = k_u z$, replace the dependent variable t by α where

$$\alpha(\zeta) := \breve{\alpha}(z, t(z)) = k_r(z - ct(z)) \; , \tag{9}$$

and replace p_z by γ. We thus obtain our basic 2D system

$$\frac{d\alpha}{d\zeta} = \frac{k_r}{k_u}(1 - \frac{m\gamma c}{p_z}) \; , \tag{10}$$

$$\frac{d\gamma}{d\zeta} = -\frac{eE_r}{k_u mc^2}\frac{p_x}{p_z}h(\alpha) \; , \tag{11}$$

where p_x and p_z must be replaced by

$$p_x = p_x(0) + mcK \cdot\Big(\cos(k_u z) - 1 + \frac{E_r}{cB_u}\frac{k_u}{k_r}[H(\alpha) - H(\alpha(0))]\Big) \; ,$$

$$p_z = \sqrt{m^2c^2(\gamma^2-1) - p_x^2} \; ,$$

$$K = \frac{eB_u}{mck_u} = \text{undulator parameter} \; .$$

Two small parameters, $1/\gamma_c$ and ε, are introduced via

$$\gamma = \gamma_c(1+\eta) = \gamma_c(1+\varepsilon\chi) \; , \tag{12}$$

where γ_c is characteristic value of γ, e.g. its mean, and ε is a characteristic spread of the energy deviation $\eta = \varepsilon\chi$ so that χ is $O(1)$. Thus χ is an $O(1)$ dependent variable replacing γ.

An asymptotic expansion, for γ_c large and ε small in (10),(11) with γ replaced by (12), yields

$$[\alpha + Q(\zeta)]' = \varepsilon K_r q(\zeta)\chi + O(\frac{1}{\gamma_c^2}) + O(\varepsilon^2) \; , \tag{13}$$

$$\chi' = -K^2\frac{\mathcal{E}}{\varepsilon\gamma_c^2}(\cos\zeta + \Delta P_{x0})h(\alpha) + O(1/\gamma_c^2) + O(1/\varepsilon\gamma_c^4) \; , \tag{14}$$

where

$$K_r = \frac{k_r}{k_u\gamma_c^2} \; , \; \mathcal{E} = \frac{E_r}{cB_u} \; , \; \Delta P_{x0} = \frac{p_x(0)}{mcK} - 1 \; ,$$

$$q(\zeta) = 1 + K^2(\cos\zeta + \Delta P_{x0})^2 \; ,$$

$$Q'(\zeta) = \frac{K_r}{2}q(\zeta) \; , \quad Q(0) = 0 \; .$$

To transform (13),(14) into a standard form for the MoA slowly varying dependent variables are needed. Clearly, $\alpha + Q(\zeta)$ is slowly varying and we anticipate that χ will be slowly varying, i.e., $\mathcal{E}/\varepsilon\gamma_c^2$ small.

Thus we define

$$\theta = \alpha + Q(\zeta) \; , \tag{15}$$

and (13),(14) become

$$\theta' = \varepsilon K_r q(\zeta)\chi + O(1/\gamma_c^2) + O(\varepsilon^2) \; , \tag{16}$$

$$\chi' = -K^2\frac{\mathcal{E}}{\varepsilon\gamma_c^2}(\cos\zeta + \Delta P_{x0})h(\theta - Q(\zeta)) + O(1/\gamma_c^2) + O(1/\varepsilon\gamma_c^4) \; , \tag{17}$$

with initial conditions as in the exact ODE's, i.e., $\theta(0,\varepsilon) = \theta_0, \chi(0,\varepsilon) = \chi_0$. To obtain a system where θ and χ interact with each other in first-order averaging we must balance the $O(\varepsilon)$ term in (16) with the $O(\mathcal{E}/\varepsilon\gamma_c^2)$ in (17). In this spirit we relate ε and γ_c by choosing

$$\varepsilon = \sqrt{\mathcal{E}}\frac{1}{\gamma_c} \; . \tag{18}$$

It is this balance that will lead to the FEL pendulum equations in the next section. In this *distinguished case* the system (16),(17) can be written

$$\theta' = \varepsilon K_r q(\zeta)\chi + O(\varepsilon^2) \; , \tag{19}$$

$$\chi' = -\varepsilon K^2(\cos\zeta + \Delta P_{x0})h(\theta - Q(\zeta)) + O(\varepsilon^2) \; , \tag{20}$$

and these are now in a standard form for the MoA.

The θ defined by (15) is a generalization of the so-called ponderomotive phase. Here it arises naturally in the process of transforming to slowly varying coordinates and finding the distinguished relation between ε and γ_c in (18). In standard treatments it is introduced heuristically to maximize energy transfer.

