How an FEL works

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<u>Some</u> Sources of information

- J.B. Murphy & C. Pelligrini, “Introduction to the Physics of the Free Electron Laser”, Laser Handbook, vol. 6 p. 9-69 (1990).
- R. Bonifacio et al, “Physics of the High-Gain Free Electron Laser & Superradiance”, Rivista del Nuovo Cimento, Vol. 13, no. 9 p1-69 (1990) [see also Rivista del Nuovo Cimento, Vol. 15, no. 11 p1-52 (1992)]
- Saldin E.L., Schneidmiller E.A., Yurkov M.V. The physics of free electron lasers. - Berlin et al.: Springer, 2000. (Advanced texts in physics, ISSN 1439-2674).
- Z. Huang & K.-J. Kim, *Review of x-ray free-electron laser theory*, Phys. Rev. ST Accel. Beams 10, 034801 (2007).
- The World Wide Web Virtual Library: Free Electron Laser research and applications http://sbfel3.ucsb.edu/www/vl_fel.html
- W.B. Colson et al., Free Electron Lasers in 2009, *Proceedings of the 31st International Free Electron Laser Conference (Liverpool, U.K.),* **WEPC43**, 591-595 (2009).

Some figures

Some figures have been ‘borrowed’ from other sources:

DESY (XFEL,FLASH) Group, Germany;

Riken/SPring-8 group, Japan;

LCLS group, USA.

What is an FEL and how does it work?

I – Spontaneous radiation

II – Coherent emission - FEL

What is a FEL?

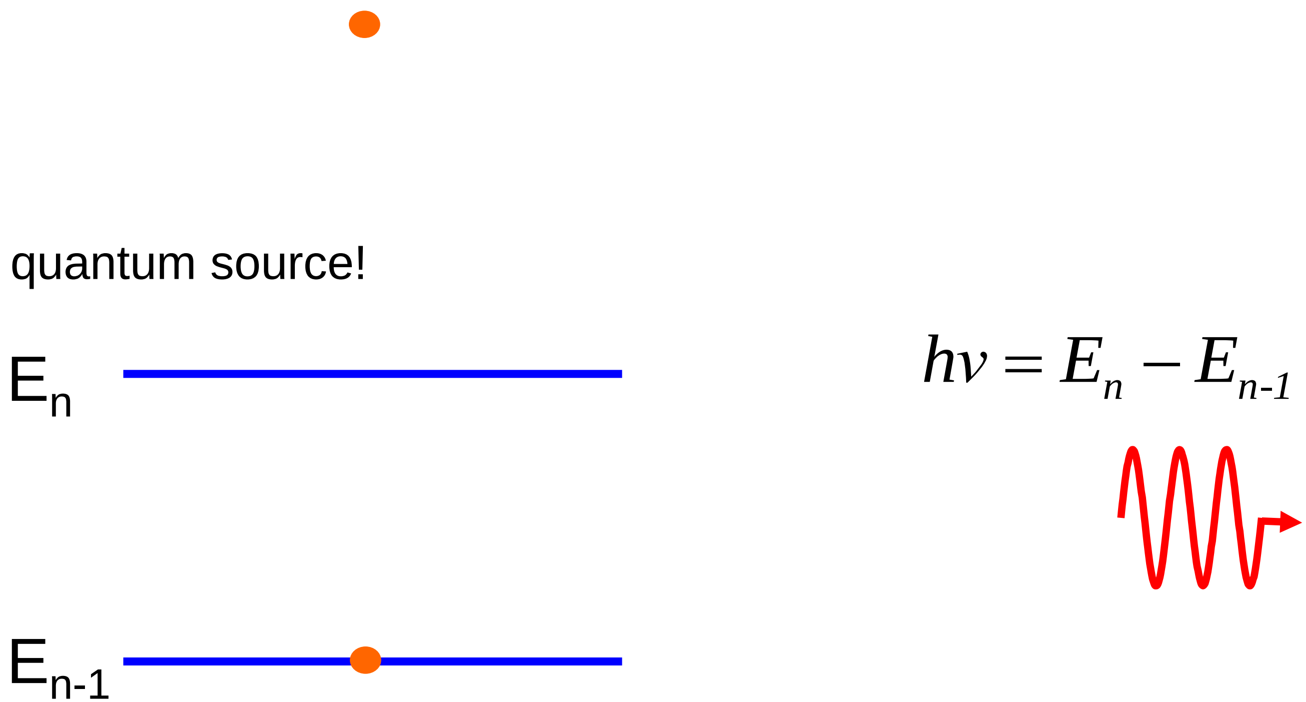
- A ***classical* source of *tuneable*, *coherent* electromagnetic radiation** due to accelerated charge (electrons)

v_z

What is a FEL?

- A ***classical* source of *tuneable*, *coherent* electromagnetic radiation** due to accelerated charge (electrons)

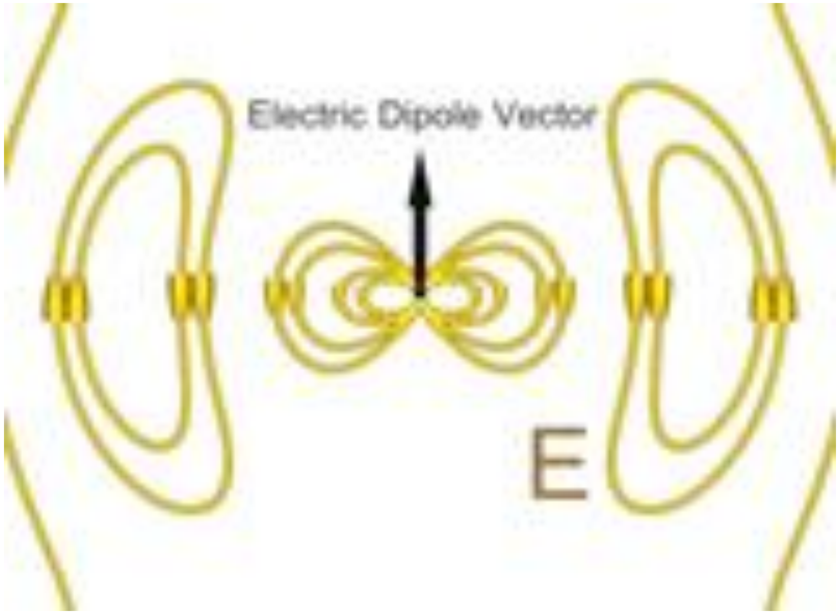
- NOT a quantum source!



$$h\nu = E_n - E_{n\text{-}1}$$

I – Spontaneous radiation:
no interaction between electrons and the light they emit

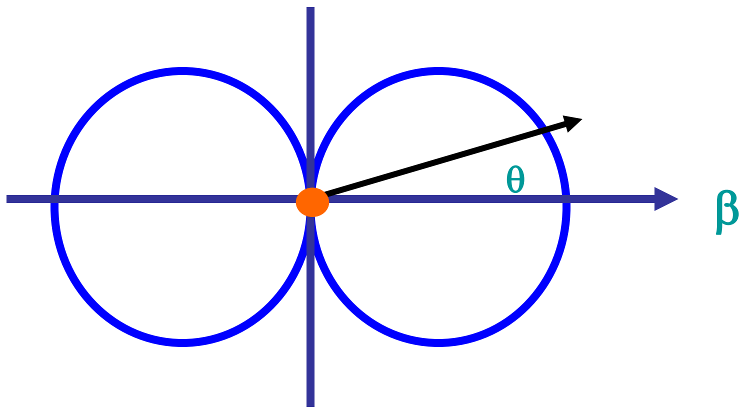
Generation of EM Radiation



Non-relativistic charge source

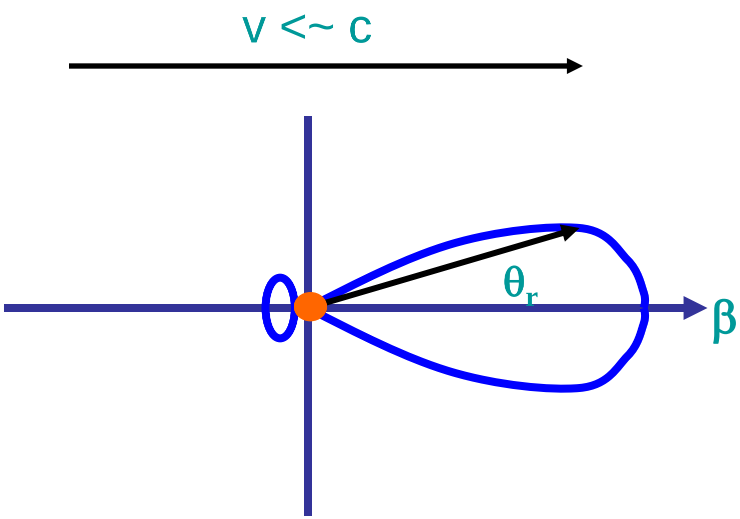
Relativistic Emission

Stationary electron



Energy emission confined to directions perpendicular to axis of oscillation

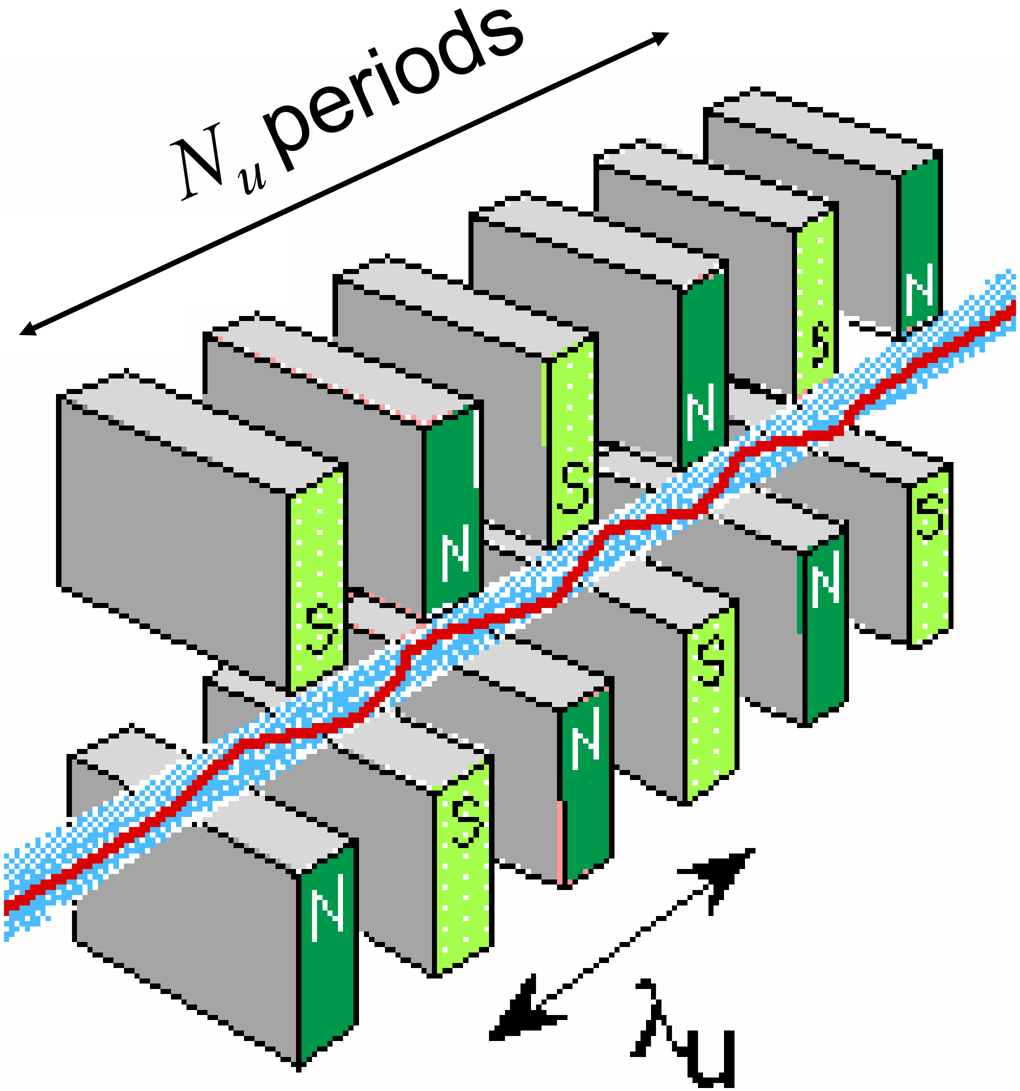
Relativistic electron



Most energy confined to the relativistic emission cone

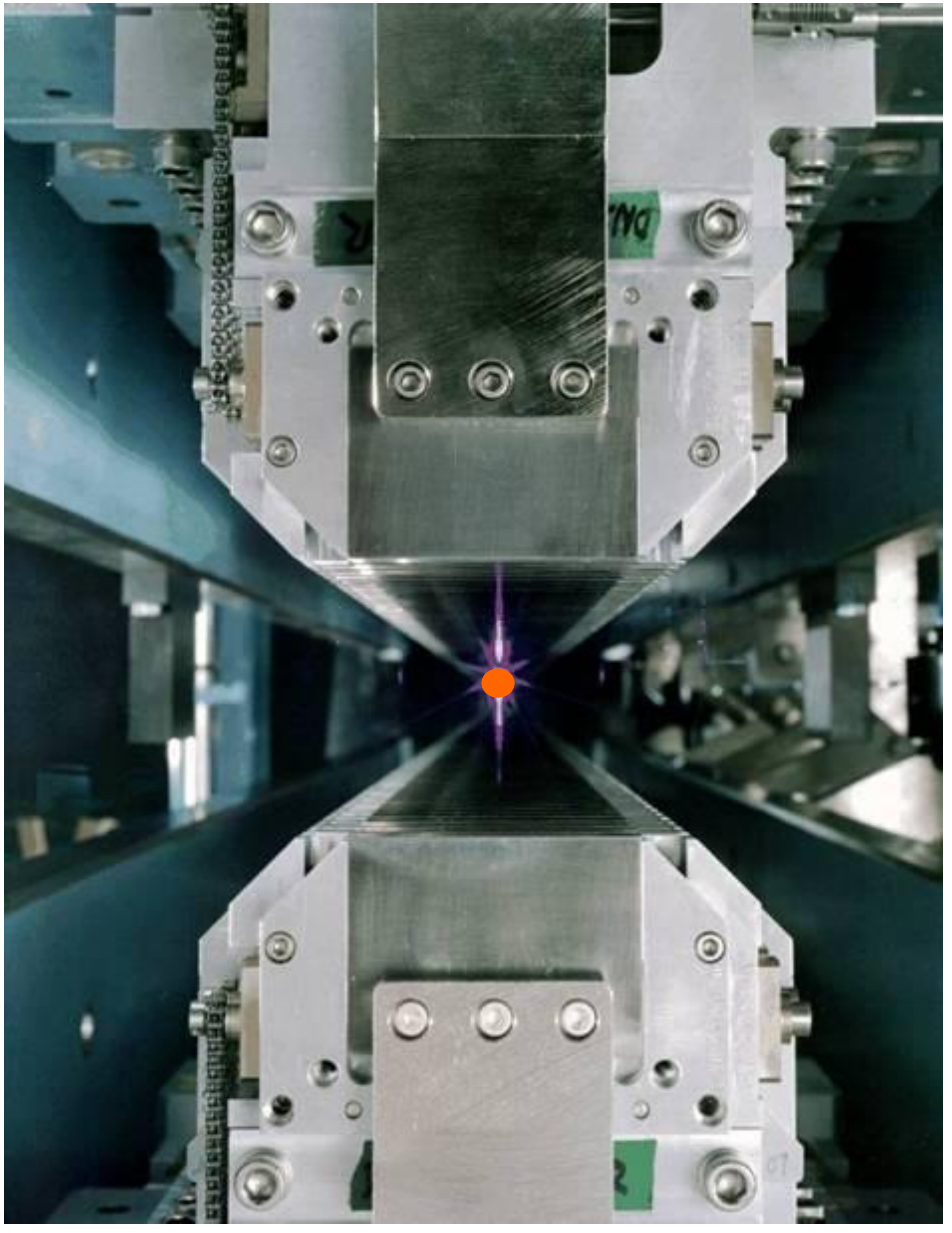
$$\theta_r = \gamma^{-1}$$

Planar Undulator or 'Wiggler'




LOWER

Looking down the axis of an undulator



The electron trajectory in an undulator

An electron trajectory

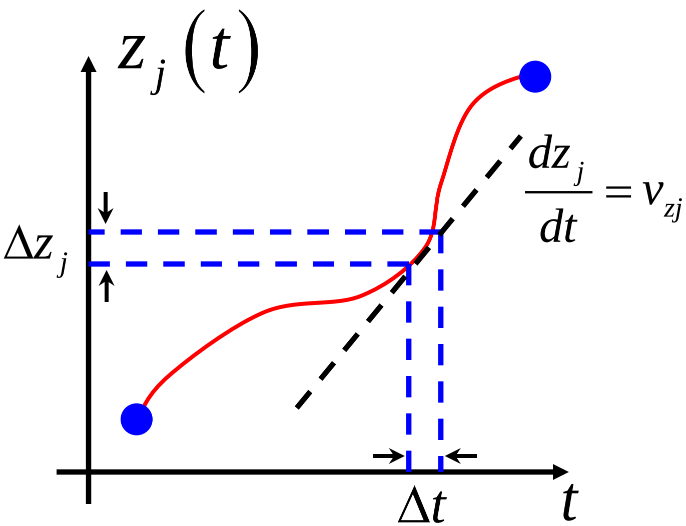
The Lorentz force equation for the j-th electron is written:
$$\frac{d\left(\gamma_j m \underset{\sim}{v}_j\right)}{dt} = -e\left(\underset{\sim}{E} + \underset{\sim}{v}_j \wedge \underset{\sim}{B}\right) \qquad \left(\gamma \equiv \frac{1}{\sqrt{1-\underset{\sim}{\beta}\cdot\underset{\sim}{\beta}}}\right)$$

The independent variable of the electron trajectory here is t, and ***only*** t:

$$\underset{\sim}{r}_j(t) = \left(x_j(t), y_j(t), z_j(t)\right) \quad \text{and so:} \quad \underset{\sim}{v}_j = \frac{d\underset{\sim}{r}_j}{dt}$$

It is convenient to change the independent variable to z to get:

$$\underset{\sim}{r}_j(z) = \left(x_j(z), y_j(z), z\right); \quad t_j(z) \quad \Rightarrow \quad \frac{d}{dt} \rightarrow ??$$



$$\frac{df}{dt} \equiv \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

Now: $f(t) = f\left(z(t)\right); \quad f(t+\Delta t) = f\left(z+\Delta z_j\right); \quad \Delta z_j = v_{zj}\Delta t$

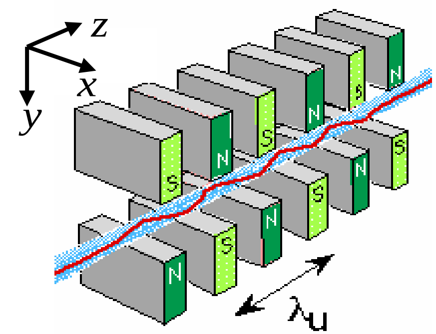
$$\Rightarrow \frac{df}{dt} = \lim_{\Delta z_j/v_{zj} \to 0} \frac{f\left(z+\Delta z_j\right) - f(z)}{\Delta z_j / v_{zj}} = v_{zj} \lim_{\Delta z_j \to 0} \frac{f\left(z+\Delta z_j\right) - f(z)}{\Delta z_j}$$

$$\Rightarrow \frac{d}{dt} = v_{zj}\frac{d}{dz} = c\beta_{zj}\frac{d}{dz} \quad \text{where: } \beta_{zj} \equiv \frac{v_{zj}}{c} \text{ and } \left|\beta_{zj}\right| < 1.$$

An electron trajectory in an undulator

Rewriting the Lorentz equation:

$$\frac{d\left(\gamma_j \underset{\sim}{\beta}_j\right)}{dz} = -\frac{e}{mc^2\beta_{zj}}\left(\underset{\sim}{E} + c\underset{\sim}{\beta}_j(t)\wedge\underset{\sim}{B}\right)$$



A planar undulator

From the Lorentz equation we can ***derive***:

$$\frac{d\gamma_j}{dt} = -\frac{e}{mc}\underset{\sim}{E}\cdot\underset{\sim}{\beta}_j \quad\Rightarrow\quad \gamma_j = \gamma_0$$, a constant, for $\underset{\sim}{E} = 0$

so that in an undulator, neglecting radiation fields & space charge:

$$\frac{d\underset{\sim}{\beta}_j}{dz} = -\frac{e}{\gamma_0 mc\beta_{zj}}\underset{\sim}{\beta}_j\wedge\underset{\sim}{B}$$

Consider a ***planar*** undulator field: $\underset{\sim}{B} = B_u\left(0, \sin\left(k_u z\right), 0\right)$.

<u>For x-component:</u>

$$\frac{d\beta_{xj}}{dz} = \frac{e}{\gamma_0 mc\beta_{zj}}\beta_{zj}B_y \quad\Rightarrow\quad \frac{d\beta_{xj}}{dz} = \frac{eB_u}{\gamma_0 mc}\sin\left(k_u z\right) \quad\Rightarrow\quad \boxed{\beta_{xj} = -\frac{a_u}{\gamma_0}\cos\left(k_u z\right)}$$

Can integrate again: $\frac{dx_j}{dz} \approx -\frac{a_u}{\gamma_0}\cos\left(k_u z\right) \quad\Rightarrow\quad \boxed{x_j \approx -\frac{a_u}{\gamma_0 k_u}\sin\left(k_u z\right)}$ $\Bigg\}$ For γ_0 >> 1, $\Rightarrow \beta_{zj} \approx 1$

where: $a_u \equiv \frac{eB_u}{mck_u}$ - Typical values $1 \lesssim a_u \lesssim 5$; $\gamma_{1GeV} \approx 2000$; $\lambda_u \approx$ 2cm

An electron trajectory in an undulator

Look at the z –component using: $\beta_{x\,j} = -\dfrac{a_u}{\gamma_0}\cos(k_u z)$ and $\gamma_0 = \dfrac{1}{\sqrt{1-\underset{\sim}{\beta}_j \cdot \underset{\sim}{\beta}_j}}$

$$\Rightarrow \quad \beta_{z\,j}^2 = 1 - \beta_{x\,j}^2 - \gamma_0^{-2} = 1 - \left(\frac{a_u}{\gamma_0}\cos(k_u z)\right)^2 - \gamma_0^{-2} \quad \Rightarrow \quad \beta_{z\,j}^2 = 1 - \frac{1}{\gamma_0^2}\left(1 + a_u^2\cos^2(k_u z)\right)$$

$$\Rightarrow \quad \beta_{z\,j}^2 = 1 - \frac{1}{\gamma_0^2}\left(1 + a_u^2\left(\frac{1+\cos(2k_u z)}{2}\right)\right) = 1 - \frac{1}{\gamma_0^2}\left(1 + \frac{a_u^2}{2} + \frac{a_u^2\cos(2k_u z)}{2}\right)$$

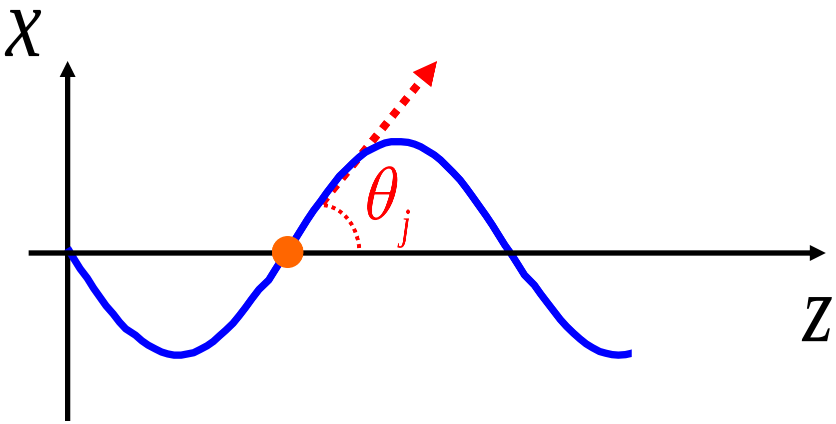
$$\Rightarrow 1 - \beta_{z\,j}^2 = \frac{1}{\gamma_0^2}\left(1 + \frac{a_u^2}{2} + \frac{a_u^2\cos(2k_u z)}{2}\right)$$

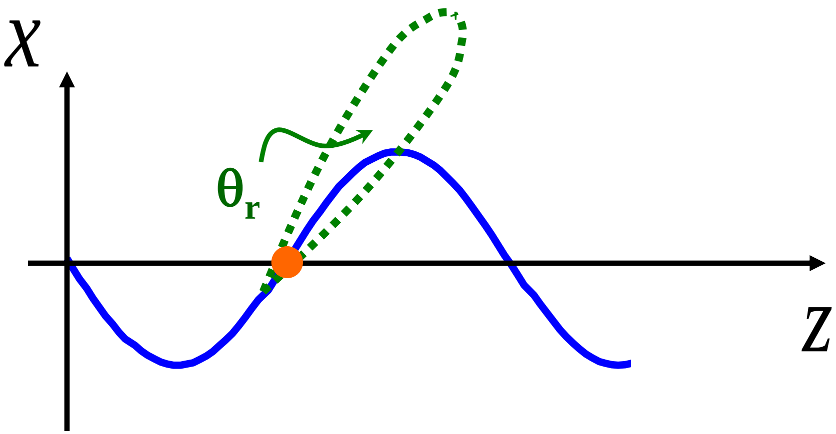
Now use $1-\beta_{z\,j}^2 = (1+\beta_{z\,j})(1-\beta_{z\,j}) \approx 2(1-\beta_{z\,j})$:

$$\Rightarrow (1-\beta_{z\,j}) \approx \frac{1}{2\gamma_0^2}\left(1 + \frac{a_u^2}{2} + \frac{a_u^2\cos(2k_u z)}{2}\right) \quad \Rightarrow \quad \beta_{z\,j} \approx 1 - \frac{1}{2\gamma_0^2}\left(1 + \frac{a_u^2}{2} + \frac{a_u^2\cos(2k_u z)}{2}\right)$$

Can be averaged to give: $\bar{\beta}_{z\,j} \approx 1 - \dfrac{1}{2\gamma_0^2}\left(1 + \dfrac{a_u^2}{2}\right)$

An electron trajectory in an undulator





The angle the electron makes with respect to the undulator axis can be approximated as:

$$\tan\theta_j = \frac{dx_j}{dz} \approx -\frac{a_u}{\gamma_0}\cos(k_u z)$$

$$\Rightarrow \theta_j \propto \frac{a_u}{\gamma_0} \quad \text{for } \gamma_0 \gg a_u$$

The radiated power is confined mainly to an angle $\theta_r \approx 1/\gamma_0$.

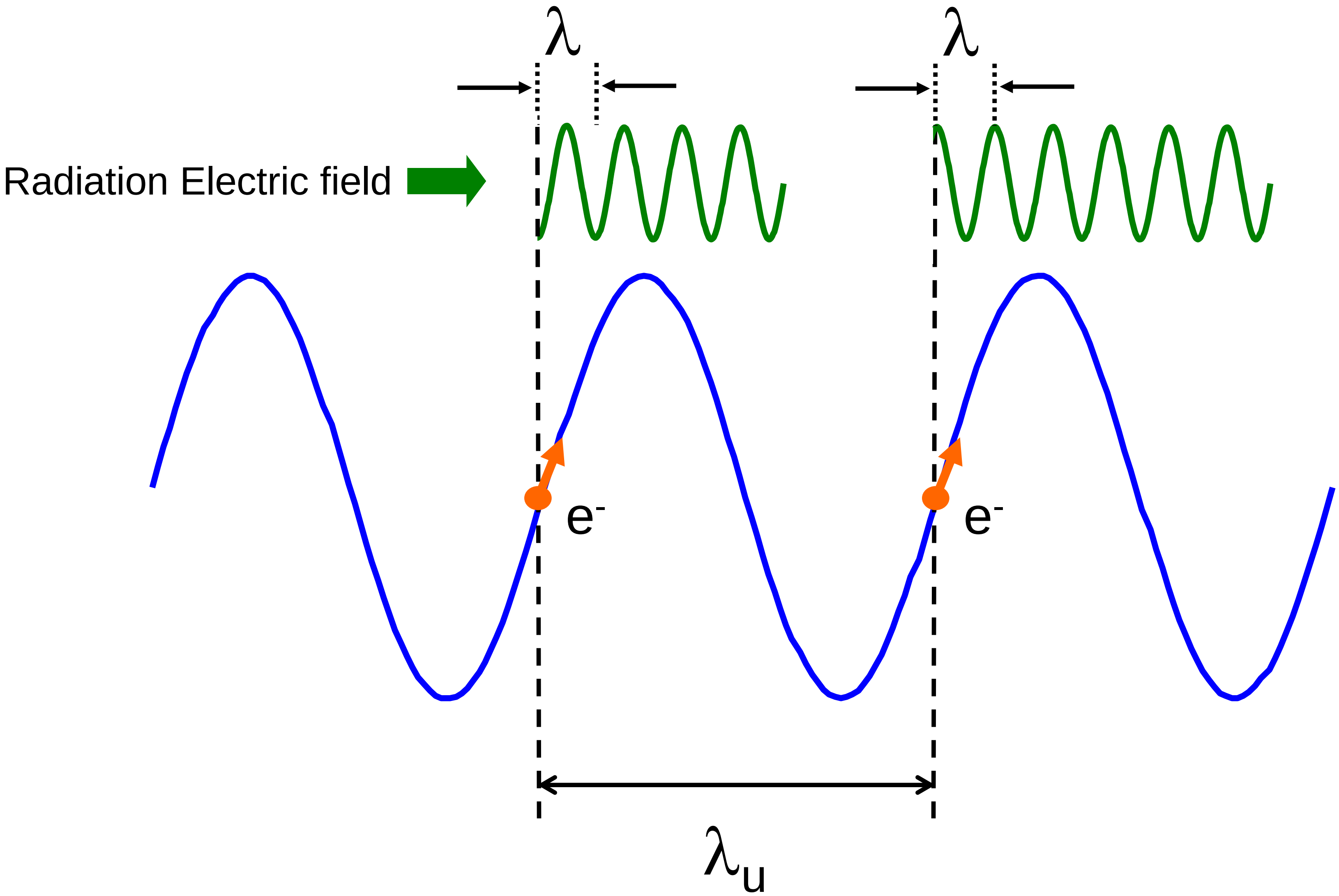
Hence if: $\theta_j \gg \theta_r$ i.e. $a_u \gg 1$,

The emitted power behaves like a 'searchlight' when viewed at end of the undulator.

$a_u \sim 1$ - 'Undulator'

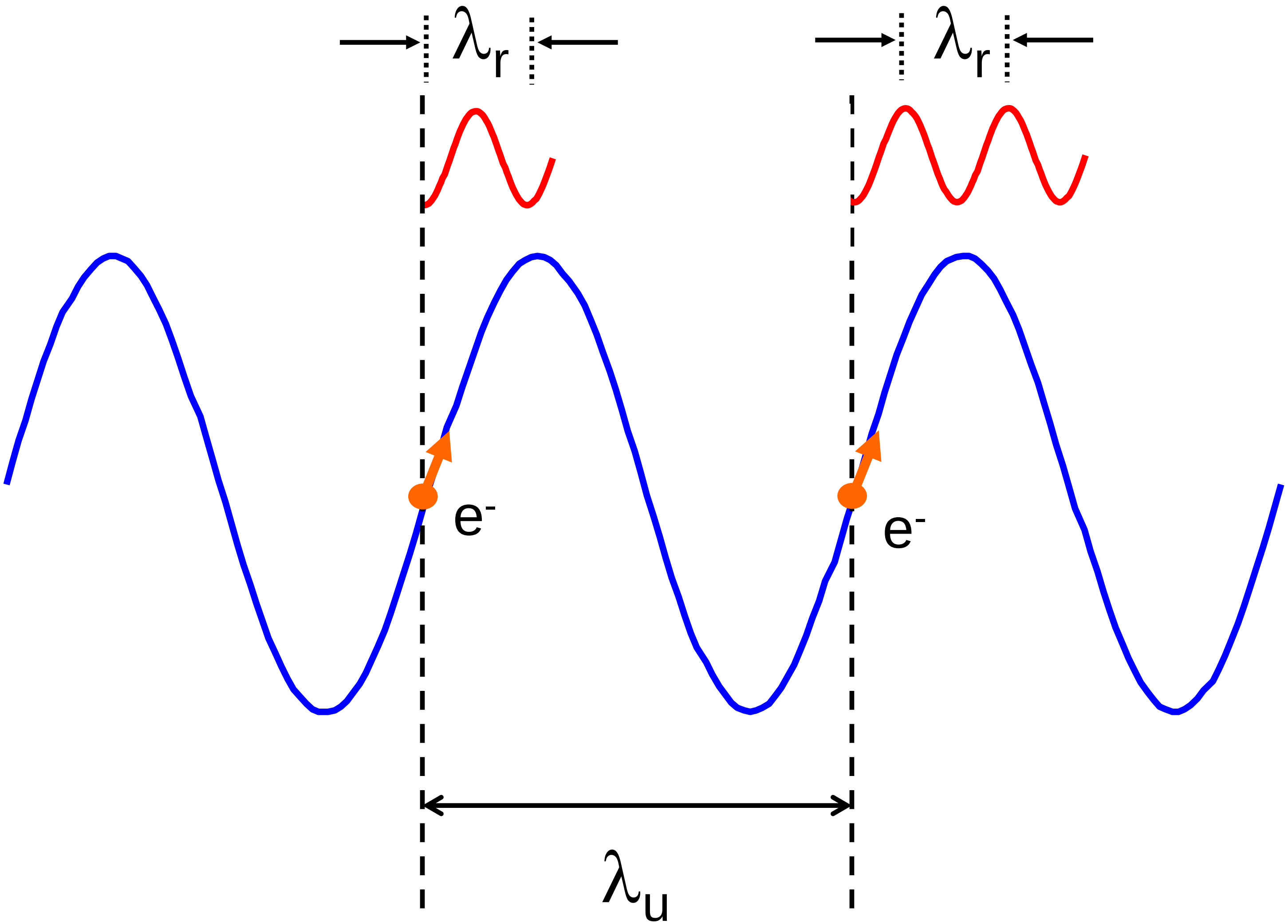
$a_u \gg 1$ - 'Wiggler'

Non-resonant emission - destructive interference

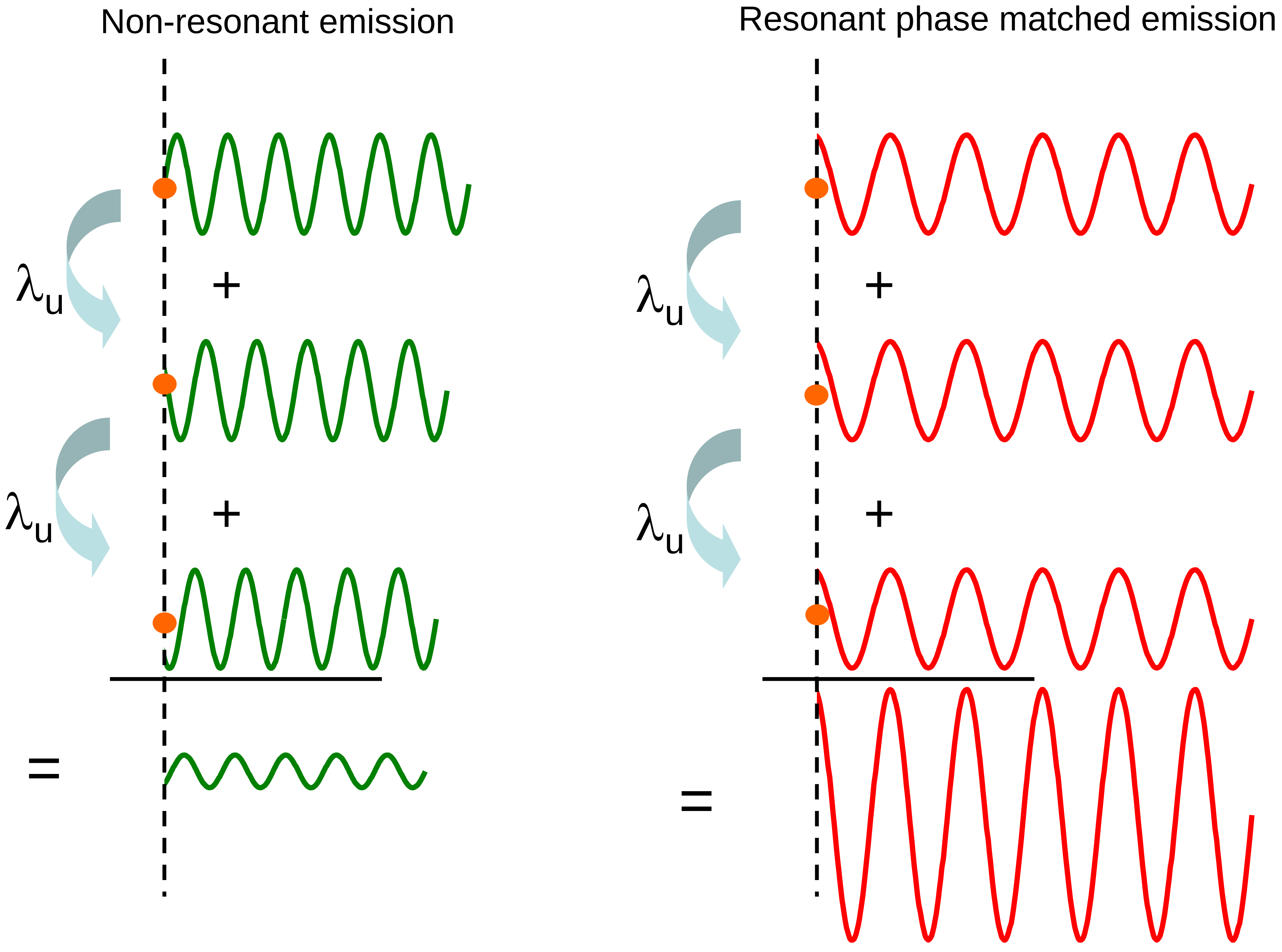


The radiation is ***not phase-matched*** to the electron trajectory.

Resonant emission - constructive interference

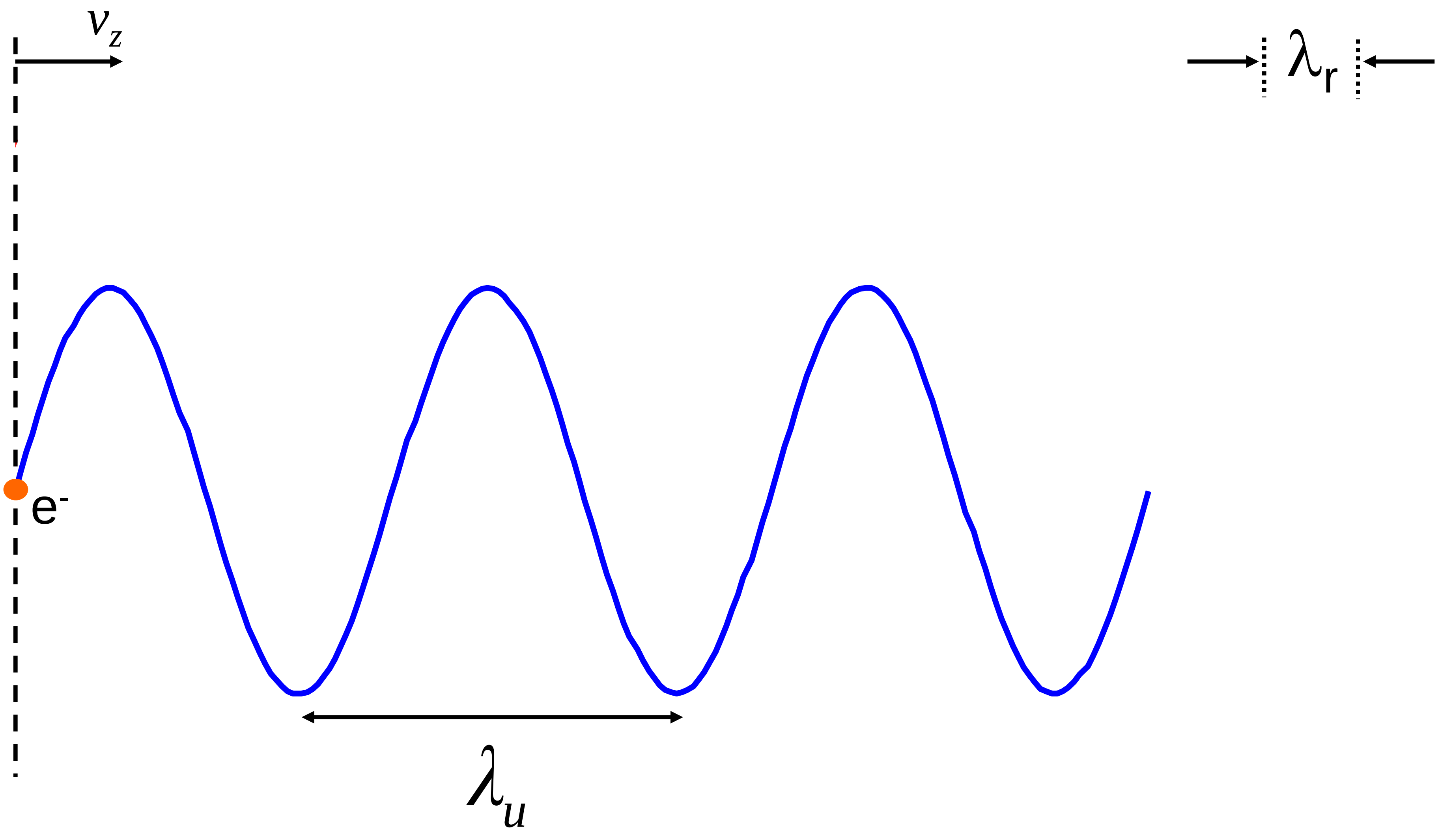


The radiation and electron trajectory are ***phase-matched***.

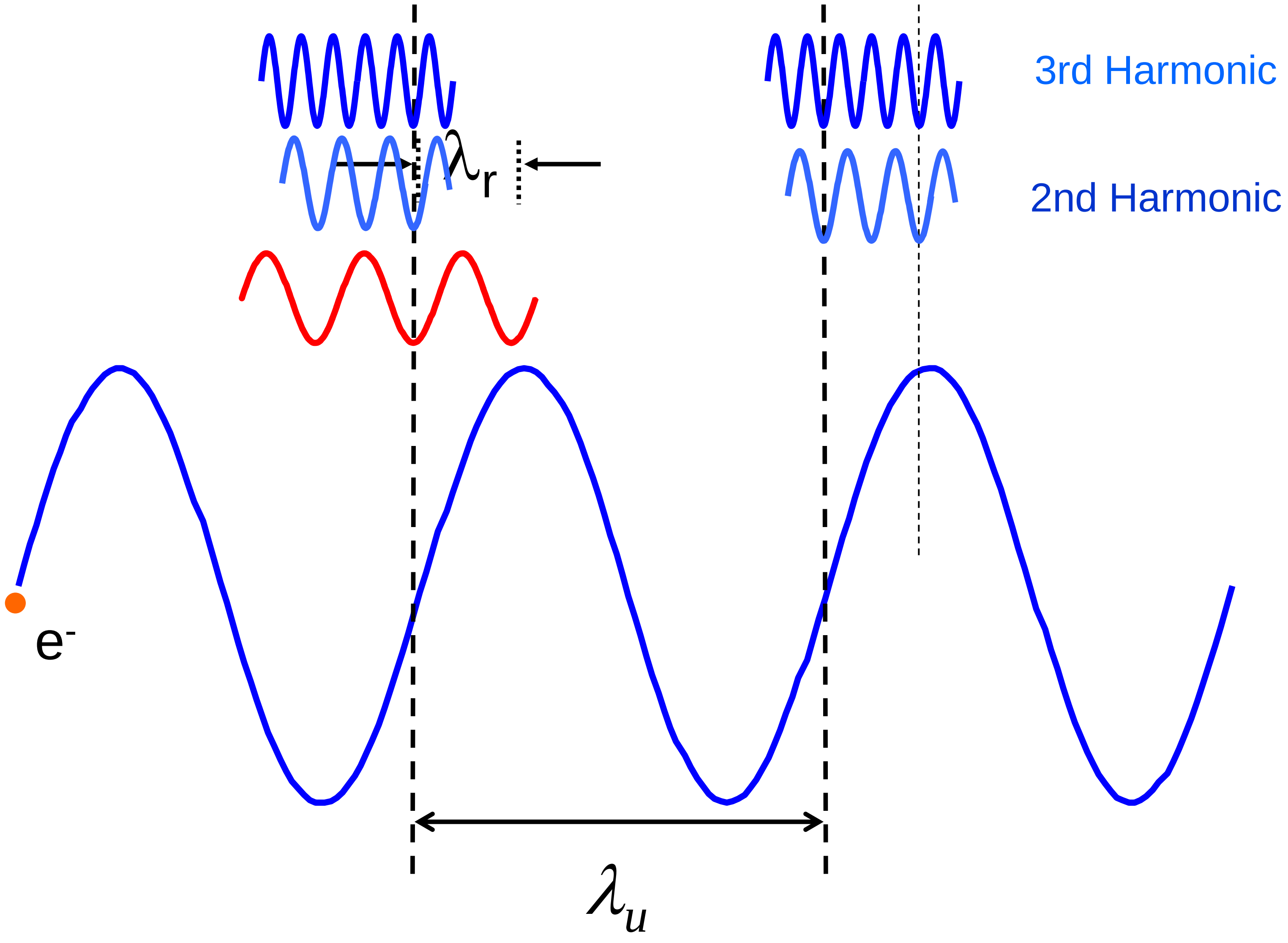


Non-resonant emission
Resonant phase matched emission
λu
+
λu
+
=
λu
+
λu
+
=

Resonant phase matched emission by an electron



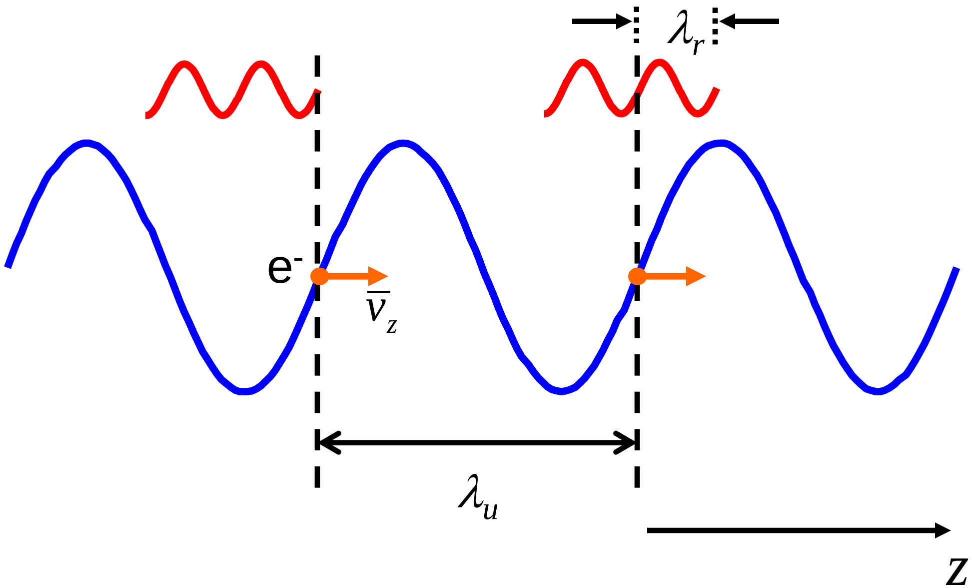
Resonant phase matched emission for harmonics



Harmonics of the fundamental are also phase-matched.