We note that $\mathcal{E}$ does not need to be small for ε to be small, thus our results may be relevant even in the high gain saturation regime.

MONOCHROMATIC CASE

The Basic ODE's for the Monochromatic Radiation Field

From now on the radiation field in (6) is monochromatic, i.e., h, H have the form

$$H(\breve{\alpha}) = (1/\nu)\sin(\nu\breve{\alpha}) \; , \quad h(\breve{\alpha}) = \cos(\nu\breve{\alpha}) \; , \tag{21}$$

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where $\nu \geq 1/2$. We also choose

$$K_r = \frac{2}{\bar{q}} \,, \tag{22}$$

where

$$\bar{q} = \lim_{T\to\infty}[\frac{1}{T}\int_0^T q(\zeta)d\zeta] = 1 + \frac{1}{2}K^2 + K^2(\Delta P_{x0})^2 \,. \tag{23}$$

The ODE's (19),(20) now become

$$\theta' = \varepsilon f_1(\chi,\zeta) + O(\varepsilon^2) \,, \tag{24}$$
$$\chi' = \varepsilon f_2(\theta,\zeta,\nu) + O(\varepsilon^2) \,, \tag{25}$$

where

$$\begin{aligned} f_1(\chi,\zeta) &= \frac{2q(\zeta)}{\bar{q}}\chi \,, \\ f_2(\theta,\zeta,\nu) &= -K^2(\cos\zeta + \Delta P_{x0}) \\ &\cdot \cos\Big(\nu\theta - \nu\zeta - \nu\Upsilon_0\sin\zeta - \nu\Upsilon_1\sin 2\zeta\Big) \\ &= -\frac{K^2}{2}e^{i\nu\theta}\sum_{n\in\mathbb{Z}}\widehat{jj}(n;\nu,\Delta P_{x0})e^{i(n-\nu)\zeta} + cc \,, \end{aligned}$$

and where

$$\Upsilon_0 = \frac{2}{\bar{q}}K^2\Delta P_{x0} \,, \quad \Upsilon_1 = \frac{\bar{q}K^2}{4} \,. \tag{26}$$

Note that $f_1(\chi,\zeta)$ and $f_2(\theta,\zeta,\nu)$ are quasiperiodic in ζ since f_1 is 2π periodic and f_2 has two base periodicities, 2π and $2\pi/\nu$.

Averages needed for the normal form analysis are:

$$\bar{f}_1(\chi) = \lim_{T\to\infty}[\frac{1}{T}\int_0^T f_1(\chi,\zeta)d\zeta] = 2\chi \,, \tag{27}$$

$$\bar{f}_2(\theta,\nu) = \lim_{T\to\infty}[\frac{1}{T}\int_0^T f_2(\theta,\zeta,\nu)d\zeta] = \begin{cases} 0 & \text{if } \nu \notin \mathbb{N} \\ -K^2\widehat{jj}(k;k,\Delta P_{x0})\cos(k\theta) & \text{if } \nu = k \in \mathbb{N} \,, \end{cases} \tag{28}$$

where $\mathbb{N}$ is the set of positive integers.

Δ-nonresonant Normal Form

The Δ-nonresonant case is an example of quasiperiodic averaging with a small divisor problem of very simple structure. This case is defined by: $\nu \in [k+\Delta, k+1-\Delta]$ with $\Delta \in (0, 0.5)$ and $k \in \mathbb{N}$.

An averaging normal form approximation (v_1, v_2) to (θ,χ) in (24),(25) is obtained by dropping the $O(\varepsilon^2)$ terms and averaging the $O(\varepsilon)$ terms over ζ by holding the slowly varying quantities θ,χ fixed. Thus if $\nu \notin \mathbb{N}$, using (27),(28), the Δ-nonresonant normal form system is

$$v_1' = \varepsilon 2v_2 \,, \quad v_2' = 0 \,, \tag{29}$$

and the same initial conditions as in the exact ODE's, i.e., $v_1(0,\varepsilon) = \theta_0$, $v_2(0,\varepsilon) = \chi_0$. The Δ-nonresonant case is natural if $|\nu - k|$ is "big ".

In [1] we obtain explicit error bounds on the normal form approximation. In fact there exists an ε independent constant $C(T)$ such that

$$|\theta(\zeta,\varepsilon) - v_1(\zeta,\varepsilon)| \leq C(T)\frac{\varepsilon}{\Delta} \,,$$
$$|\chi(\zeta,\varepsilon) - v_2(\zeta,\varepsilon)| \leq C(T)\frac{\varepsilon}{\Delta} \,,$$

for $0 \leq \zeta \leq T/\varepsilon$ with ε sufficiently small. Note that in the "nonresonant case", i.e., when $\nu \geq 1/2$ with $\nu \notin \mathbb{N}$, there exists $\Delta \in (0, 0.5)$ and $k \in \mathbb{N}$ such that $\nu \in [k+\Delta, k+1-\Delta]$. However the error bound increases as $\Delta \to 0$, i.e., as ν moves toward resonance.