What are properties of radiation from an undulator ?

Resonant emission - constructive interference



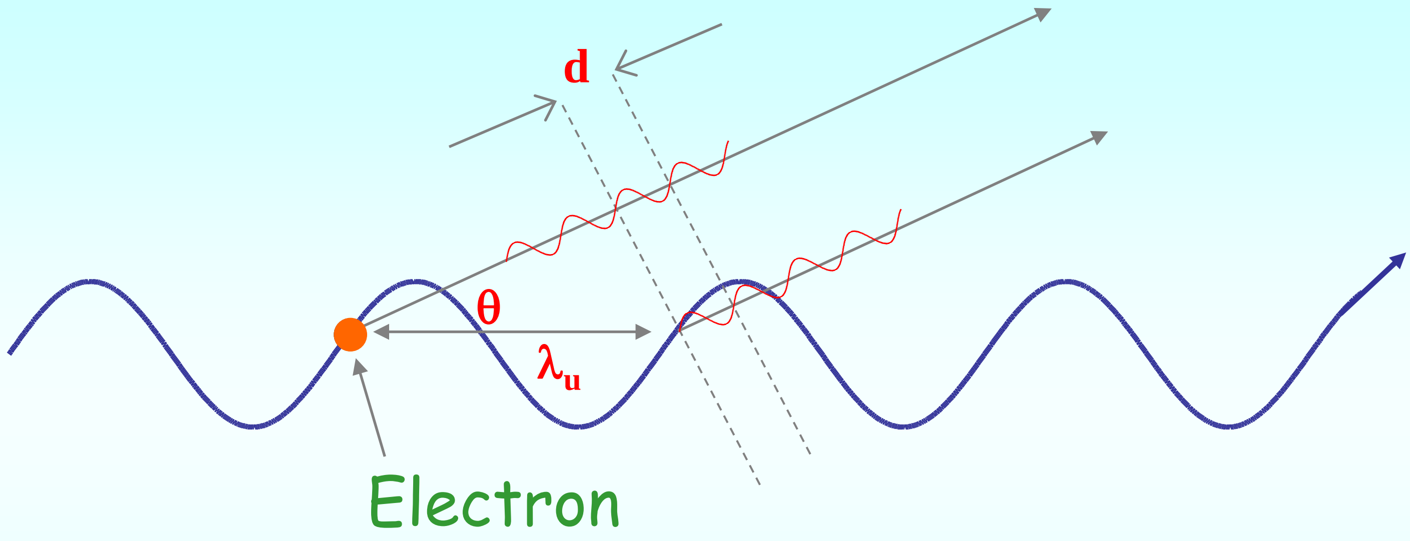
The time taken for the electron to travel one undulator period:

A resonant radiation wavefront will have travelled $\Rightarrow$

Equating:

Resonant emission - constructive interference including harmonics and angle from undulator axis

Condition for constructive interference: $d = n\lambda = \dfrac{\lambda_u}{\beta_{zj}} - \lambda_u \cos\theta$



Where: $n = 1, 2, 3, \ldots$ is an integer representing the harmonic number

Undulator Equation

Substituting in for the average longitudinal velocity of the electron, $\bar{\beta}_z$, for the earlier planar case:

Substitute $\bar{\beta}_{zj} \approx 1 - \frac{1}{2\gamma_0^2}\left(1 + \frac{a_u^2}{2}\right)$ into $n\lambda_r = \frac{\lambda_u}{\bar{\beta}_{zj}} - \lambda_u \cos\theta$

$$\Rightarrow \quad \lambda_r = \frac{\lambda_u}{2n\gamma_0^2}\left(1 + \bar{a}_u^2 + \theta^2\gamma_0^2\right)$$

Including angular dependence

$\bar{a}_u = \frac{e\lambda_u B_u^{RMS}}{2\pi mc}$ is the RMS “wiggler/undulator parameter”
- In this form also valid for helical undulators

For a 3 GeV electron passing through a 5 cm period undulator with $\bar{a}_u$ = 3, the wavelength of the first harmonic (n = 1) on axis ($\theta = 0$) is ~ 4 nm

Spontaneous spectrum

We can calculate the spontaneous emission spectrum in the frequency range $d\omega$ and solid angle $d\Omega$ around the observation direction $\underset{\sim}{n}$ *by inserting the expression for the electron trajectory into the standard formula for far-field emission* from an accelerated charged particle (see e.g. "Classical Electrodynamics" by Jackson , ch. 14)

$$\frac{d^2 I}{d\omega\, d\Omega}(\underset{\sim}{n},\omega) = \frac{e^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \underset{\sim}{n}\times\left(\underset{\sim}{n}\times\underset{\sim}{\beta}\right)e^{i\omega\left(t-\underset{\sim}{n}\cdot\underset{\sim}{r}/c\right)}\,dt\right|^2$$

where $\underset{\sim}{r}(t)$ is the electron position at time t, and $\underset{\sim}{\beta} = \frac{\underset{\sim}{v}}{c}$

Spontaneous spectrum

If we do this we find that the fundamental spectrum on axis ($\underset{\sim}{n} = \underset{\sim}{\hat{z}}$) is :

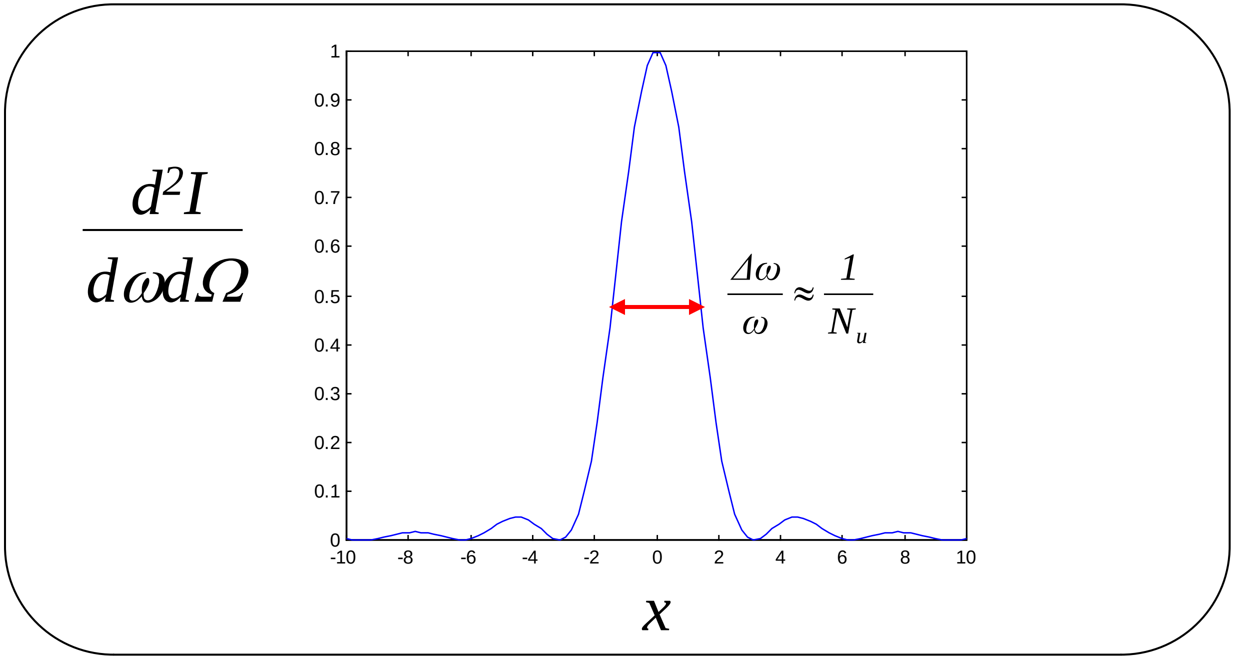
$$\boxed{\frac{d^2 I}{d\omega \, d\Omega}(\underset{\sim}{n} = \underset{\sim}{\hat{z}}, \omega) \propto \frac{\sin^2 x}{x^2}}$$

Where: $x = \pi N_u \dfrac{\omega - \omega_r}{\omega_r}$

N_u is the number of undulator periods

$\omega_r = \dfrac{2ck_u\gamma_0^2}{1+\bar{a}_u^2}$ is the central (resonance) frequency

On axis fundamental spontaneous spectrum therefore looks like :



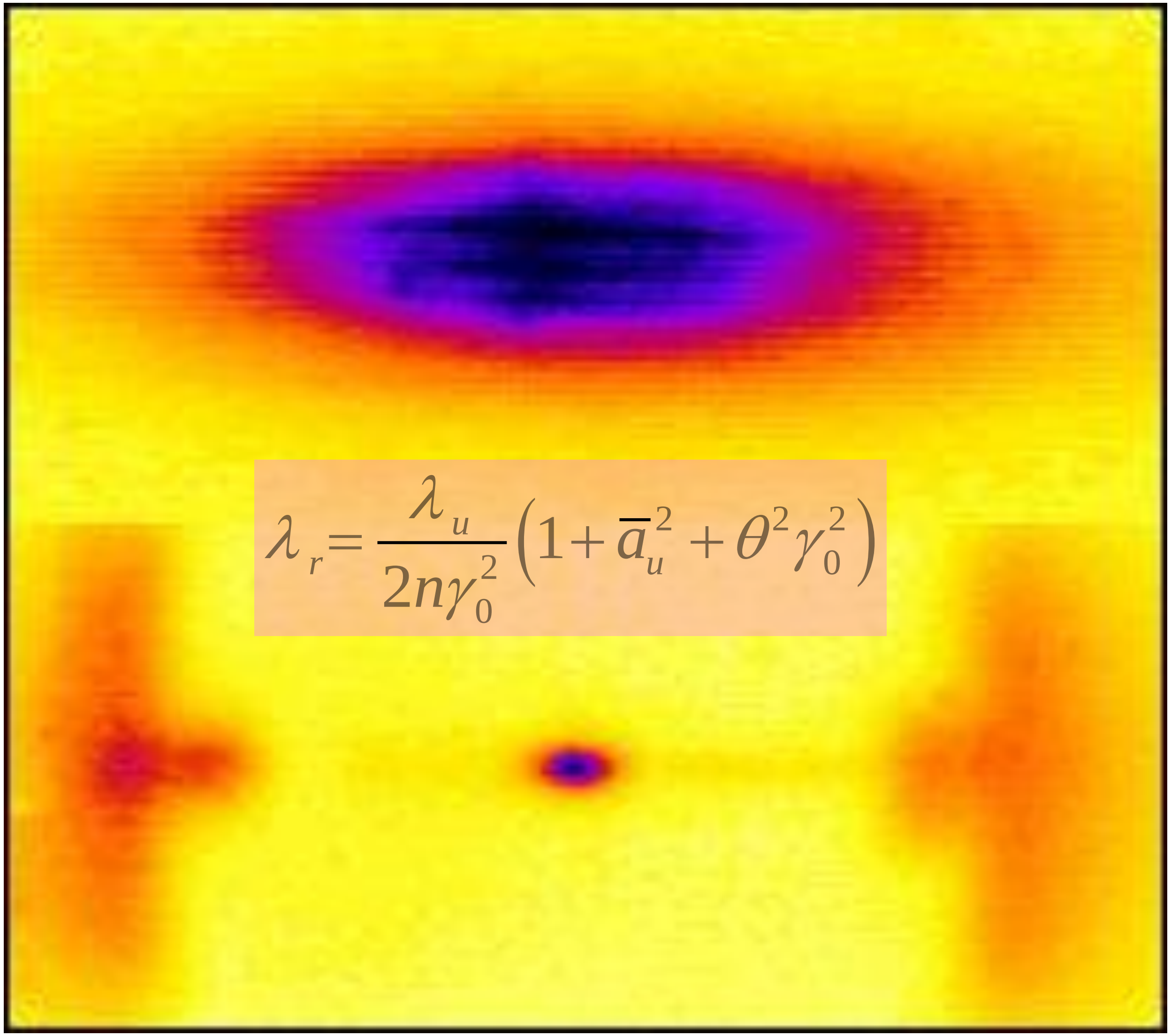
Main features :

- Spectrum strongly peaked at frequency ω_r
  i.e. at wavelength

$$\lambda_r = \frac{2\pi c}{\omega_r} = \lambda_u \left(\frac{1+\bar{a}_u^2}{2\gamma_0^2} \right)$$

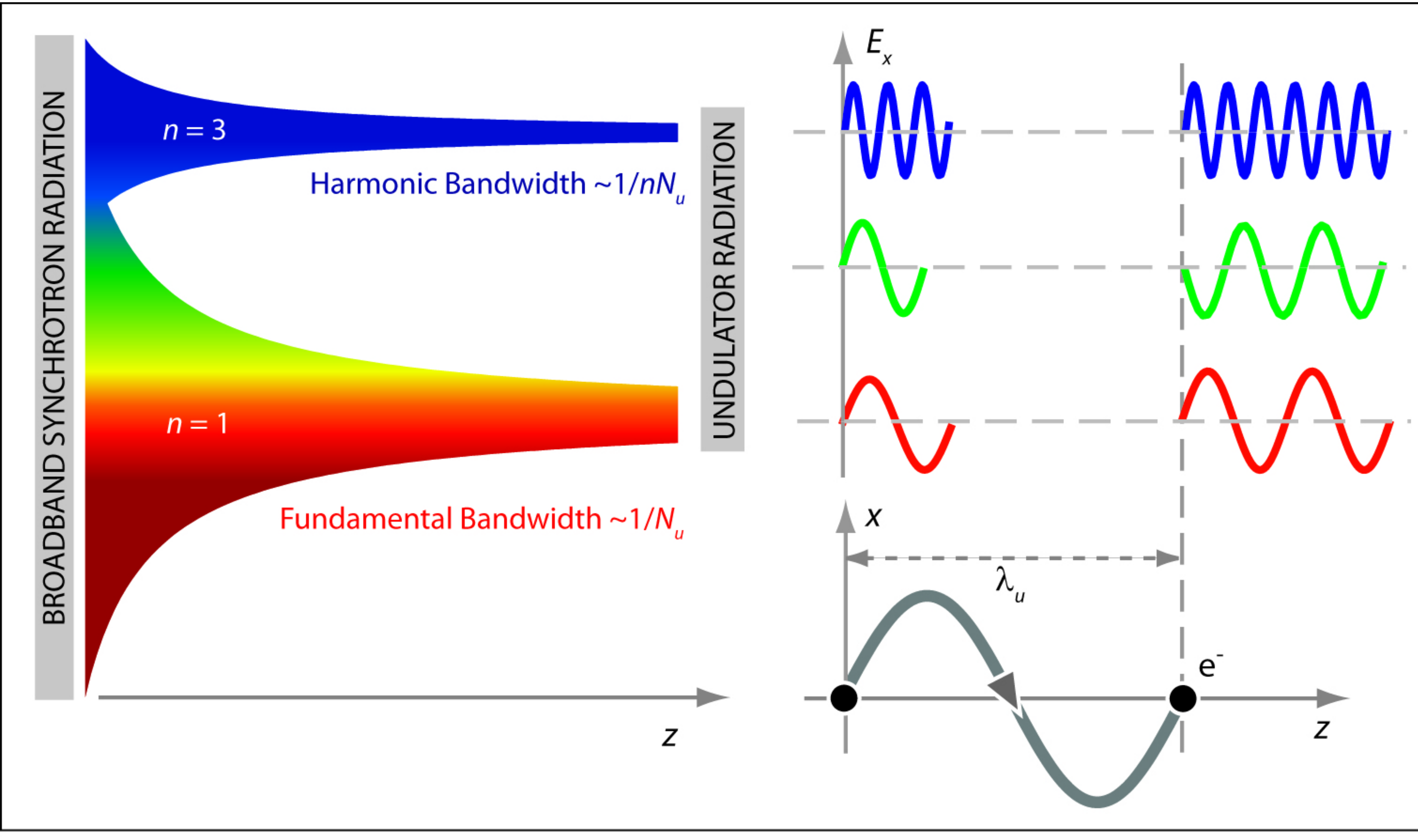
- Width of spectrum $\quad \frac{\Delta\omega}{\omega} \approx \frac{1}{N_u}$

Undulator Radiation

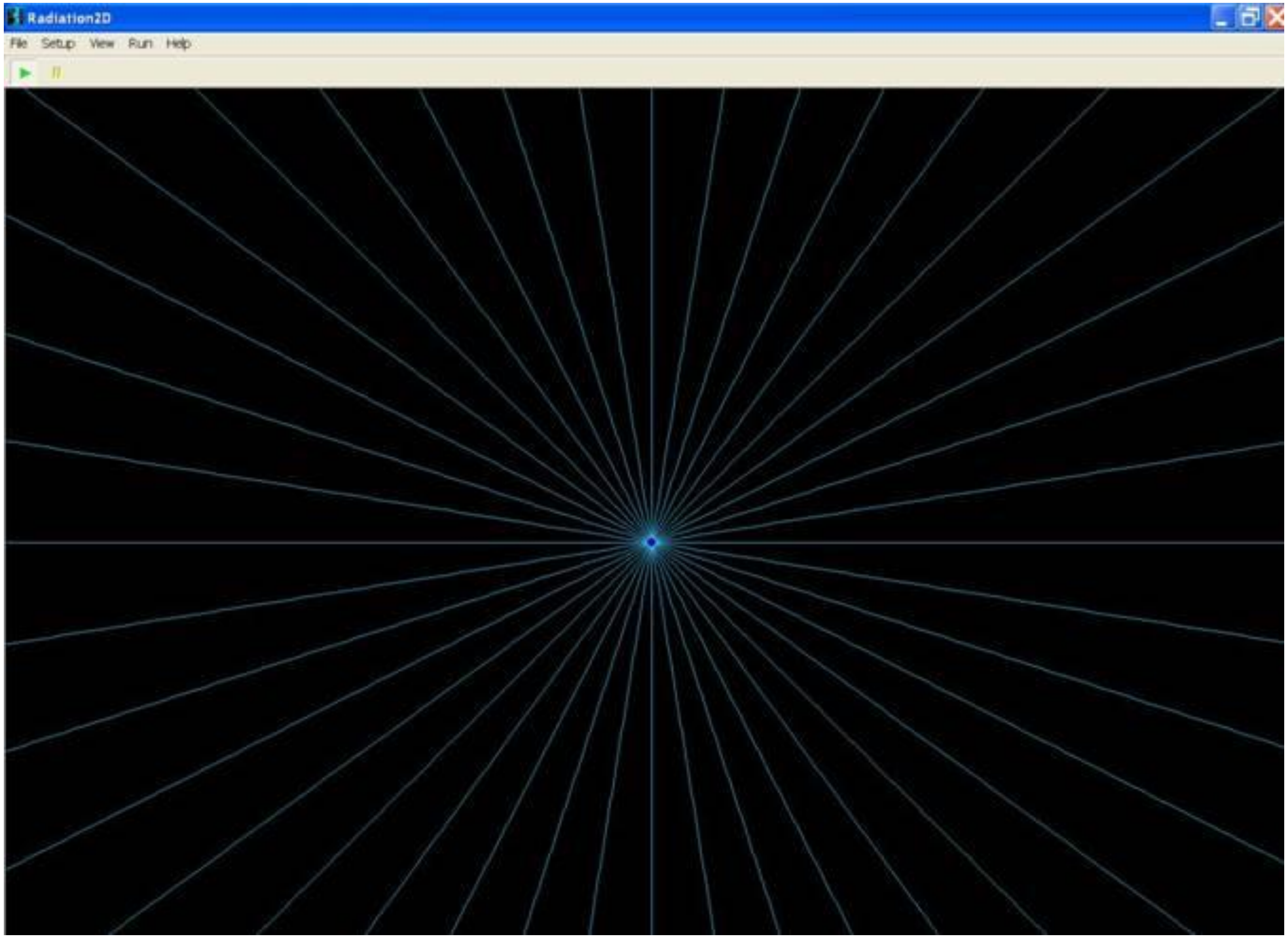


Undulator radiation (top) focused on a spot (bottom) by a refractive lens.

Summarising



Undulator radiation
Setup>trajectory>undulator



Code available at: http://www-xfel.spring8.or.jp/

Electron bunching in a ***fixed*** radiation field

The electron-radiation interaction

The Lorentz force (electron dynamics)

$$\underset{\sim}{F}_j = q\left[\underset{\sim}{E} + \underset{\sim}{v}_j \times \underset{\sim}{B}\right]$$

Maxwell wave equation* (radiation evolution)

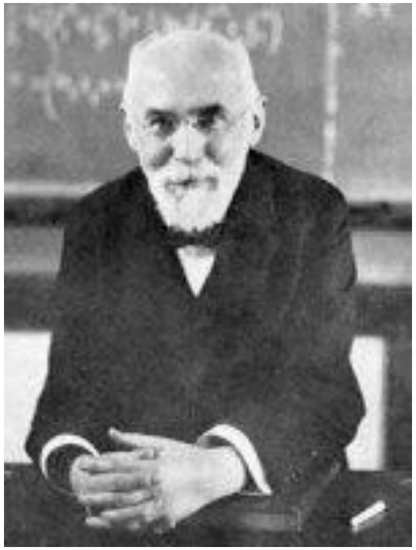
$$\cancel{\nabla^2 \underset{\sim}{E} - \frac{1}{c^2}\frac{\partial^2 \underset{\sim}{E}}{\partial t^2} = \mu_0 \frac{\partial \underset{\sim}{J}}{\partial t}}$$

Both equations must be solved together simultaneously

(self-consistently) to fully describe the FEL interaction

*Neglect static fields (space charge effects) – Compton limit

How the electron is effected by the resonant radiation



Hendrick Antoon Lorentz

The Lorentz Force Equation:

$$\underset{\sim}{F}_j = \frac{d\left(\gamma_j m_0 \underset{\sim}{v}_j\right)}{dt} = -|e|\left[\underset{\sim}{E} + \underset{\sim}{v}_j \times \underset{\sim}{B}\right]$$

Can calculate

The rate of change of electron energy

$$\frac{d\left(\gamma_j m_0 c^2\right)}{dt} = -|e|\underset{\sim}{E} \cdot \underset{\sim}{v}_j$$

Slow energy exchange

The rate of change of electron energy: $$\frac{d\left(\gamma_j m_0 c^2\right)}{dt} = -\left|e\right| \underset{\sim}{E} \cdot \underset{\sim}{v}_j$$

Consider plane-wave field: $\underset{\sim}{E} = \underset{\sim}{\hat{x}}\, E_0 \sin\left(k_r z - \omega_r t\right)$

Interacting with an electron on trajectory: $\beta_{x\,j} = -\dfrac{a_u}{\gamma_0}\cos\left(k_u z_j\right)$

Assuming: $\gamma_j \approx \gamma_0$

$$\Rightarrow \quad \frac{d\left(\gamma_j m_0 c^2\right)}{dt} = -\left|e\right| \underset{\sim}{E} \cdot \underset{\sim}{v}_j = -\left|e\right| E_0 \sin\left(k_r z_j - \omega_r t\right)\left(-\frac{a_u}{\gamma_0}\cos\left(k_u z_j\right)\right)$$

$$\Rightarrow \quad \frac{d\gamma_j}{dt} = \frac{\left|e\right| a_u E_0}{\gamma_0 m_0 c^2}\sin\left(k_r z_j - \omega_r t\right)\cos\left(k_u z_j\right)$$

$$= \frac{\left|e\right| a_u E_0}{\gamma_0 m_0 c^2}\frac{1}{2}\left(\sin\left(\left(k_r + k_u\right) z_j - \omega_r t\right) + \sin\left(\left(k_r - k_u\right) z_j - \omega_r t\right)\right)$$

Slow energy exchange

$$\frac{d\gamma_j}{dt} = \frac{|e| a_u E_0}{\gamma_0 m_0 c^2} \frac{1}{2}\left(\sin\left(\left(k_r + k_u\right) z_j - \omega_r t\right) + \sin\left(\left(k_r - k_u\right) z_j - \omega_r t\right)\right)$$

The first sin term on RHS is a wave with speed in z direction of:

$$v_z = \frac{\omega_r}{k_r + k_u} = \frac{ck_r}{k_r + k_u} \quad \Rightarrow \quad \beta_z = \frac{k_r}{k_r + k_u}$$

Recall previous result for resonance:

$$\lambda_r = \frac{1 - \bar{\beta}_{zj}}{\bar{\beta}_{zj}} \lambda_u \quad \Rightarrow \quad \bar{\beta}_{zj} = \frac{\lambda_u}{\lambda_r + \lambda_u} = \frac{k_r}{k_r + k_u}$$

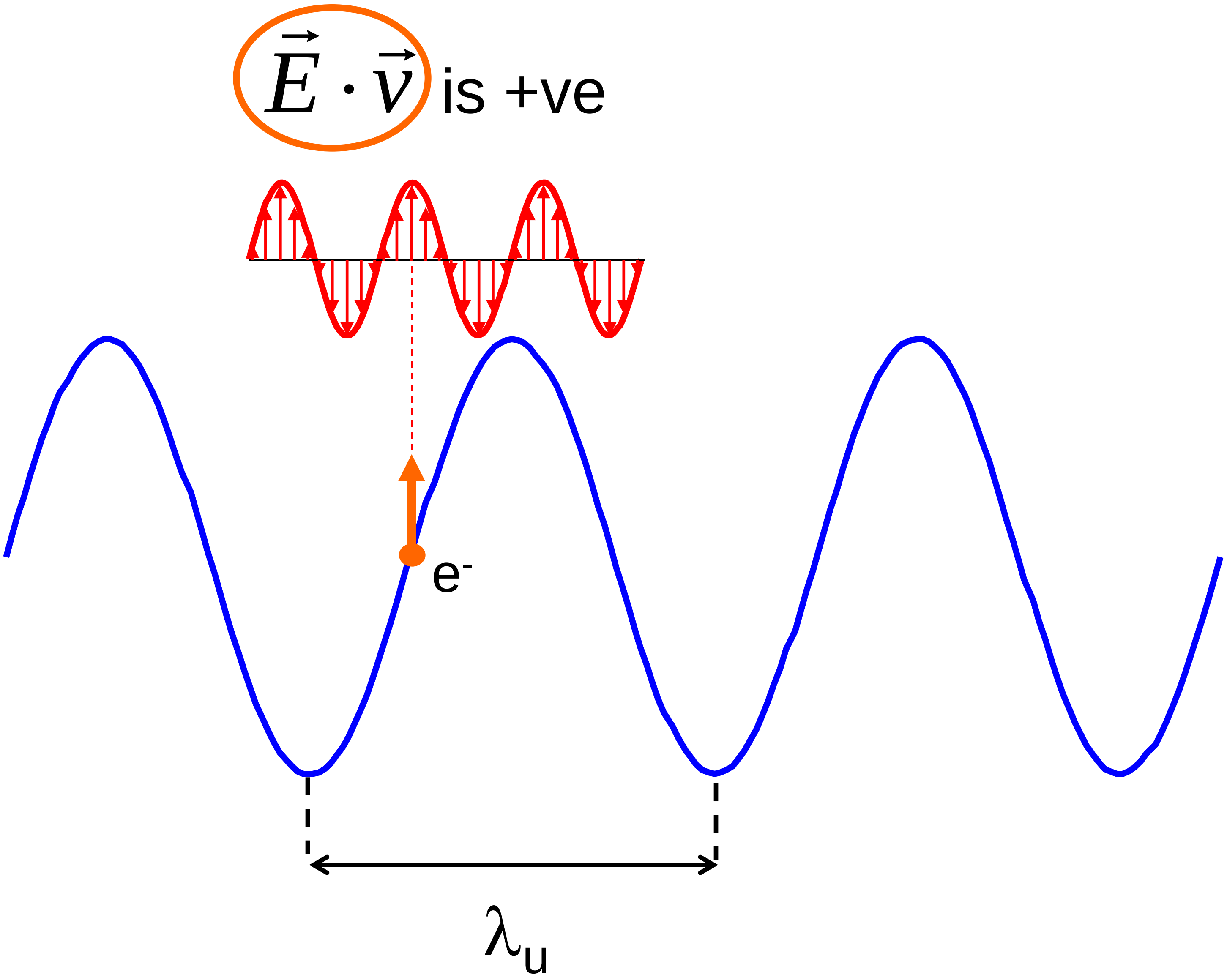
So, a resonant electron with average speed $\bar{\beta}_{zj}$ will have $\frac{d\gamma_j}{dt} \approx$ constant

The second sin term on RHS is a wave with speed in z direction of:

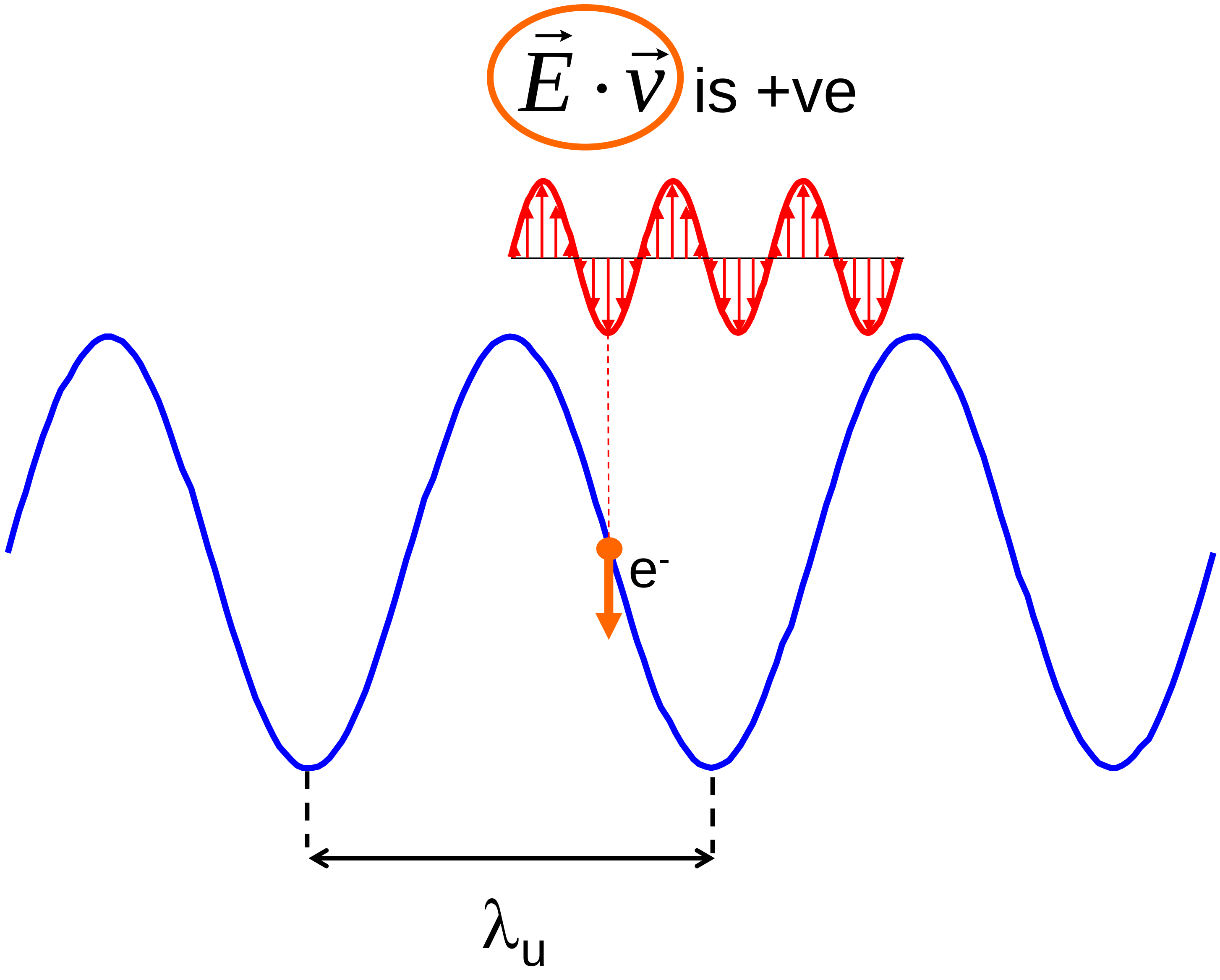
$$v_z = \frac{\omega_r}{k_r - k_u} = \frac{ck_r}{k_r - k_u} \quad \Rightarrow \quad \beta_z = \frac{k_r}{k_r - k_u} > 1$$

– fast, non-resonant, phase variation.

Resonant emission – electron energy change

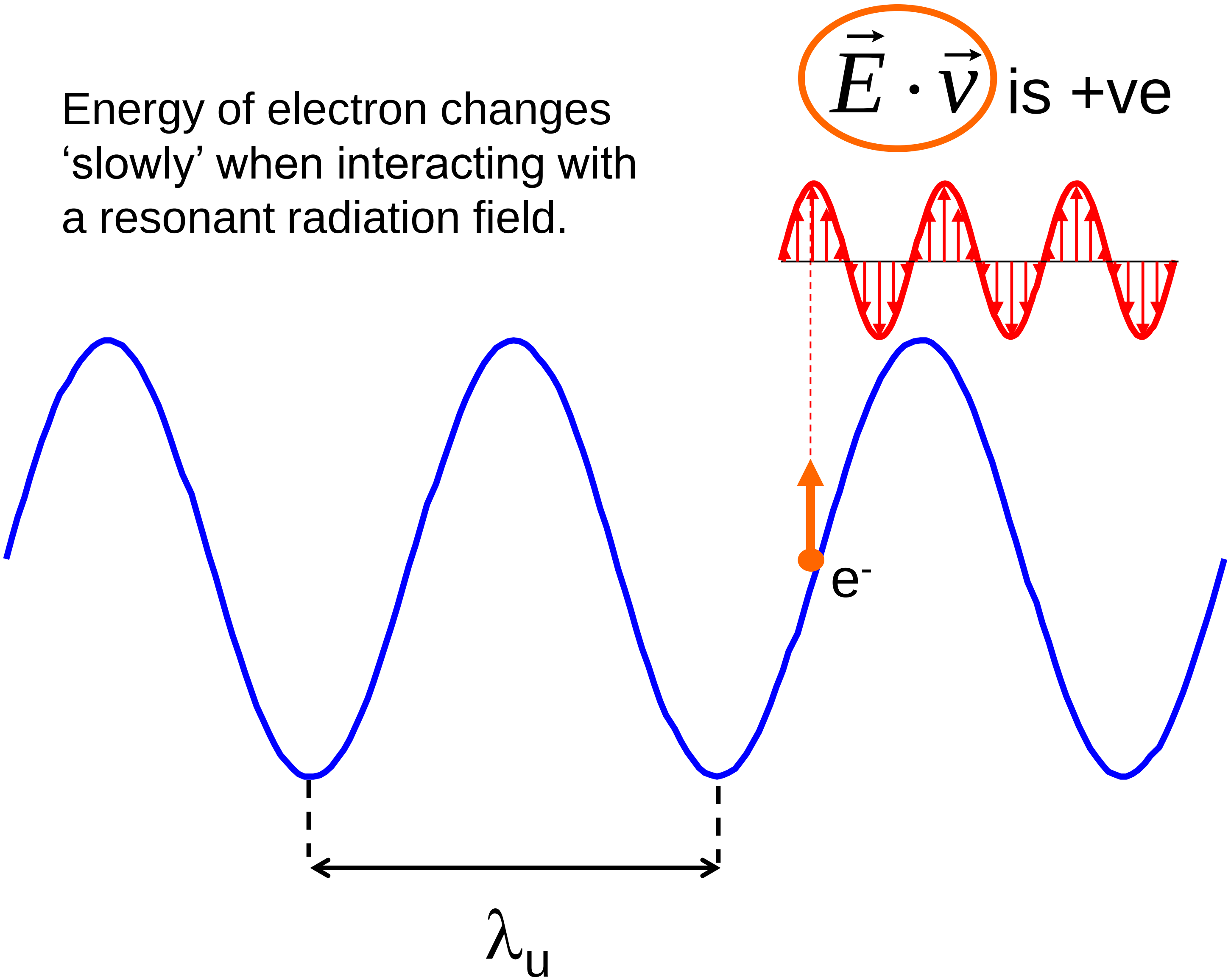


Resonant emission – electron energy change



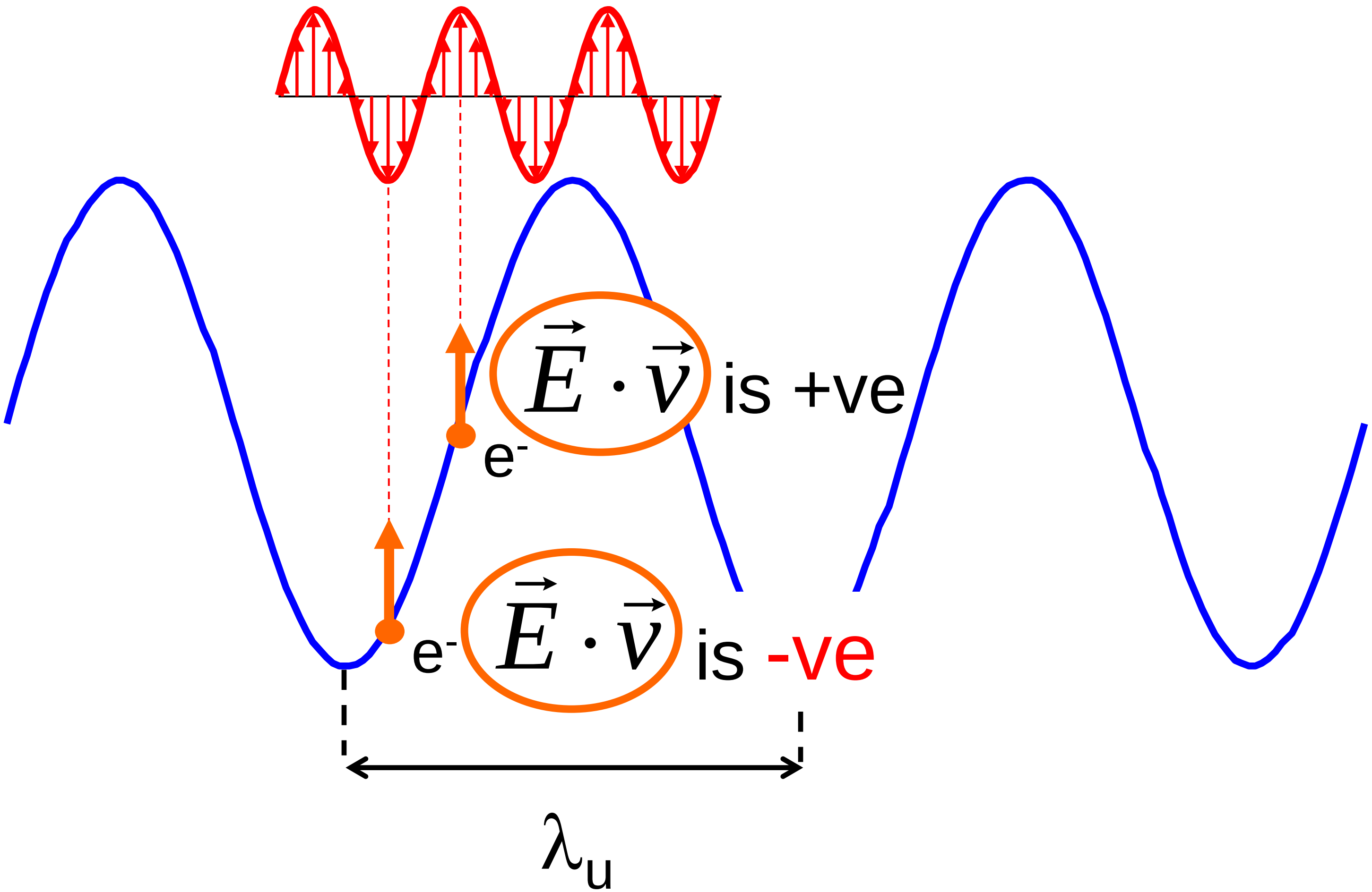
Resonant emission – electron energy change

Energy of electron changes 'slowly' when interacting with a resonant radiation field.



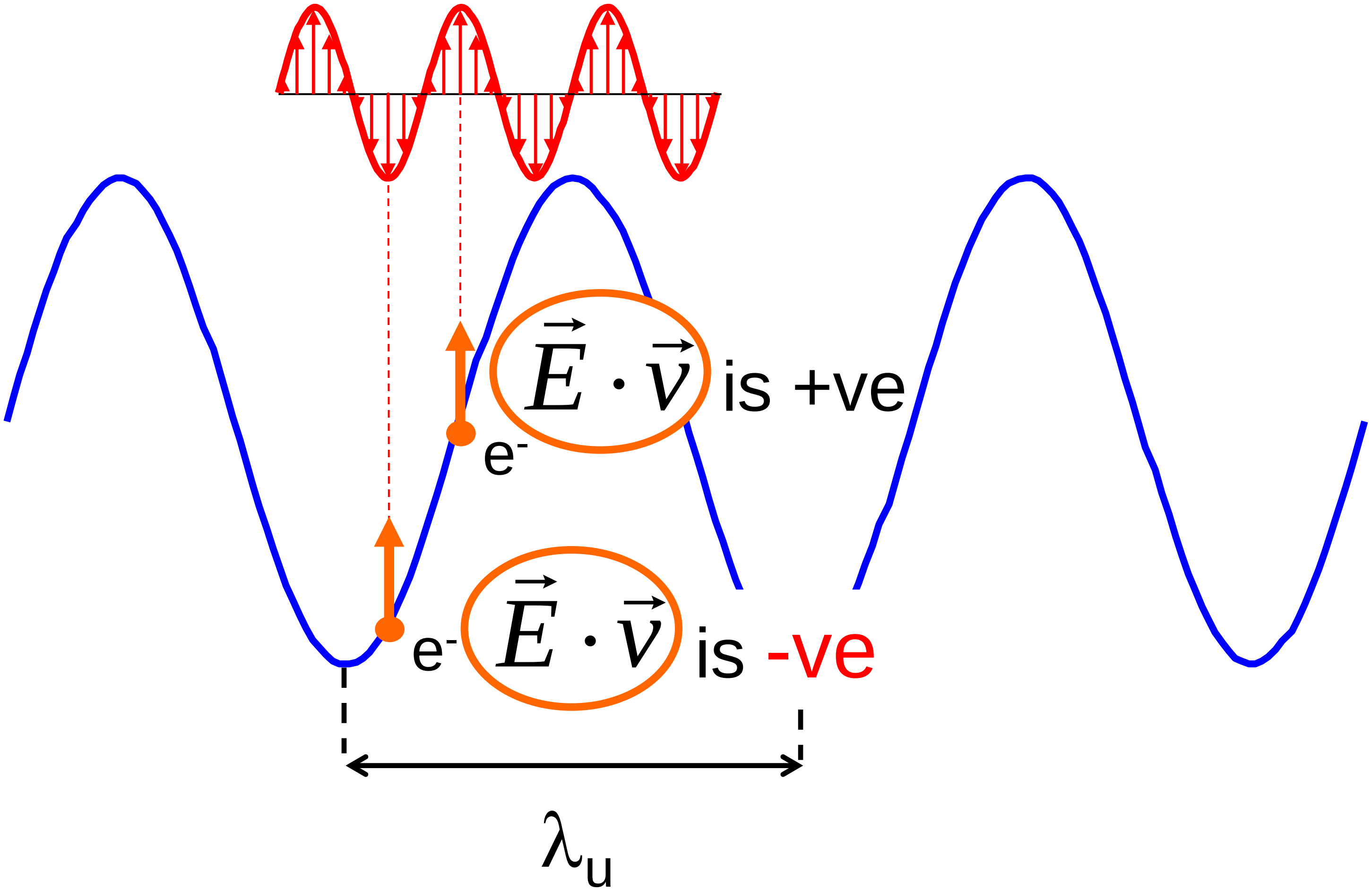
Resonant emission – electron energy change

For an electron with a different phase with respect to radiation field:



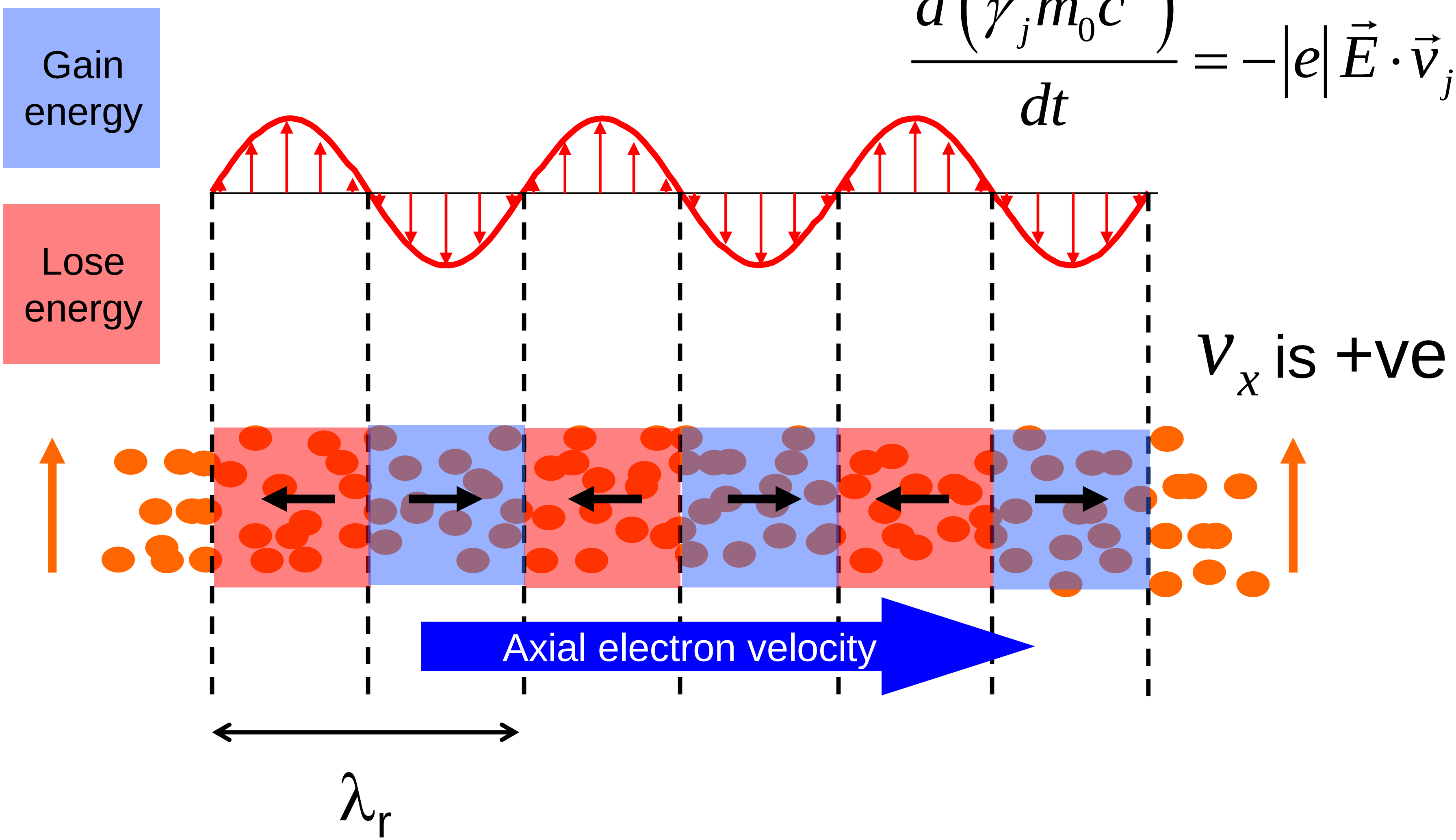
Resonant emission – electron energy change