Near-to-resonant Normal Form

In this case we explore $O(\varepsilon)$ neighborhoods of the $\nu = k$ resonances. We write

$$\nu = k + \varepsilon a, \tag{30}$$

where $k \in \mathbb{N}, a \in [-1/2, 1/2]$ and derive a "near-to-resonant" normal form system. Here a is a measure of the distance of ν from k.

The $O(\varepsilon)$ neighborhood of k is natural in first-order averaging, since if $|\nu - k|$ is too small then the normal form will be close to the resonant normal form of (36),(37) and if $|\nu - k|$ is too big then ν will be in the Δ-nonresonant regime. Equation (30) clearly includes the resonant case for $a = 0$.

We write (24),(25) as:

$$\theta' = \varepsilon f_1(\chi,\zeta) + O(\varepsilon^2) \,, \tag{31}$$
$$\chi' = \varepsilon f_2^R(\theta,\varepsilon\zeta,\zeta,k,a) + O(\varepsilon^2) \,, \tag{32}$$
$$\begin{aligned} f_2^R(\theta,\tau,\zeta,k,a) &= -K^2(\cos\zeta + \Delta P_{x0}) \\ &\cdot \cos\Big(k[\theta - \zeta - \Upsilon_0\sin\zeta - \Upsilon_1\sin 2\zeta] - a\tau\Big) \\ &= -\frac{K^2}{2}\exp(i[k\theta - a\tau]) \\ &\cdot \sum_{n\in\mathbb{Z}}\widehat{jj}(n;k,\Delta P_{x0})e^{i\zeta[n-k]} + cc \,. \end{aligned} \tag{33}$$

Note that $f_1(\chi,\zeta)$, $f_2^R(\theta,\tau,\zeta,k,a)$ are 2π periodic in ζ. The near-to-resonant normal form ODE's are obtained from (31),(32) by dropping the $O(\varepsilon^2)$ terms and averaging the righthand sides over ζ holding the slowly varying quantities $\theta,\chi,\varepsilon a\zeta$ fixed. We thus obtain

$$v_1' = 2\varepsilon v_2 \,, \tag{34}$$
$$v_2' = -\varepsilon K^2\widehat{jj}(k;k,\Delta P_{x0})\cos(kv_1 - \varepsilon a\zeta) \,, \tag{35}$$

and the same initial conditions as in the exact ODE's, i.e., $v_1(0,\varepsilon) = \theta_0$, $v_2(0,\varepsilon) = \chi_0$. For $a = 0$, (34),(35) become the so-called "resonant" normal form

$$v_1' = 2\varepsilon v_2 \,, \tag{36}$$
$$v_2' = -\varepsilon K^2\widehat{jj}(k;k,\Delta P_{x0})\cos(kv_1) \,. \tag{37}$$

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For $\Delta P_{x0} = a = 0$, (34),(35) are the standard FEL pendulum equations whence θ generalizes the so-called ponderomotive phase. Note also that for $\Delta P_{x0} = 0$:

$$\widehat{jj}(k;k,0) = \begin{cases} \frac{1}{2}(-1)^n[J_n(x_n) - J_{n+1}(x_n)] & \text{if } k = 2n+1 \\ 0 & \text{if } k \text{ even ,} \end{cases}$$

where $x_n = (2n+1)\Upsilon_1$ and $n = 0, 1, ...$ with J_m=m-th-order Bessel function of first kind. In the special case when $K^2\widehat{jj}(k;k,\Delta P_{x0}) = 0$ the ODE's (34),(35) are the same as the nonresonant equations (29) and this occurs, e.g., when $\Delta P_{x0} = 0$ and k even.

In [1] we obtain explicit error bounds on the normal form approximations. We find that there exists an ε independent constant $C_R(T)$ such that

$$|\theta(\zeta,\varepsilon) - v_1(\zeta,\varepsilon)| \le C_R(T)\varepsilon \ ,$$
$$|\chi(\zeta,\varepsilon) - v_2(\zeta,\varepsilon)| \le C_R(T)\varepsilon \ , \quad (38)$$

for $0 \le \zeta \le T/\varepsilon$ with ε sufficiently small.