Rate of electron energy change is 'slow' but changes periodically with respect to the radiation phase



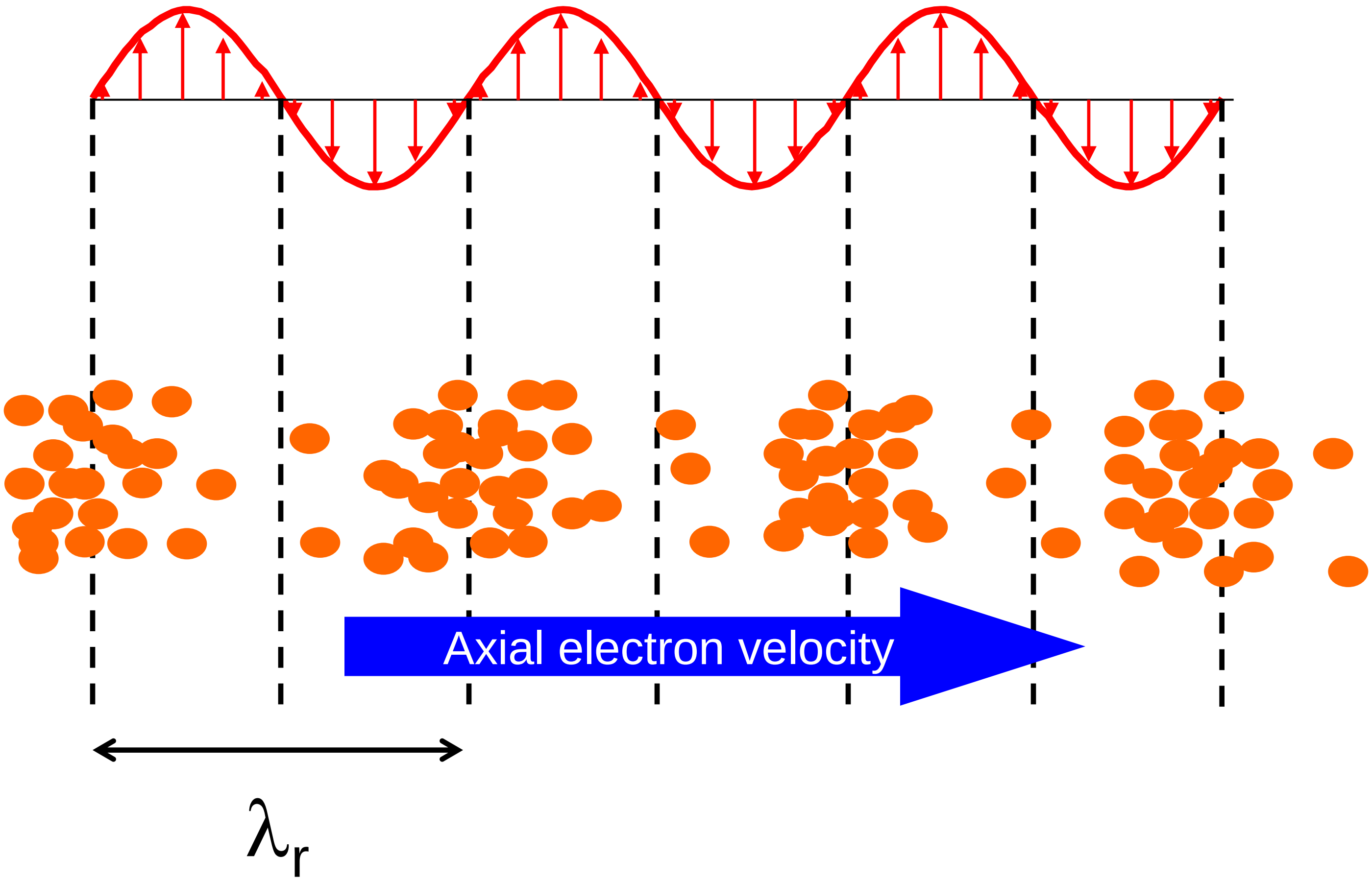
Resonant interaction – electron bunching

$$\frac{d\left(\gamma_j m_0 c^2\right)}{dt} = -|e|\vec{E}\cdot\vec{v}_j$$

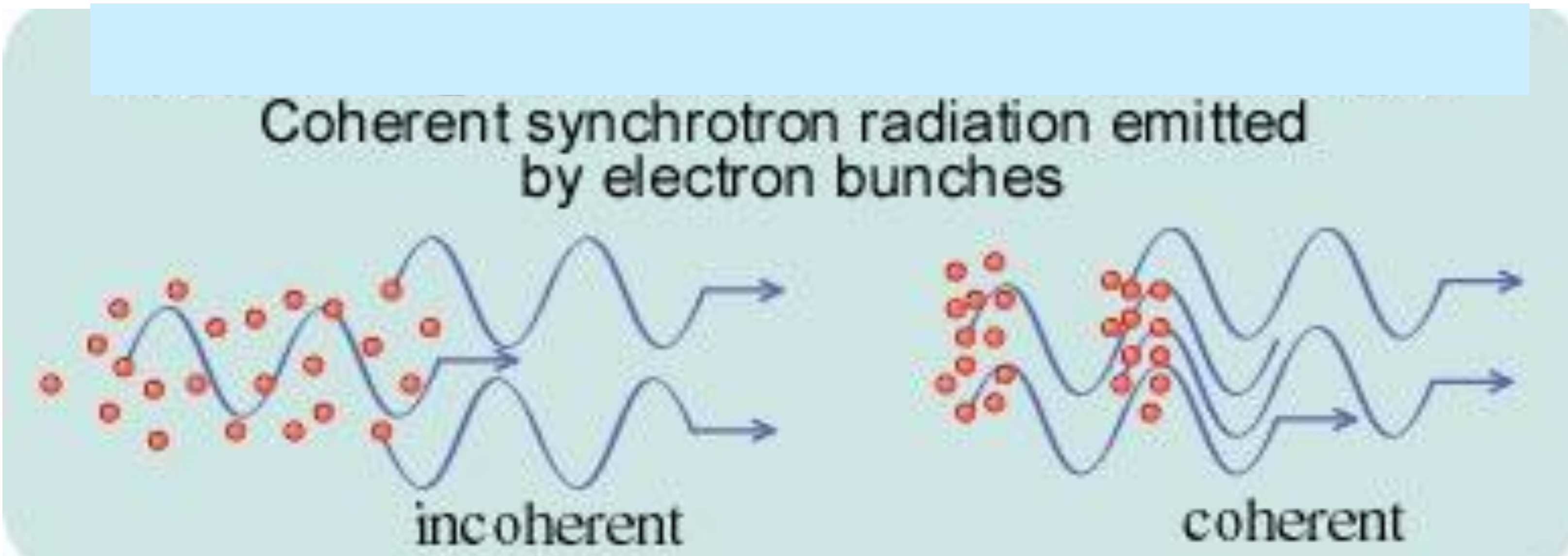


Resonant interaction – electron bunching

Electrons bunch at resonant radiation wavelength – coherent process



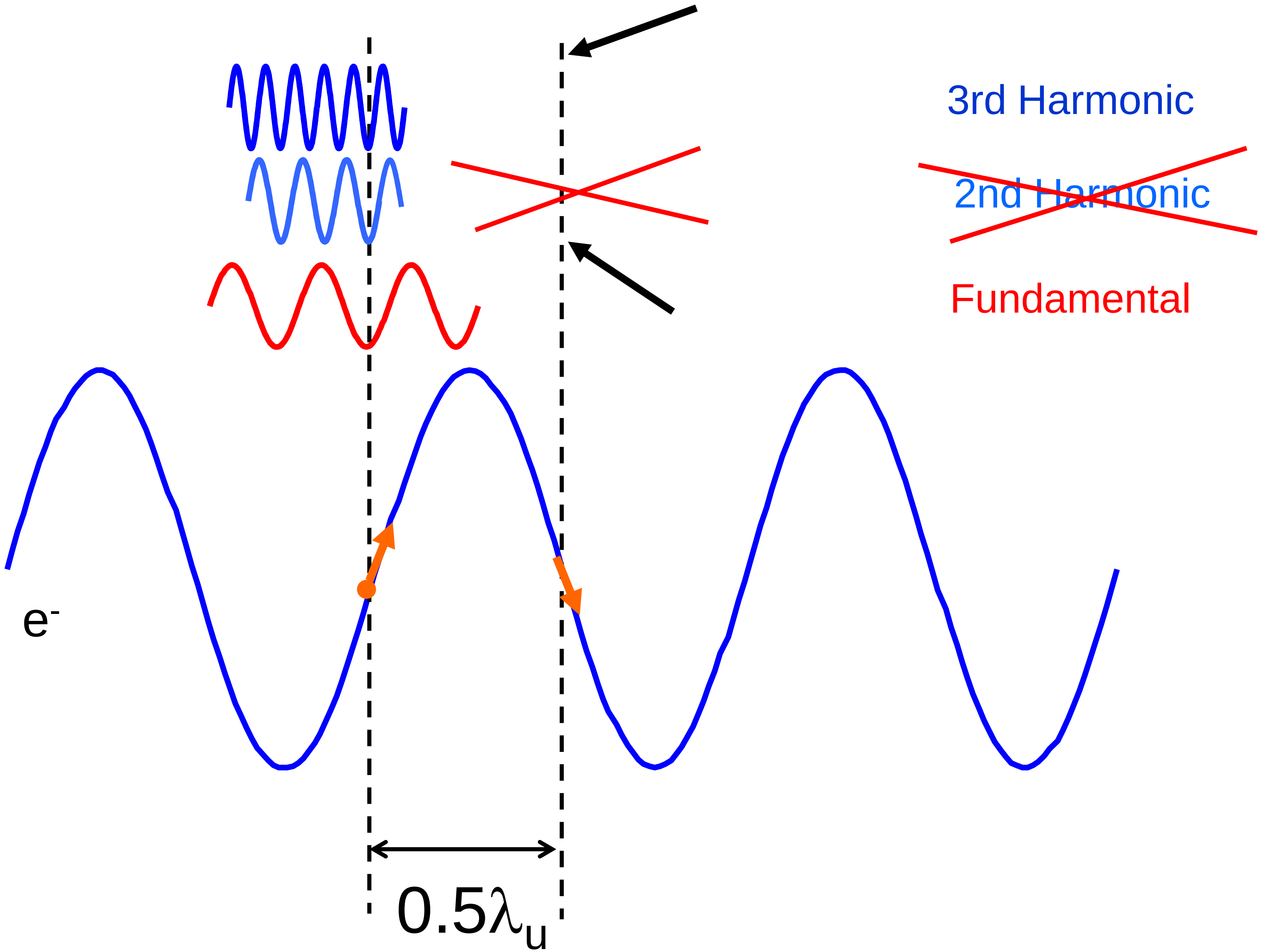
Bunched electrons can exchange energy coherently with radiation



Radiation power $\propto \left(\sum_{j=1}^{N} E_j \, e^{i\phi_j} \right)^2 = \sum_{j=1}^{N} E_j^2 + \sum_{\substack{j=1 \\ j \neq k}}^{N} \sum_{k=1}^{N} E_j E_k^* \, e^{i(\phi_j - \phi_k)}$

If $\phi_j \approx \phi_k \ \forall \ j$ then the 2nd term >> 1st term as there are N^2 of them and results in coherent emission.

Resonant interaction – ***Even*** harmonics

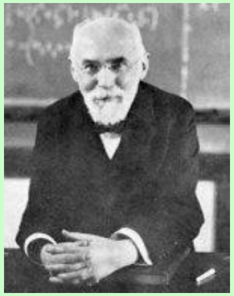


Even harmonics do not allow a slow exchange of energy

Electron bunching in a ***self-consistent*** radiation field: The FEL mechanism

Basic FEL mechanism

Radiation field bunches electrons



$$\underset{\sim}{F}_j = -|e|\left[\underset{\sim}{E} + \underset{\sim}{v}_j \times \underset{\sim}{B}\right]$$

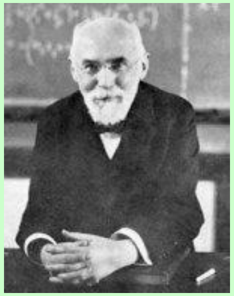
Bunched electrons drive radiation

$$\nabla^2 \underset{\sim}{E} - \frac{1}{c^2}\frac{\partial^2 \underset{\sim}{E}}{\partial t^2} = \mu_0 \frac{\partial \underset{\sim}{J}_\perp}{\partial t}$$

$$\underset{\sim}{J}_\perp \equiv -|e| \sum_{j=1}^{N} \underset{\sim}{v}_\perp \, \delta\left(\underset{\sim}{r} - \underset{\sim}{r}_j(t)\right)$$ The transverse current density

Basic FEL mechanism

Radiation field bunches electrons



$$\frac{d\theta_j}{d\bar{z}} = p_j$$

$$\frac{dp_j}{d\bar{z}} = -\left(Ae^{i\theta_j} + c.c.\right)$$

$A(\bar{z},\bar{t}) \propto E(\bar{z},\bar{t})\cancel{\sin(k_r z-\omega_r t+\phi)}$ – Radiation envelope

Bunched electrons drive radiation

$$\frac{\partial A}{\partial \bar{z}} + \frac{\partial A}{\partial \bar{t}} = b(\bar{z},\bar{t})$$

$$b(\bar{z},\bar{t}) = \frac{I(\bar{t})}{I_{pk}}\left(\frac{1}{N}\sum_{j=1}^{N} e^{-i\theta_j(\bar{z})}\right)\Bigg|_{\bar{t}}$$

$$\theta_j \equiv (k_r + k_u) z_j - \omega_r t$$

$$p_j \equiv \frac{\gamma_j - \gamma_r}{\rho\gamma_r}$$

$$\rho|A|^2 \equiv \frac{P_{rad}}{P_{beam}}$$

$$\rho = \frac{1}{\gamma_r}\left(\frac{\bar{a}_w \omega_p f_B}{4ck_w}\right)^{2/3}.$$

$$\omega_p = \left(e^2 n_{pk}/\epsilon_0 m\right)^{1/2}$$

$$f_B = J_0(\zeta) - J_1(\zeta)$$

$$\zeta = \bar{a}_w^2/2(1+\bar{a}_w^2)$$

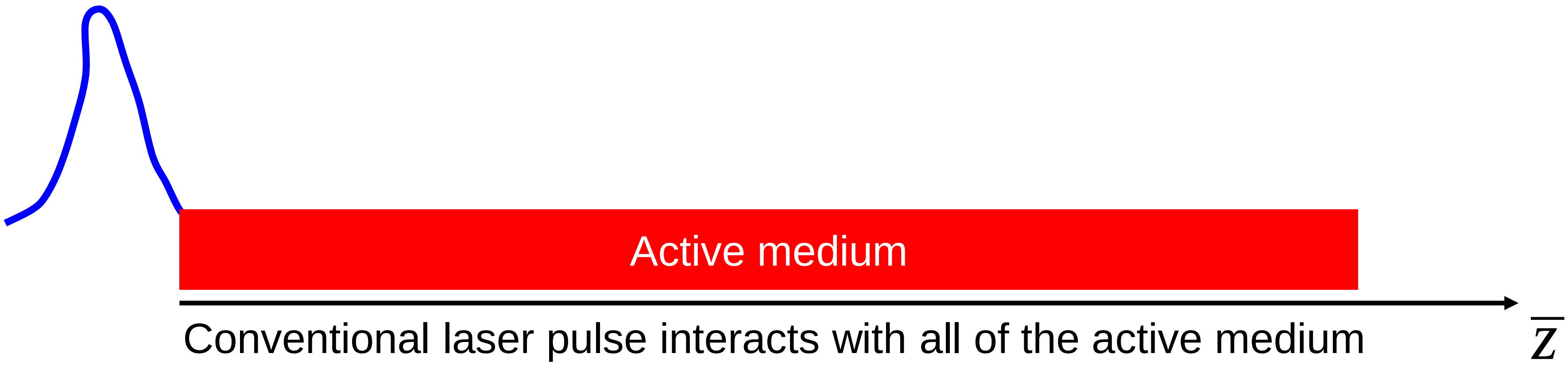
$$\bar{z} = \frac{z}{l_g}, \quad \bar{t} = \frac{z - c\bar{\beta}_z t}{l_c}$$

$$l_g = \lambda_w/4\pi\rho$$

$$l_c = \lambda_r/4\pi\rho$$

These equations are assumed ‘slowly varying’ i.e. any evolution is assumed slow with respect to the radiation/undulator period. They can be subsequently averaged over a radiation/undulator period.

Conventional laser Vs FEL pulses



Linear analysis

$$\frac{d\theta_j}{d\bar{z}} = p_j$$

$$\frac{dp_j}{d\bar{z}} = -\left(Ae^{i\theta_j} + c.c.\right)$$

$$\frac{\partial A}{\partial \bar{z}} + \cancel{\frac{\partial A}{\partial \bar{t}}} = b\left(\bar{z}, \cancel{\bar{t}}\right)$$

Steady-state approx.: “No pulses”

Assume that:

$$A_0 \ll 1,$$

$$p_{j_0} = \delta$$

$\theta_j = \theta_{0j} + \theta_{1j}$ etc. where: $\theta_{1j} \ll 1$

$$\left\langle e^{-i\theta_0} \right\rangle = 0; \quad \theta_{0j} = U\left(0, 2\pi\right]$$

Using: $e^x \approx 1 + x + \frac{x^2}{2} + \ldots$

$$\Rightarrow e^{i\left(\theta_{0j}+\theta_{1j}\right)} = e^{i\theta_{0j}} e^{i\theta_{1j}} \approx e^{i\theta_{0j}} \left(1 + i\theta_{1j}\right)$$

$$\left\langle x \right\rangle \equiv \frac{1}{N} \sum_{j=1}^{N} x_j$$

$$\Rightarrow \quad \frac{db}{d\bar{z}} = -iP - i\delta b$$

$$\frac{dP}{d\bar{z}} = -A + i\delta P$$

$$\frac{dA}{d\bar{z}} = b$$

Where:

$$b \equiv \left\langle -i\theta_1 e^{-i\theta_0} \right\rangle$$

$$P \equiv \left\langle p_1 e^{-i\theta_0} \right\rangle$$

The steady-state approximation can be thought of as the continuous e$^-$ beam limit where the electron ‘pulse’ has no beginning or end. In this case one can see that the radiation field can only be a function of the distance through the undulator and no pulse effects can be present.

Linear analysis

First assume resonance: $\delta = 0$

Differentiating linear equations:

$$\frac{d^2A}{d\bar{z}^2} = \frac{db}{d\bar{z}} = -iP$$

$$\Rightarrow \frac{d^3A}{d\bar{z}^3} = -i\frac{dP}{d\bar{z}} = iA$$

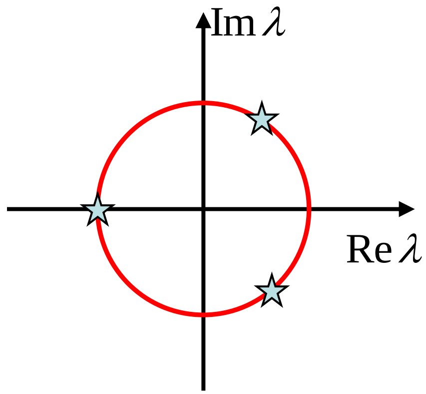
Look for solutions:

$$A(\bar{z}) = A_0 e^{i\lambda\bar{z}}$$

$$\Rightarrow -i\lambda^3 = i$$

$$\Rightarrow \lambda^3 = -1$$

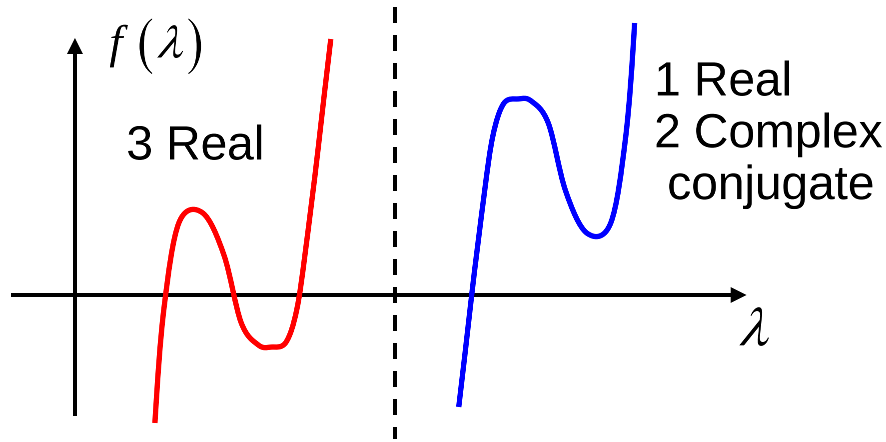
$$\Rightarrow \lambda = -1; \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right); \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$



Away from resonance: $\delta \neq 0$

the dispersion relation is:

$$f(\lambda) = \lambda^3 - \delta\lambda^2 + 1 = 0$$



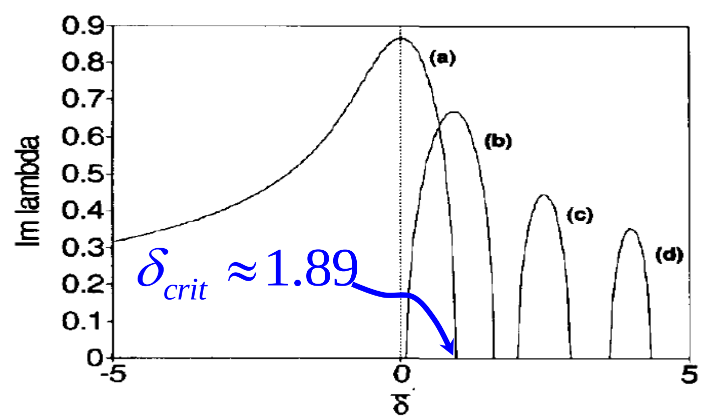


Fig. 3. Im λ as a function of δ for (a) $\sigma_\epsilon = 0$; (b) $\sigma_\epsilon = 2.0$; (c) $\sigma_\epsilon = 5.0$; (d) $\sigma_\epsilon = 8.0$.

Linear analysis

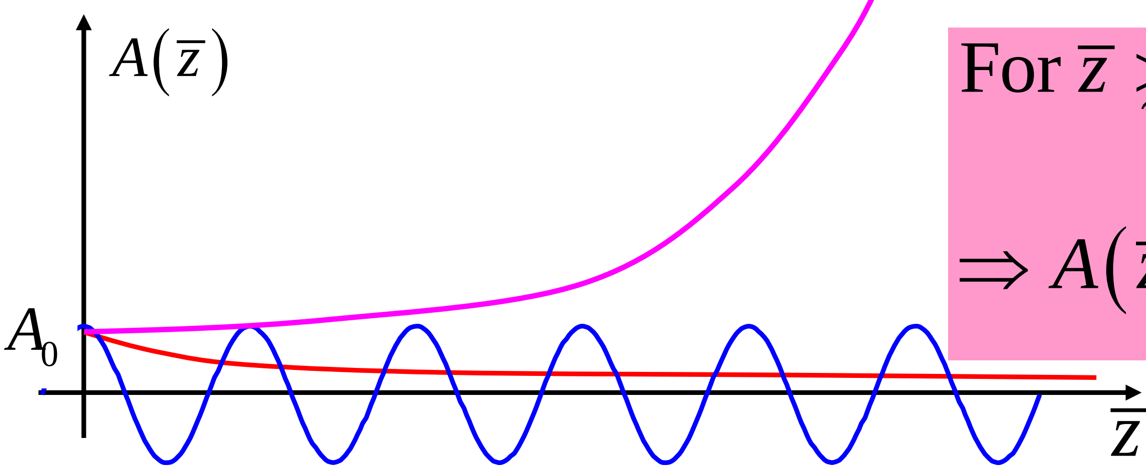
Solutions for $\delta = 0$:

$$A(\bar{z}) = \frac{A_0}{3}\sum_j c_j e^{i\lambda_j \bar{z}} \quad \text{for} \quad \lambda_j = \left[-1;\ \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right);\ \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\right]$$

Real parts give oscillatory solutions.

Imaginary parts give exponential growth: $\left(-i\frac{\sqrt{3}}{2}\right)$

and exponential decay: $\left(+i\frac{\sqrt{3}}{2}\right)$

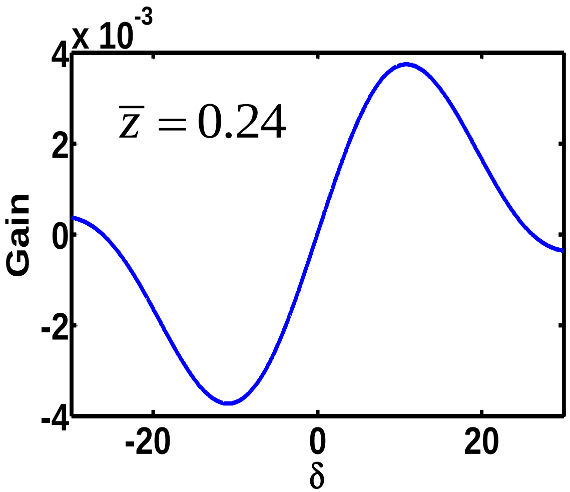


For $\bar{z} \gtrsim 1$ $\quad l_g = \lambda_w / 4\pi\rho$

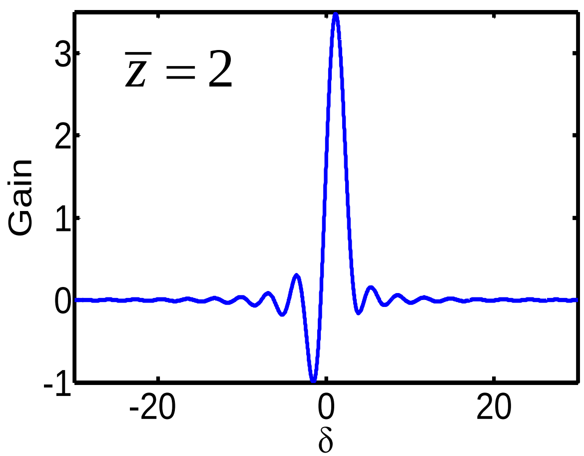
$$\Rightarrow A(\bar{z}) \approx \frac{A_0}{3} e^{\frac{\sqrt{3}}{2}\bar{z}} = \frac{A_0}{3} e^{\frac{\sqrt{3}}{2}\frac{z}{l_g}}$$

Gain as a function of detuning from resonance

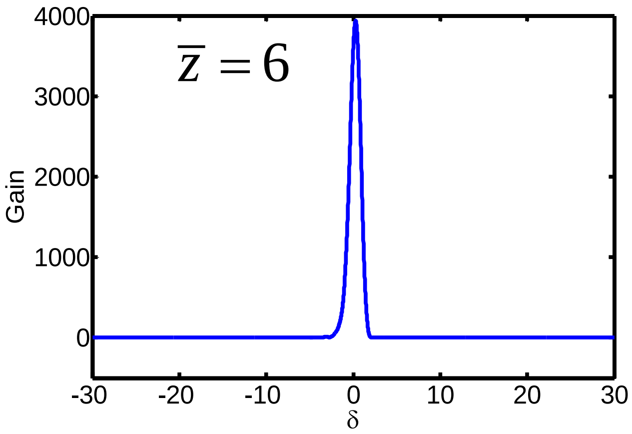
$$\text{Gain} = \frac{\left|A(\bar{z})\right|^2 - \left|A_0\right|^2}{\left|A_0\right|^2}$$



Oscillatory terms dominate



Oscillatory & exponential



+ve exponential term dominates

$$\delta \equiv p_{j0} = \frac{\gamma_{j0} - \gamma_r}{\rho \gamma_r}$$

Constants of motion

$$\frac{d\theta_j}{d\bar{z}} = p_j$$

$$\frac{dp_j}{d\bar{z}} = -\left(Ae^{i\theta_j} + c.c.\right)$$

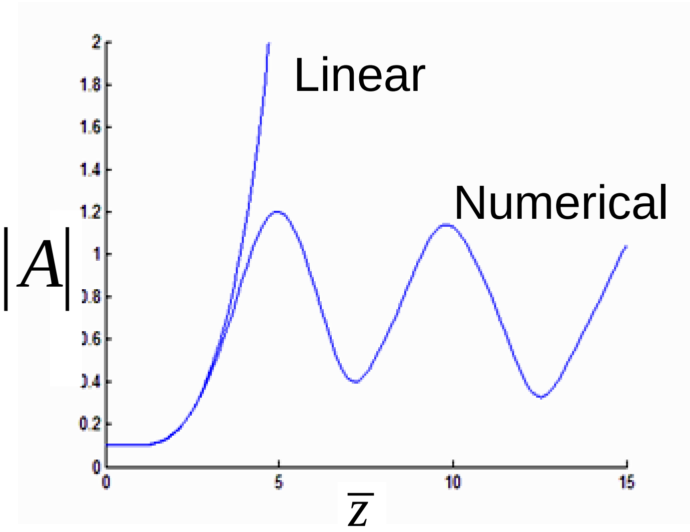
$$\frac{dA}{d\bar{z}} = b\left(\bar{z}, \bar{t}\right)$$

Two constants of motion can be obtained from these equations in the steady-state limit:

$$|A|^2 + \langle p \rangle = \text{ constant}$$

$$\frac{\langle p^2 \rangle}{2} + i\left(A^* b - Ab^*\right) - \delta |A|^2 = \text{ constant}$$

Where the constant is the variables' initial values.

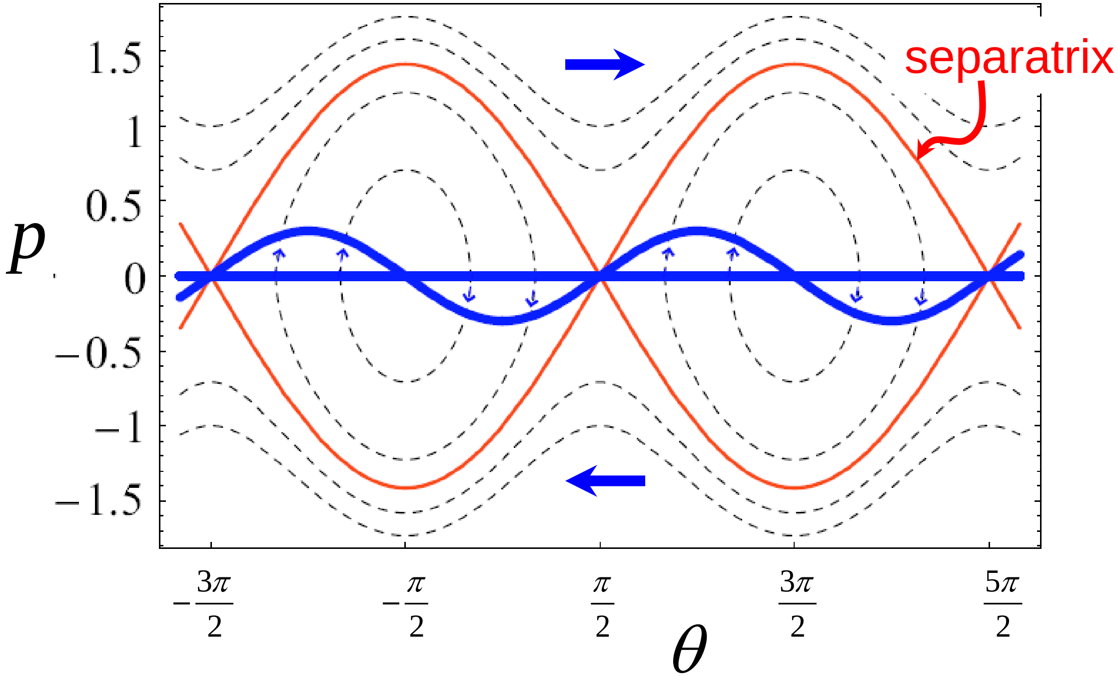


The first constant above corresponds to conservation of energy. The second, incorporating phase dependent terms is related to the Hamiltonian of the system. Opposite is plotted the linear and non-linear (numerical) solutions of the equations for a resonant interaction ($\delta = 0$). From the definition of :

$$\rho |A|^2 \equiv \frac{P_{rad}}{P_{beam}}$$

and the saturated scaled field $|A_{sat}|$~1, it is seen that ρ is a measure of the efficiency if the interaction.

The pendulum equation and phase-space



$$\frac{d\theta_j}{d\bar{z}} = p_j$$

$$\frac{dp_j}{d\bar{z}} = -2a\cos\left(\theta_j + \phi\right)$$

$$\frac{da}{d\bar{z}} = \left\langle \cos\left(\theta + \phi\right)\right\rangle$$

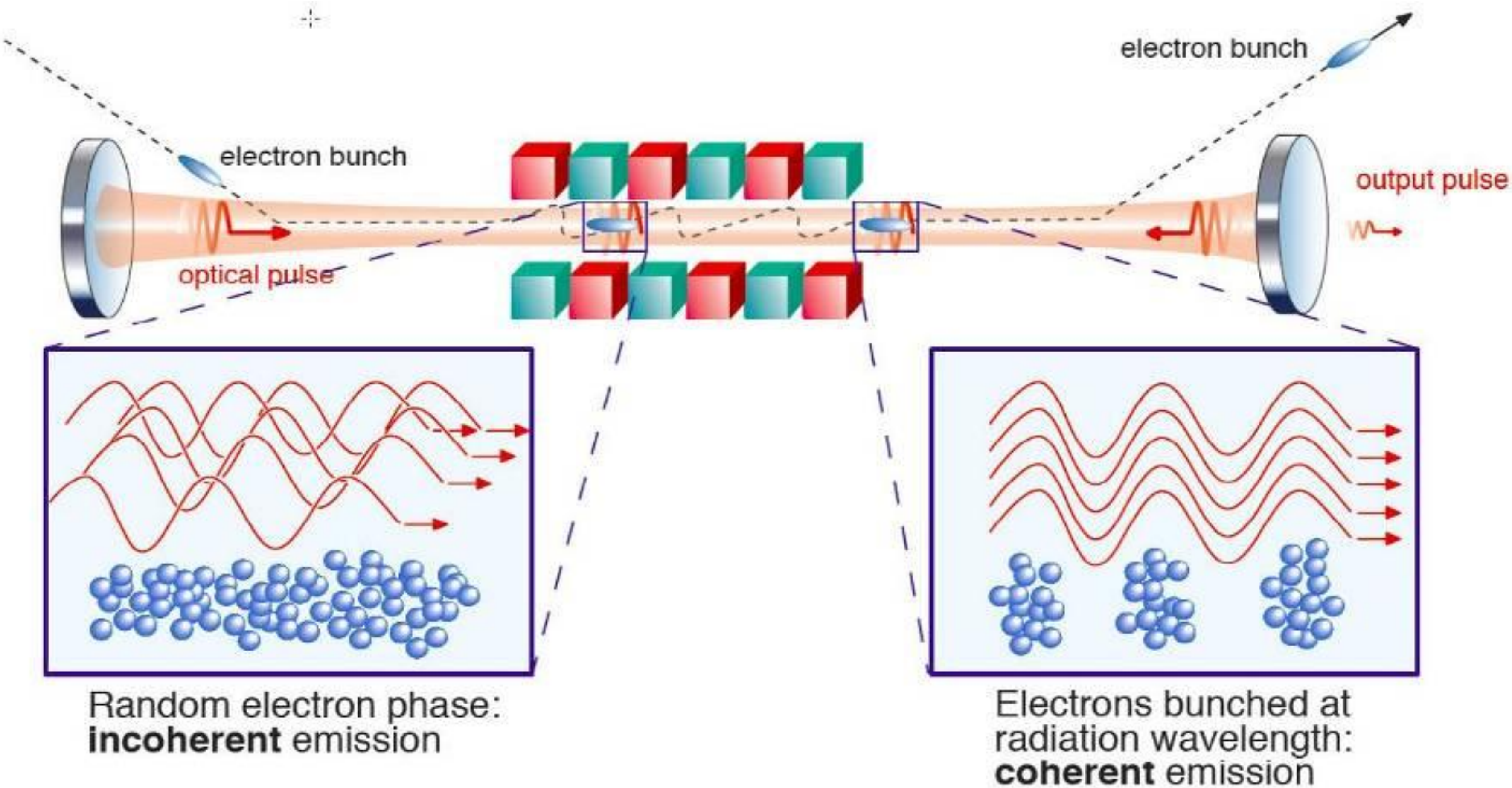
$$\frac{d\phi}{d\bar{z}} = -\frac{1}{a}\left\langle \sin\left(\theta + \phi\right)\right\rangle$$

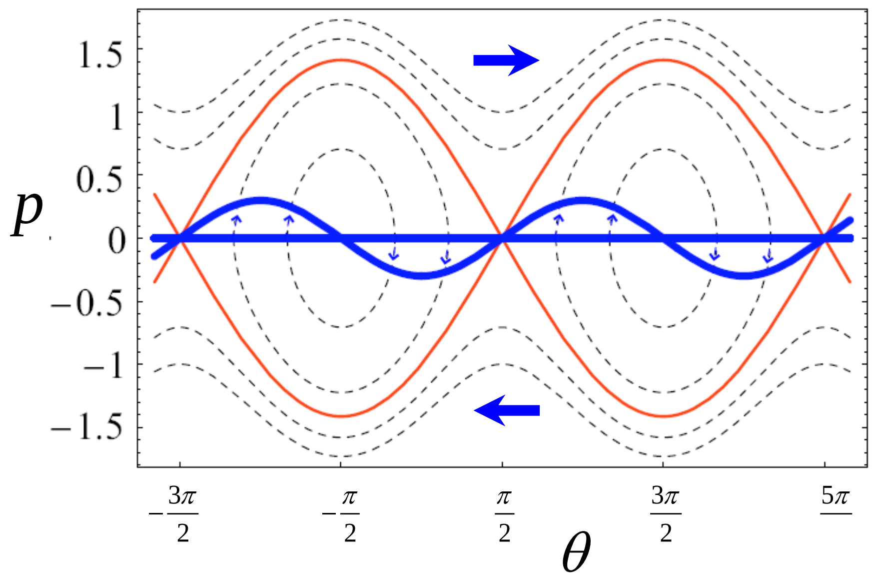
The electrons can be thought of as a collection of pendula initially distributed over a range of angles with respect to the vertical. The radiation field is analogous to the gravitational field. The separartrix defines the boundary between pendula that librate and rotate. Of course in the FEL equations above, unlike a gravitational field, the radiation field can evolve in both amplitude a, and phase ϕ.

Low Gain mechanism

$$\bar{z} \lesssim 1$$

Low Gain – needs cavity feedback





Here we consider the low gain limit with $\overline{z} = 0.24$

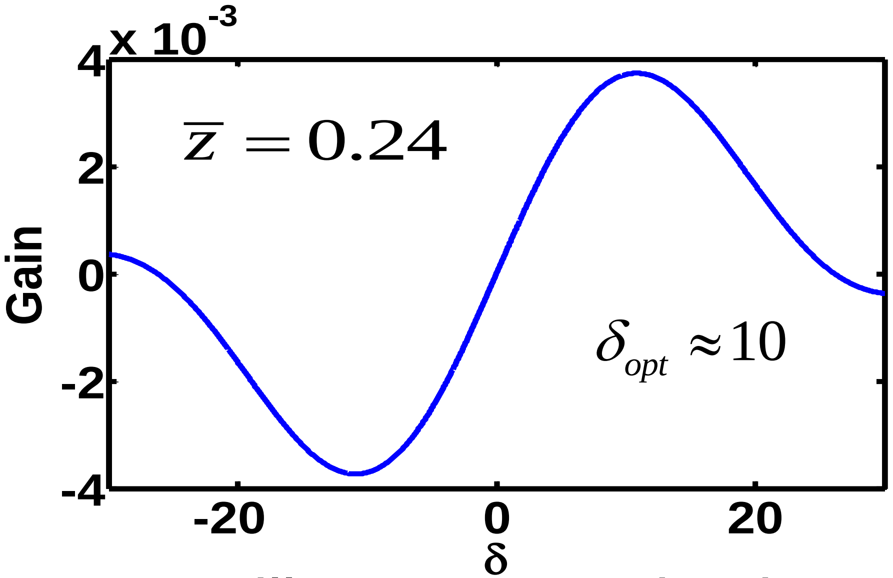
The phase-space representation of opposite will be used to look at what happens with the electrons and radiation during the interaction.

$$\text{Gain} = 4\overline{z}^3 \frac{1}{\Delta^3}\left(1 - \cos(\Delta) - \frac{\Delta}{2}\sin(\Delta)\right)$$

where $\Delta = \delta\overline{z}$

The Gain is a maximum for $\Delta \approx 2.6$

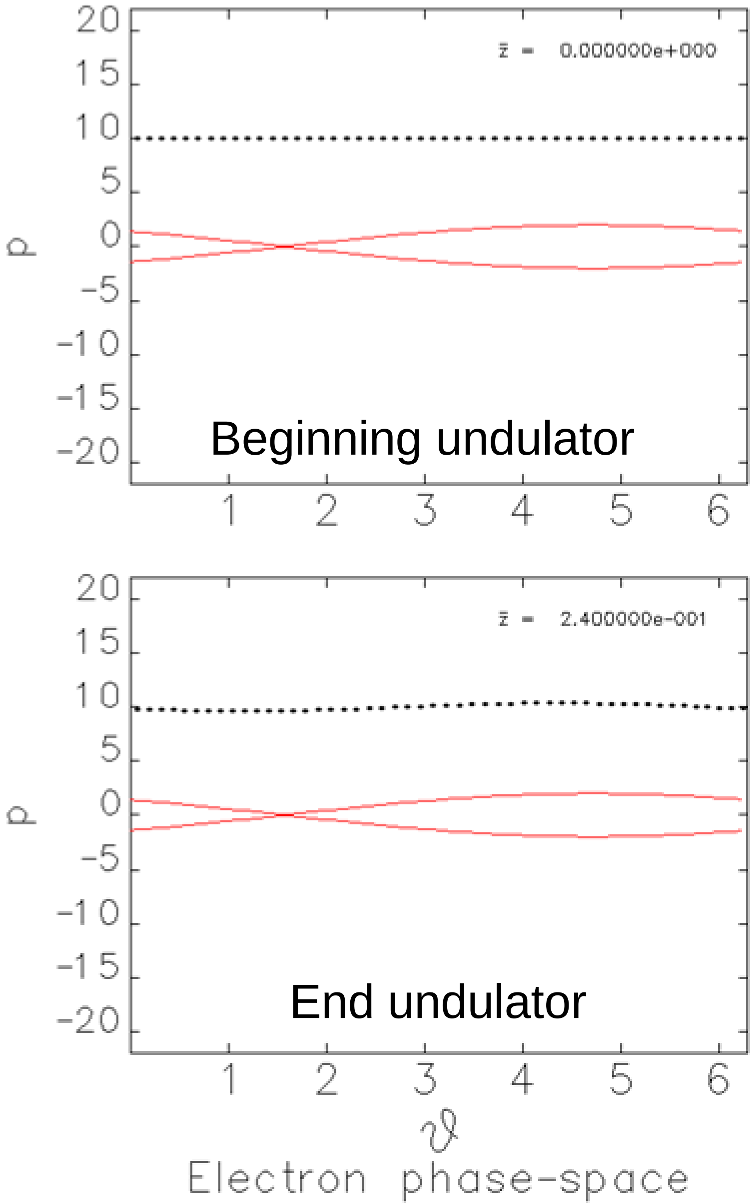
$\Rightarrow \quad \delta_{opt} \approx \dfrac{2.6}{\overline{z}}$ in the low-gain limit.