A phase plane portrait for the system (34), (35) is shown in Figs. 1 and 2 with $k = 1$ and $K^2\widehat{jj}(k;k,\Delta P_{x0}) = 2$. For the resonant case, $a = 0$, we clearly see a pendulum behavior exhibiting four types of motion (libration, separatrix motion, rotation, and fixed point) and in the subcase $a = 0, \Delta P_{x0} = 0$ this is the standard FEL pendulum structure. To help understand the near-to-resonance behavior we have superposed orbits for $a = 1/3$ resp. $a = 1/6$ for four initial conditions with $v_1(0) = -3\pi/2$. For $v_2(0) = a/2$, v_2 is constant, for $v_2(0)$ starting on top of the libration curve we see the spiral motion moving to the right, for $v_2(0)$ starting on the lower rotation curve we see a modification of the resonant rotation moving to the left and for $v_2(0)$ starting on the upper rotation curve we see a modification of the resonant rotation moving to the right. The orbits for $a = 1/3$ resp. $a = 1/6$ are computed from (34),(35) with Matlab's ode45 solver. Details of the near-to-resonant behavior are discussed in [1] by writing the solution of (34),(35) in terms of solutions of the simple pendulum equation.

COMMENTS AND FUTURE WORK

In the collective case there is a continuous range of frequencies and so it is natural to ask, "what happens in the noncollective case considered in this paper if there is a continuous range of frequencies?". In this situation h can be modeled as

$$h(\alpha) = \int_{-\infty}^{\infty} \tilde{h}(\xi)\exp(-i\xi\alpha)d\xi \ . \quad (39)$$

For $\tilde{h}(\xi) = [\delta(\xi-\nu)+\delta(\xi+\nu)]/2$, where δ is the delta distribution, (39) gives $h(\alpha) = \cos(\nu\alpha)$ as in the monochromatic case of (21), and there are resonances for integer ν.

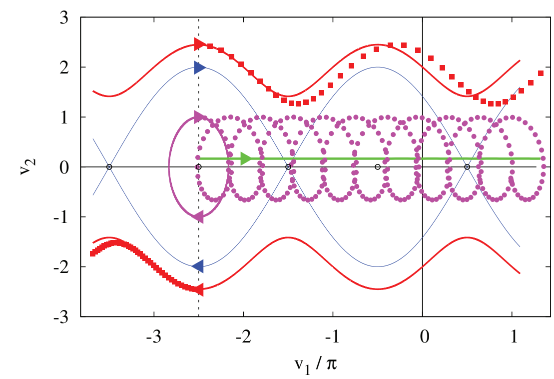

Figure 1: Phase plane orbits on resonance ($a = 0$: solid magenta, blue, red curves and five black fixed points) and near-to-resonance ($a = 1/3$: green solid and dotted magenta and red curves). $k = 1, \mathcal{A} = 2$.

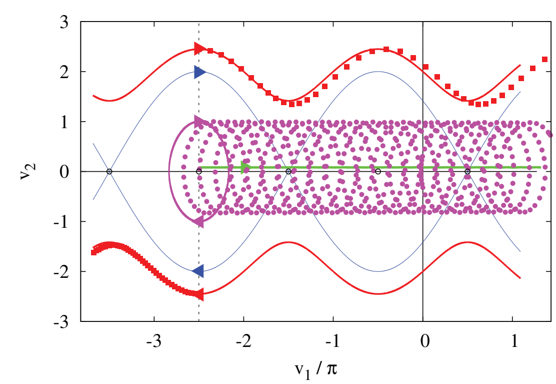

Figure 2: Phase plane orbits on resonance ($a = 0$: solid magenta, blue, red curves and five black fixed points) and near-to-resonance ($a = 1/6$: green solid and dotted magenta and red curves). $k = 1, \mathcal{A} = 2$.

However we have found that for continuous $\tilde{h}$ the average of $(\cos\zeta + \Delta P_{x0})h(\theta - Q(\zeta))$ is zero, i.e.,

$$\lim_{T\to\infty}[\frac{1}{T}\int_0^T (\cos\zeta + \Delta P_{x0})h(\theta - Q(\zeta))d\zeta] = 0 \ . \quad (40)$$

Thus the averaging normal form for (19),(20) is just the nonresonant normal form and thus a continuous $\tilde{h}(\xi)$, localized for example near the $\nu = 1$ (monochromatic) resonance, washes out the effect of that resonance in the first-order averaging normal form. This does not mean that there is no resonant behavior near $\nu = 1$ because we have not yet proved that the normal form gives a good approximation, i.e., it may not be possible to prove an averaging theorem. We are pursuing this. However, even if an averaging theorem can be proven there might still be an effect in second-order averaging.

Secondly it would be interesting to include the y dynamics, but not assuming the zero initial conditions in y, thus treating the full 3D dynamics.

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Thirdly, it would be interesting to study the helical undulator as we have done here for the planar undulator, i.e., via first-order averaging.

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