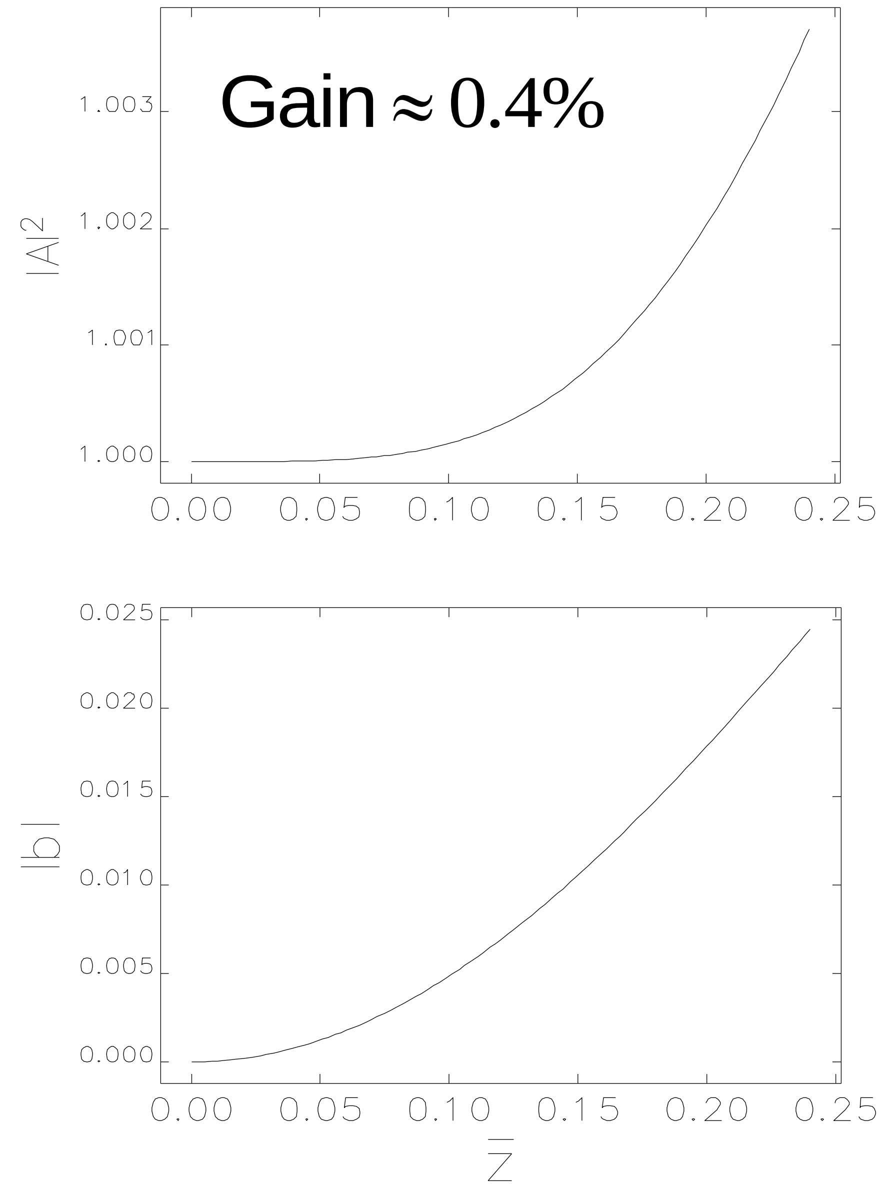


Oscillatory terms dominate

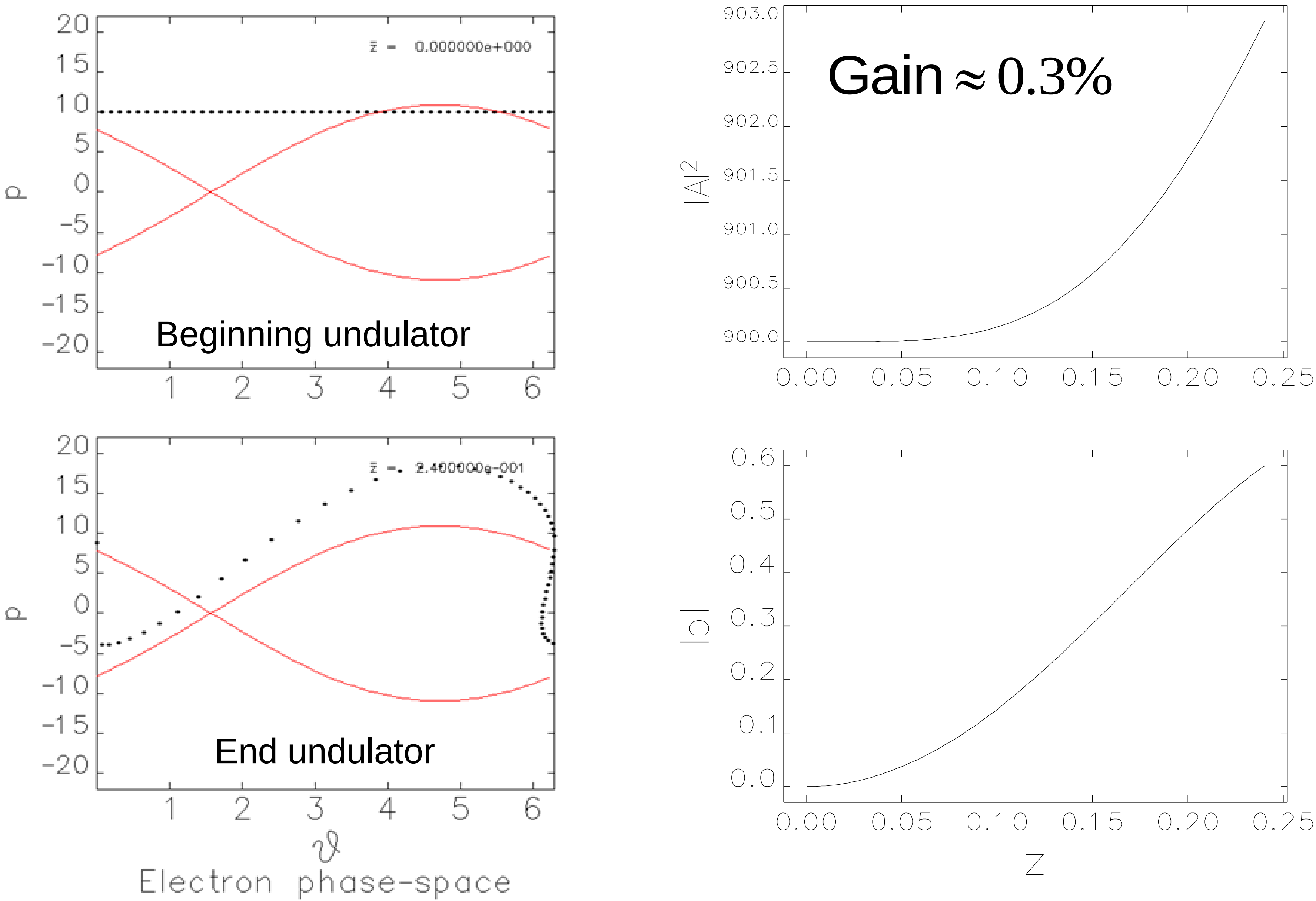
FEL saturates when Gain=Cavity Losses

Low Gain – early stages

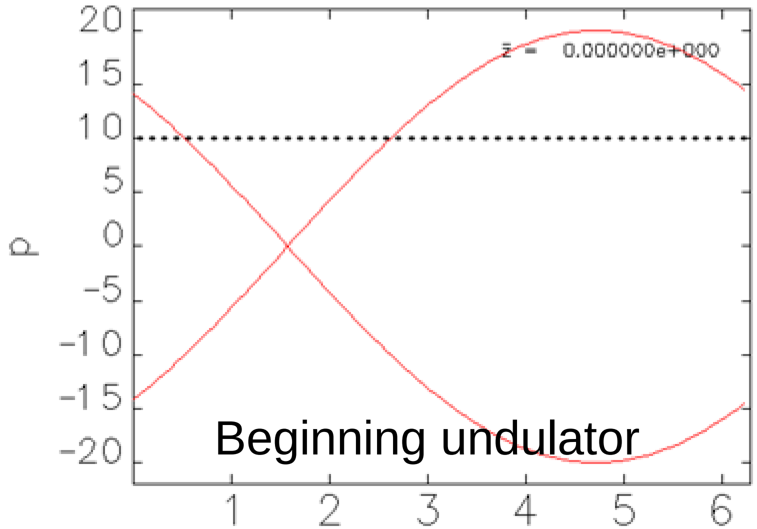


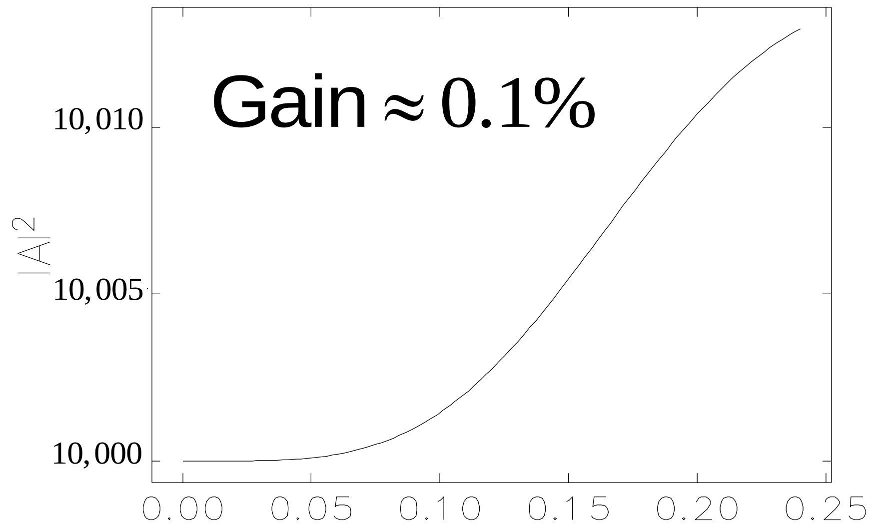


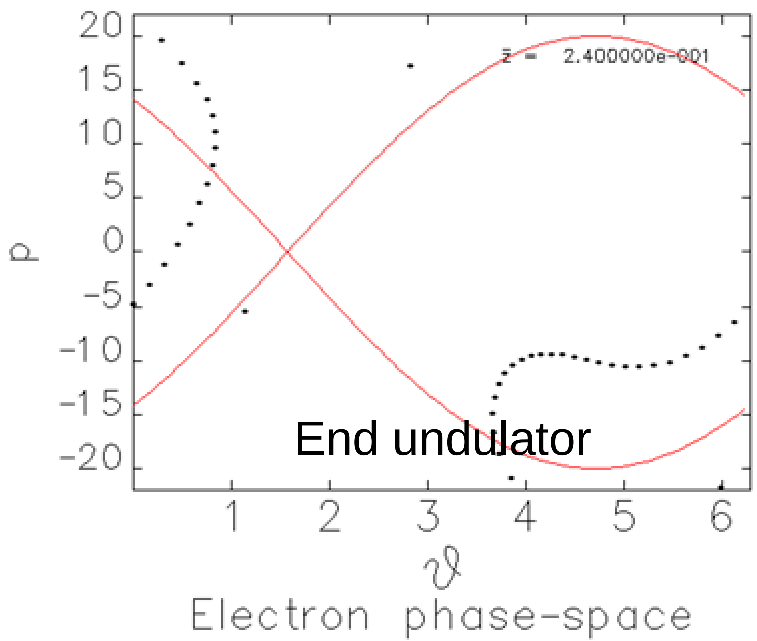
Low Gain – intermediate

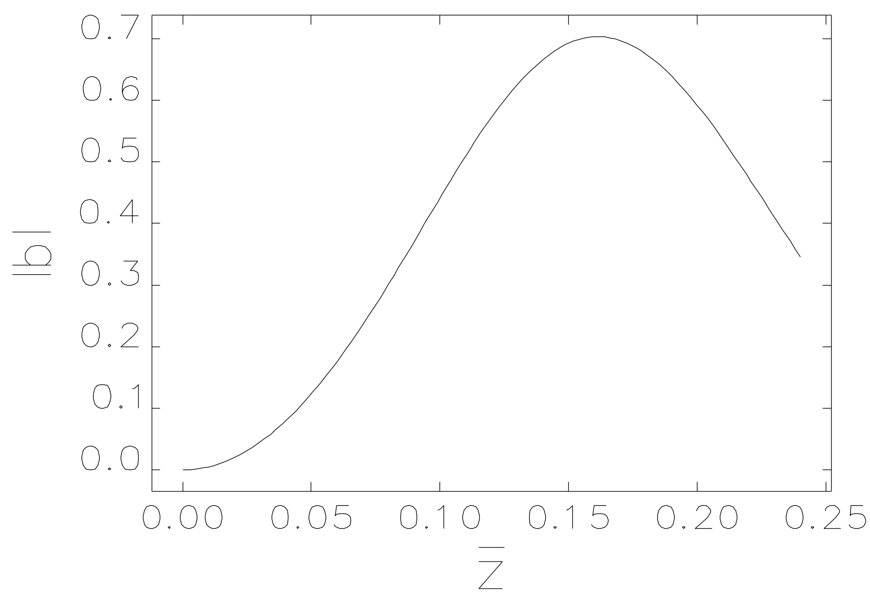


Low Gain – saturated





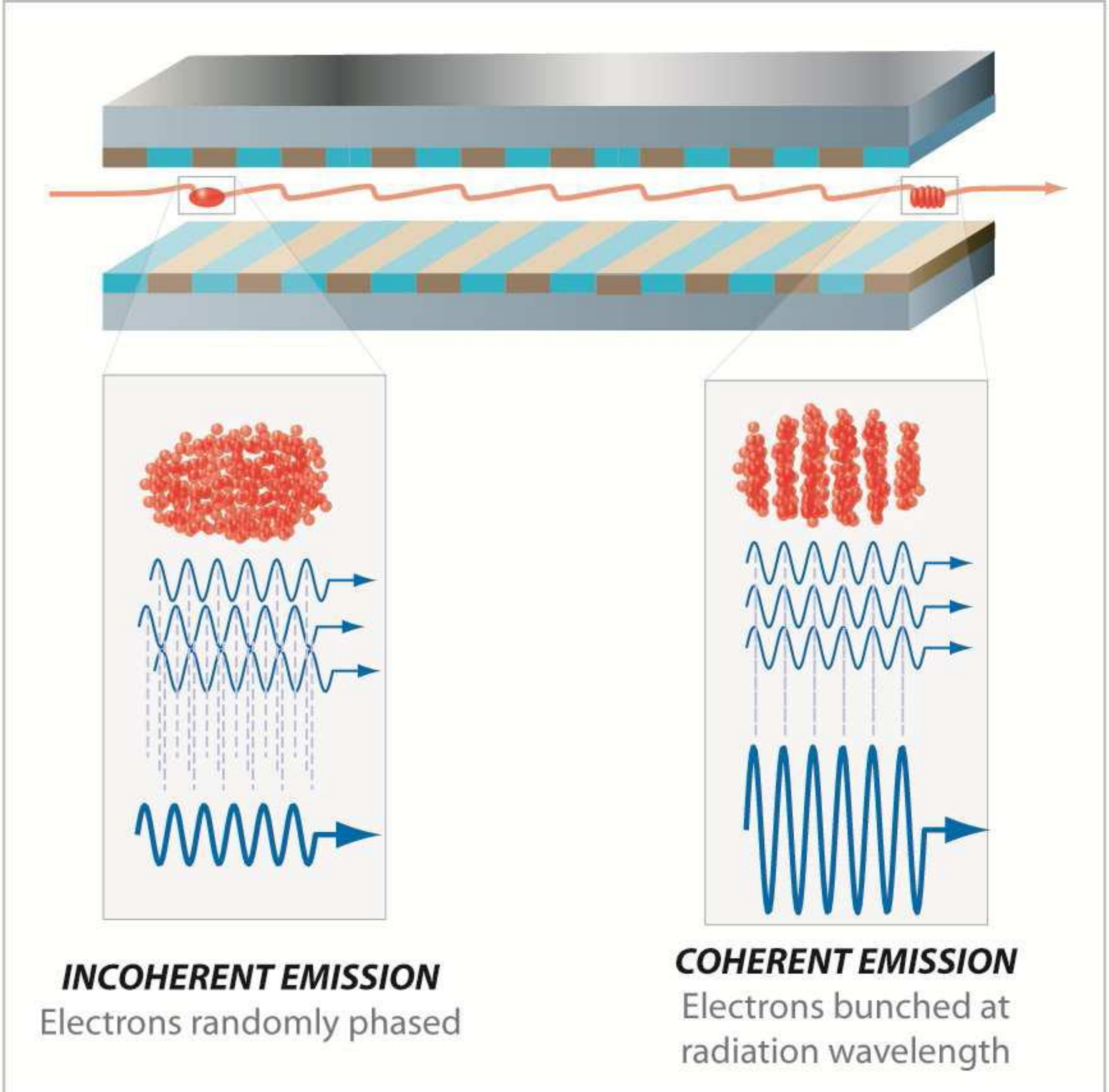


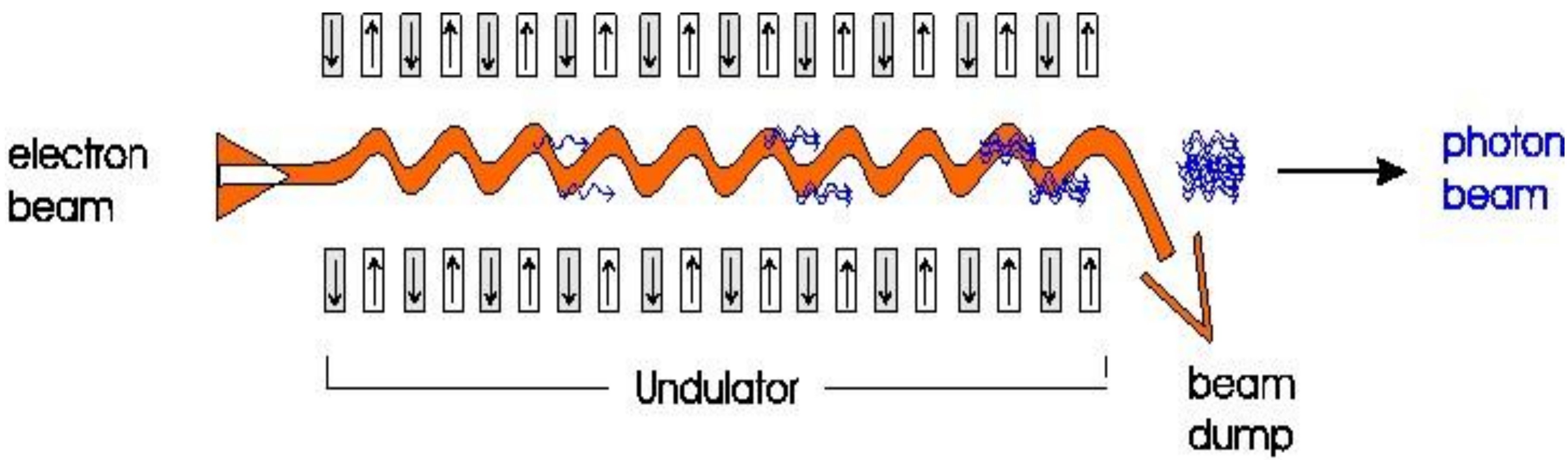


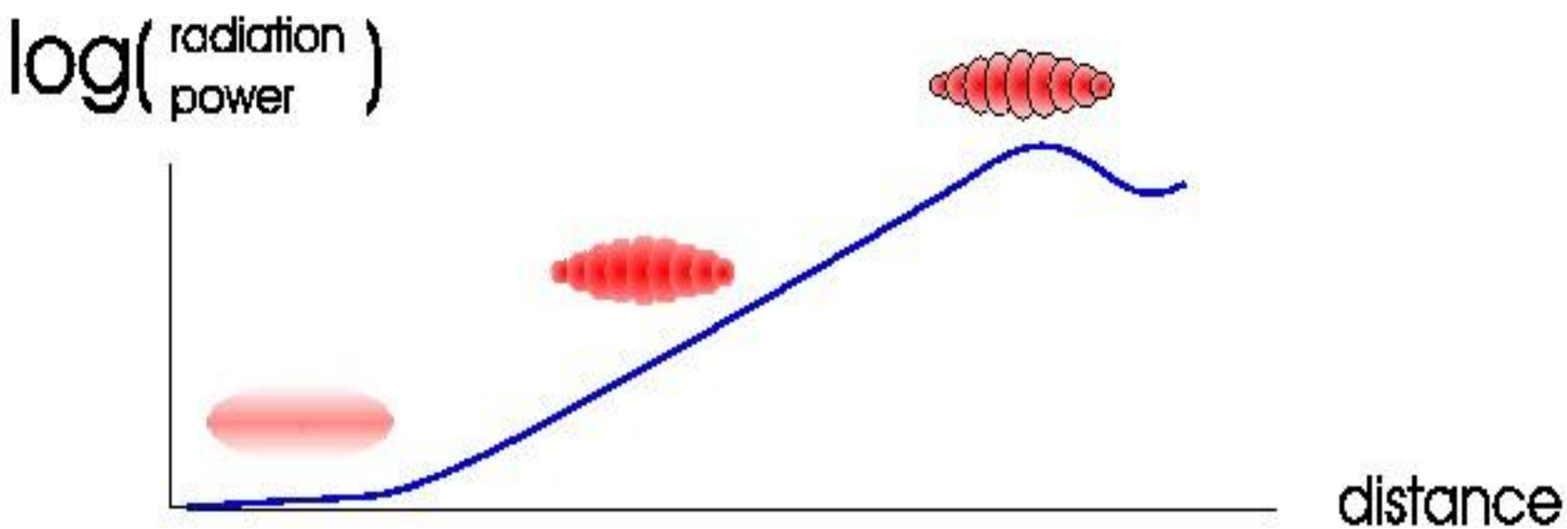
High Gain mechanism

$$\bar{z} > 1$$

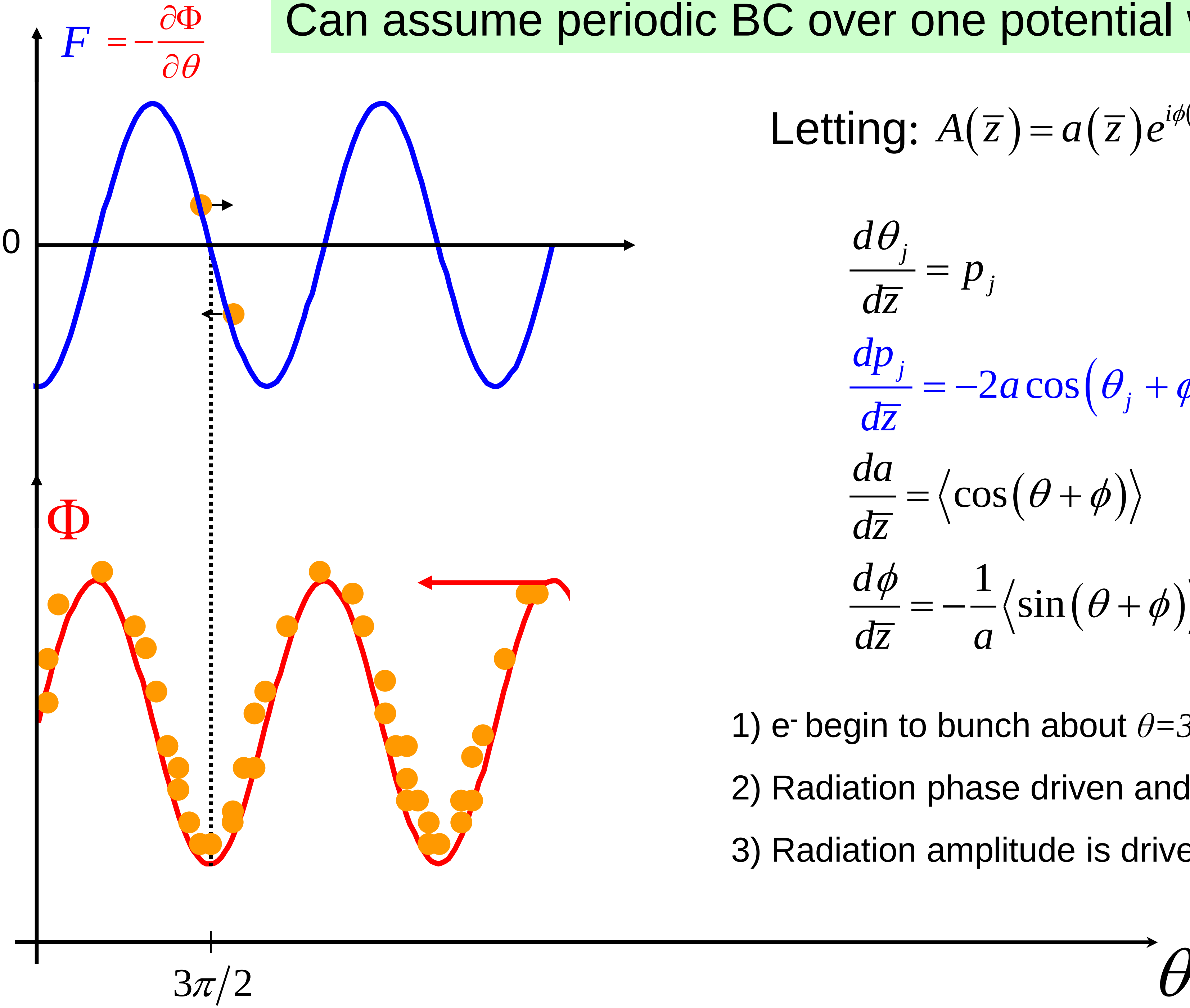
$$A(\bar{z}) \approx \frac{A_0}{3} e^{\frac{\sqrt{3}}{2}\frac{z}{l_g}}$$


INCOHERENT EMISSION
Electrons randomly phased
COHERENT EMISSION
Electrons bunched at
radiation wavelength


electron
beam
photon
beam
Undulator
beam
dump


log(radiation power)
distance

Can assume periodic BC over one potential well:



Letting: $A(\bar{z}) = a(\bar{z})e^{i\phi(\bar{z})} \Rightarrow$

$$\frac{d\theta_j}{d\bar{z}} = p_j$$

$$\frac{dp_j}{d\bar{z}} = -2a\cos(\theta_j + \phi)$$

$$\frac{da}{d\bar{z}} = \langle \cos(\theta + \phi) \rangle$$

$$\frac{d\phi}{d\bar{z}} = -\frac{1}{a}\langle \sin(\theta + \phi) \rangle$$

1) e^- begin to bunch about $\theta = 3\pi/2$

2) Radiation phase driven and shifts

3) Radiation amplitude is driven

Can assume periodic BC over one potential well:

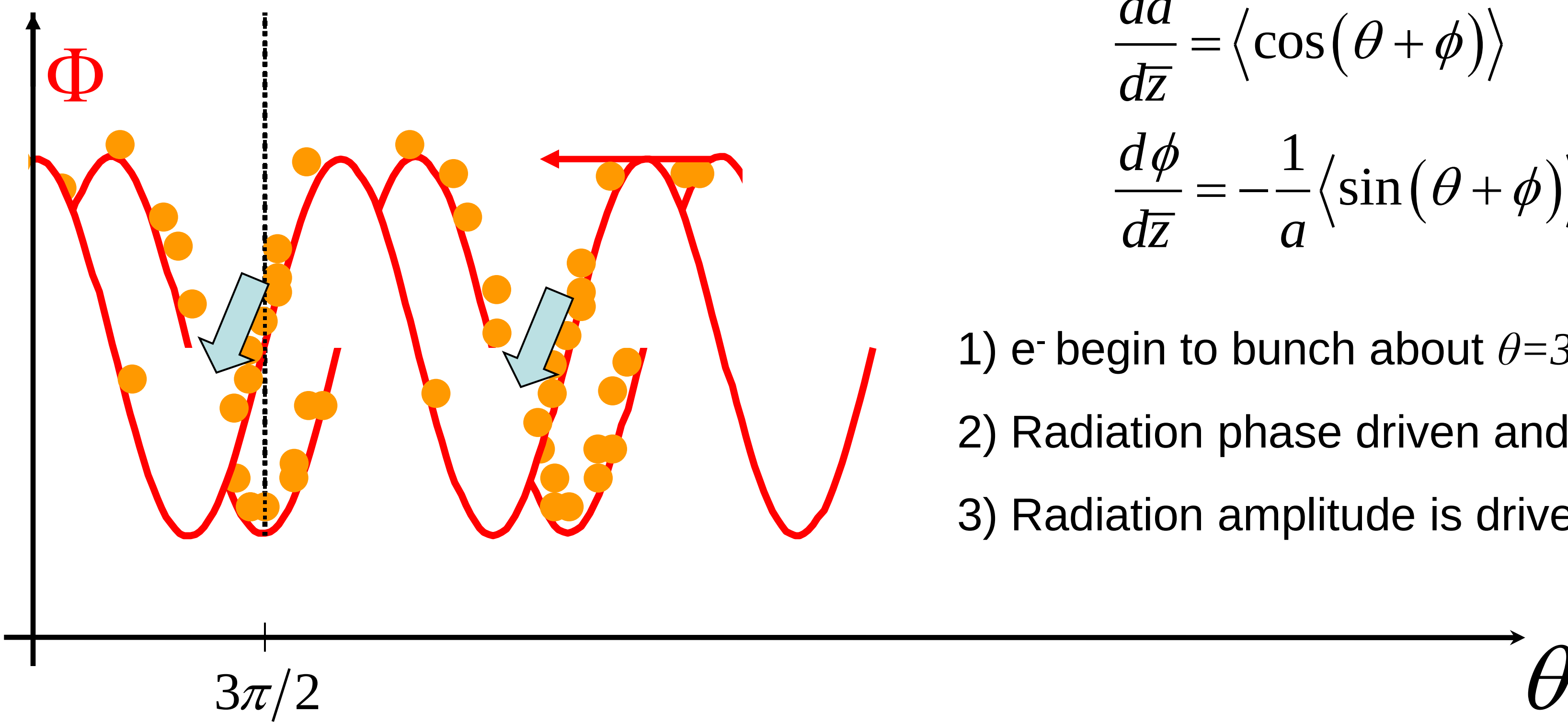
Letting: $A(\bar{z}) = a(\bar{z})e^{i\phi(\bar{z})} \Rightarrow$

$$\frac{d\theta_j}{d\bar{z}} = p_j$$

$$\frac{dp_j}{d\bar{z}} = -2a\cos(\theta_j + \phi)$$

$$\frac{da}{d\bar{z}} = \langle \cos(\theta + \phi) \rangle$$

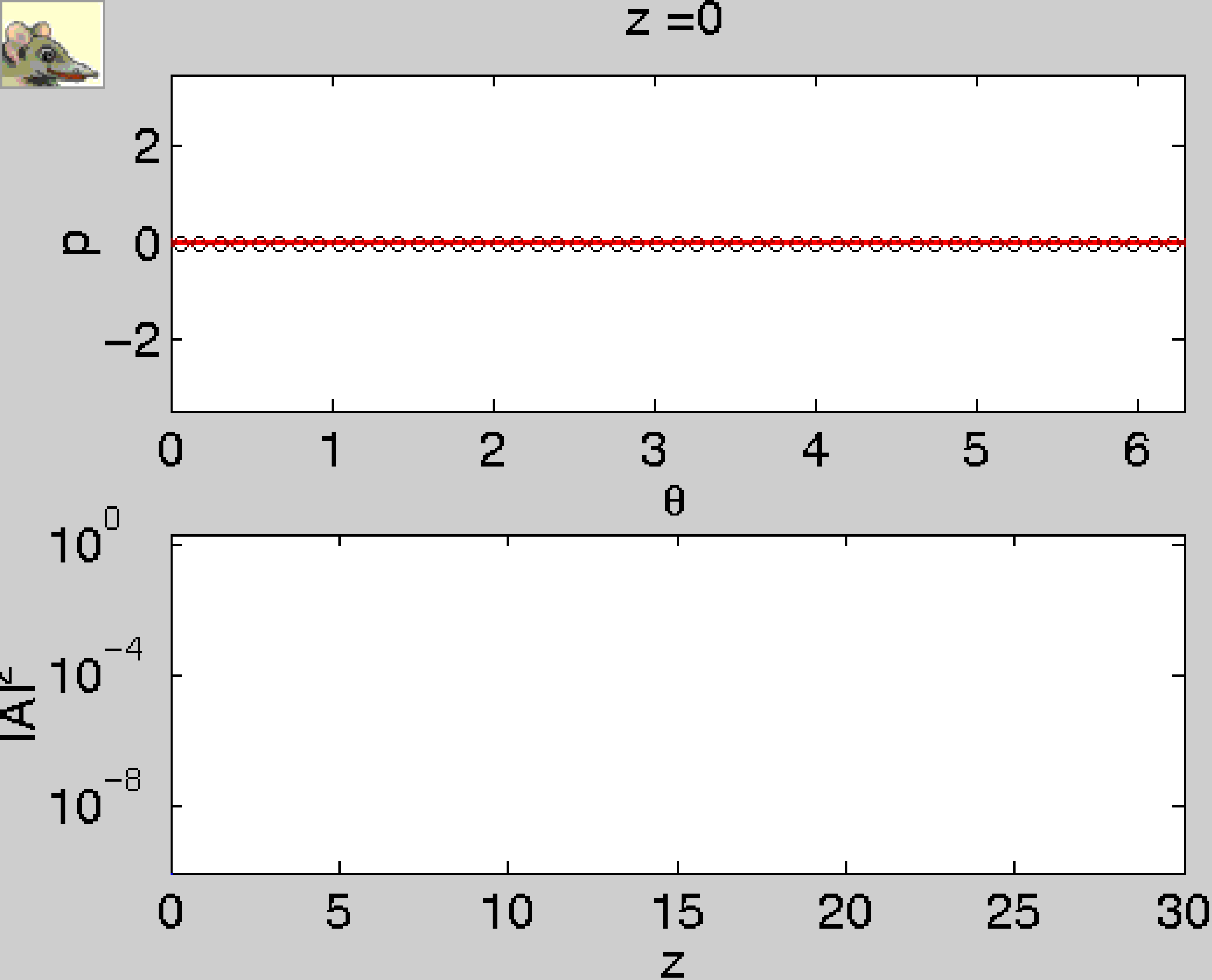
$$\frac{d\phi}{d\bar{z}} = -\frac{1}{a}\langle \sin(\theta + \phi) \rangle$$



1) e^- begin to bunch about $\theta = 3\pi/2$

2) Radiation phase driven and shifts

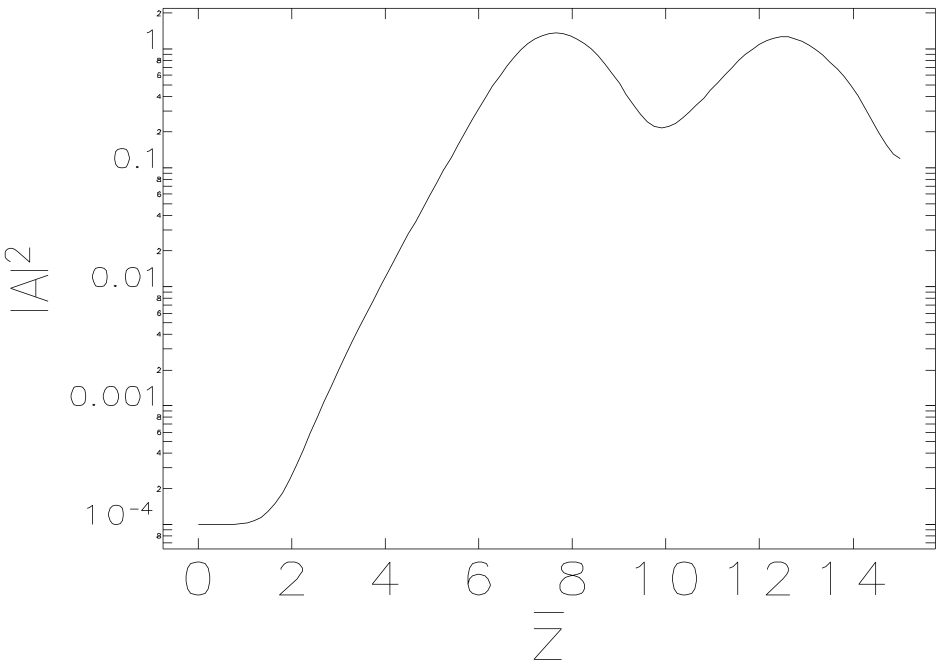
3) Radiation amplitude is driven


z =0
p
θ
|A|²
z

Some resources

If you want to download some Fortran and Matlab codes that solve the FEL equations and plot their solutions you can obtain them from:

http://phys.strath.ac.uk/eurofel/rebs/rebs.htm



Scaled intensity

Because the equations are universally scaled (depend only on initial conditions) and because $|A_{sat}|^2 \approx 1$, can see from scaling of A that the rho parameter is a measure of the efficiency of the high-gain FEL amplifier:

$$\rho |A|^2 \equiv \frac{P_{rad}}{P_{beam}} \quad \Rightarrow \quad \rho \approx \frac{P_{rad}^{(sat)}}{P_{beam}}$$

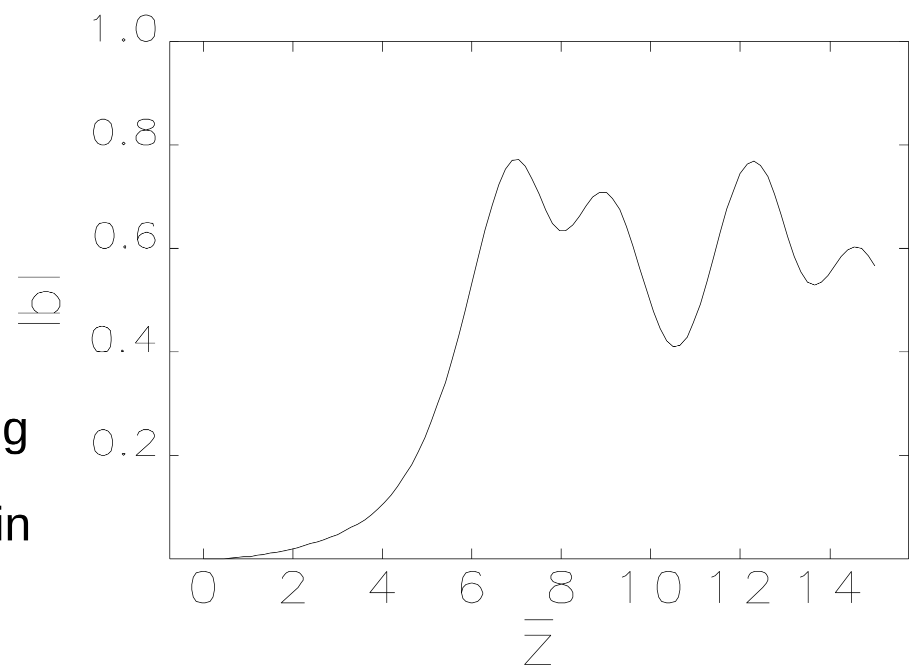
Also see that the saturated radiated power:

$$P_{rad}^{(sat)} \approx \rho P_{beam}$$

As: $P_{beam} \propto I,$ and: $\rho \propto I^{1/3},$

see that: $P_{rad}^{(sat)} \propto I^{4/3},$ demonstrating

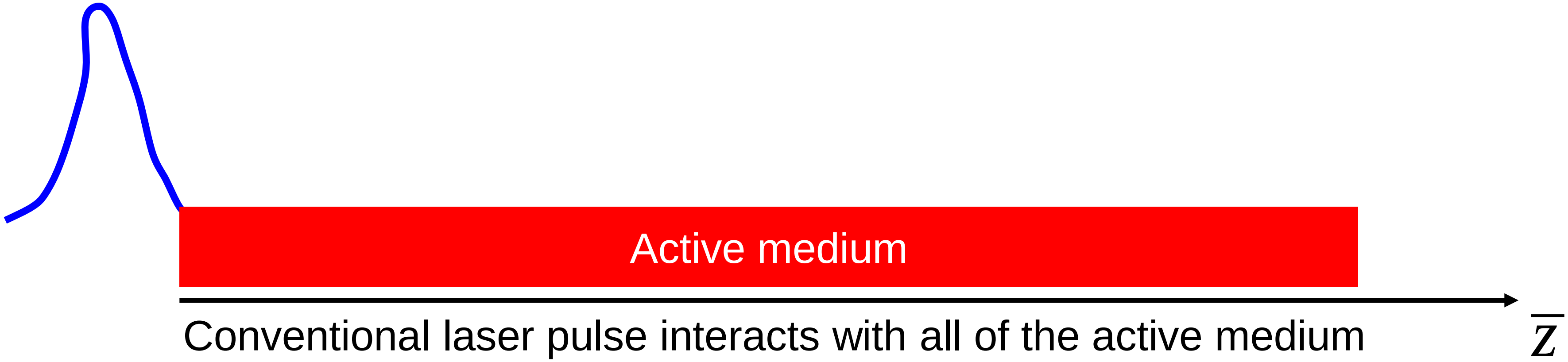
that, in the steady-state, the high-gain FEL interaction is a ***cooperative*** or ***collective*** interaction.



Bunching parameter

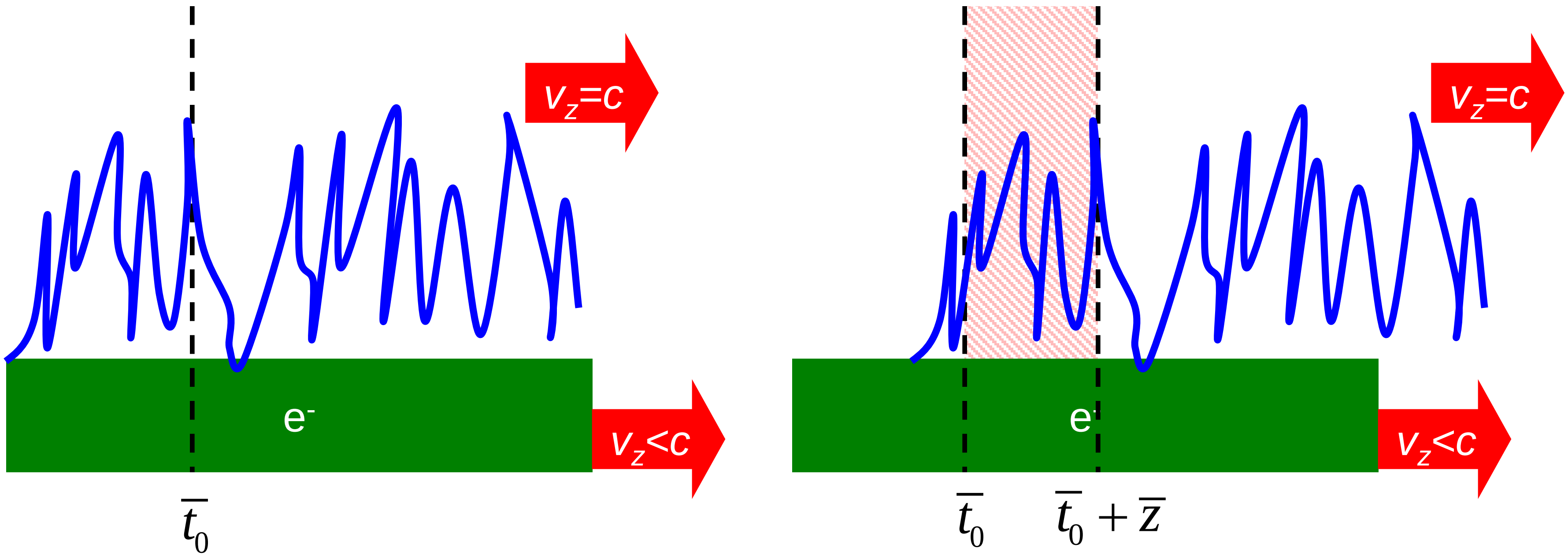
Pulse effects in the high gain

Conventional laser Vs FEL pulses



An important parameter that defines how the FEL evolves within the pulses is the cooperation length $l_c = \lambda / 4\pi\rho$ - the relative electron/radiation slippage in a gain length l_g

FEL pulses starting from ***noise*** in a High-Gain amplifier (SASE)



(Note: in the scaled variables a radiation wavefront propagates a distance $\bar{z}$ with respect to the electron rest frame.)

Regions of radiation pulse separated by $\bar{t} \sim 2\pi l_c$ evolve independently of other regions. Hence there can be many regions that evolve independently from different initial source terms due to noise* $b(\bar{z}=0,\bar{t}) \neq \text{ constant}$. This leads to a noisy temporal and spectral radiation pulse.

VOLUME 73, NUMBER 1 PHYSICAL REVIEW LETTERS 4 JULY 1994

Spectrum, Temporal Structure, and Fluctuations in a High-Gain Free-Electron Laser Starting from Noise

R. Bonifacio,[1,2] L. De Salvo,[1] P. Pierini,[2] N. Piovella,[1] and C. Pellegrini[3]

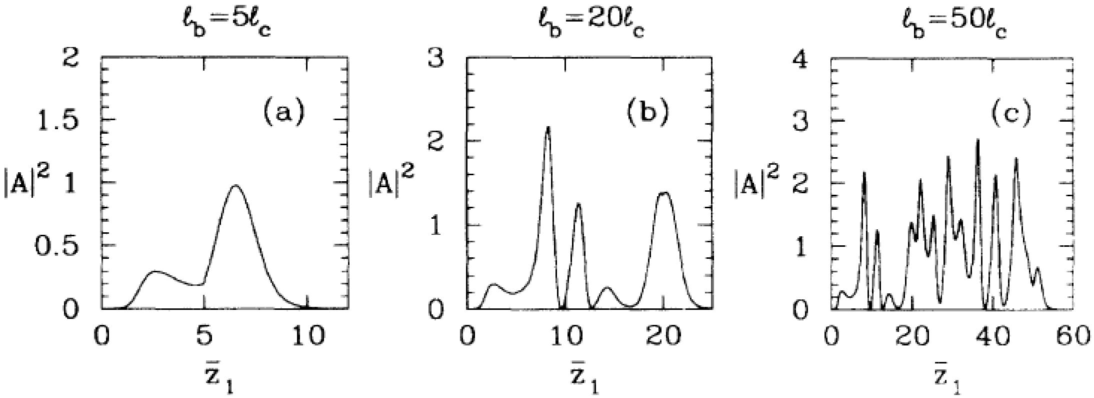


FIG. 1. Results of the numerical model: temporal structure of the radiated pulse, $|A|^2$ vs $\bar{z}_1$, at the first saturation, for three values of the electron bunch length, at $z = 14\ell_g$ and for $\langle|b_0|^2\rangle = 10^{-6}$: (a) $\ell_b = 5\ell_c$, (b) $\ell_b = 20\ell_c$, and (c) $\ell_b = 50\ell_c$. The temporal scale is in units of $z_1 = (z - v_{\parallel}t)/\ell_c$.

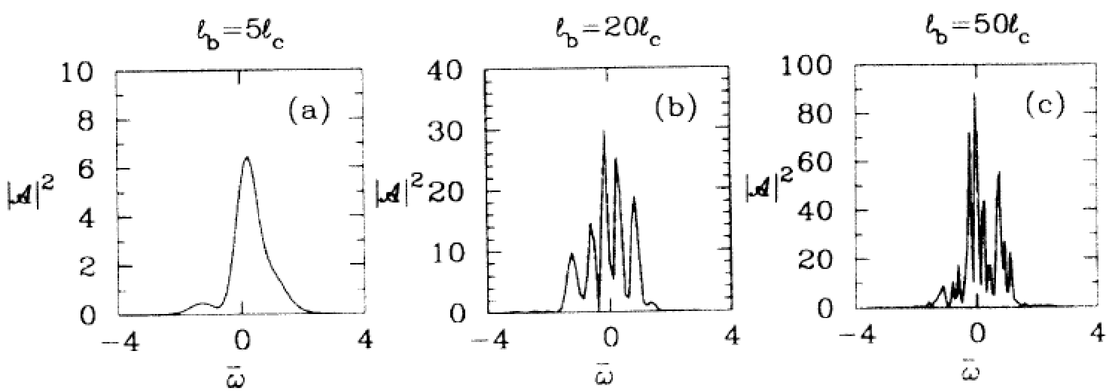
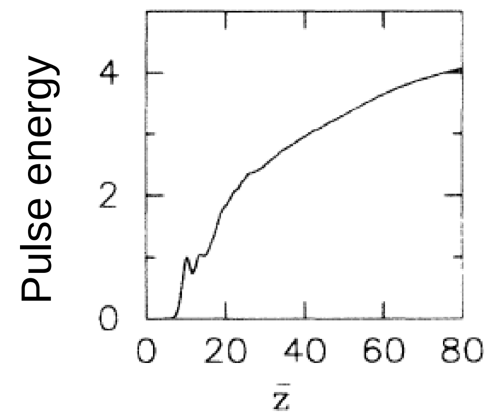
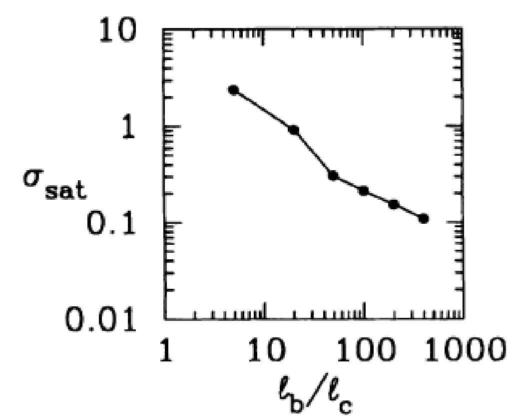


FIG. 3. Spectrum of the radiated pulses, for the same cases

The spike widths are $\lesssim 2\pi l_c$





Saturation length – shot-to-shot fluctuations.

Self Amplified Spontaneous Emission (SASE) in the x-ray

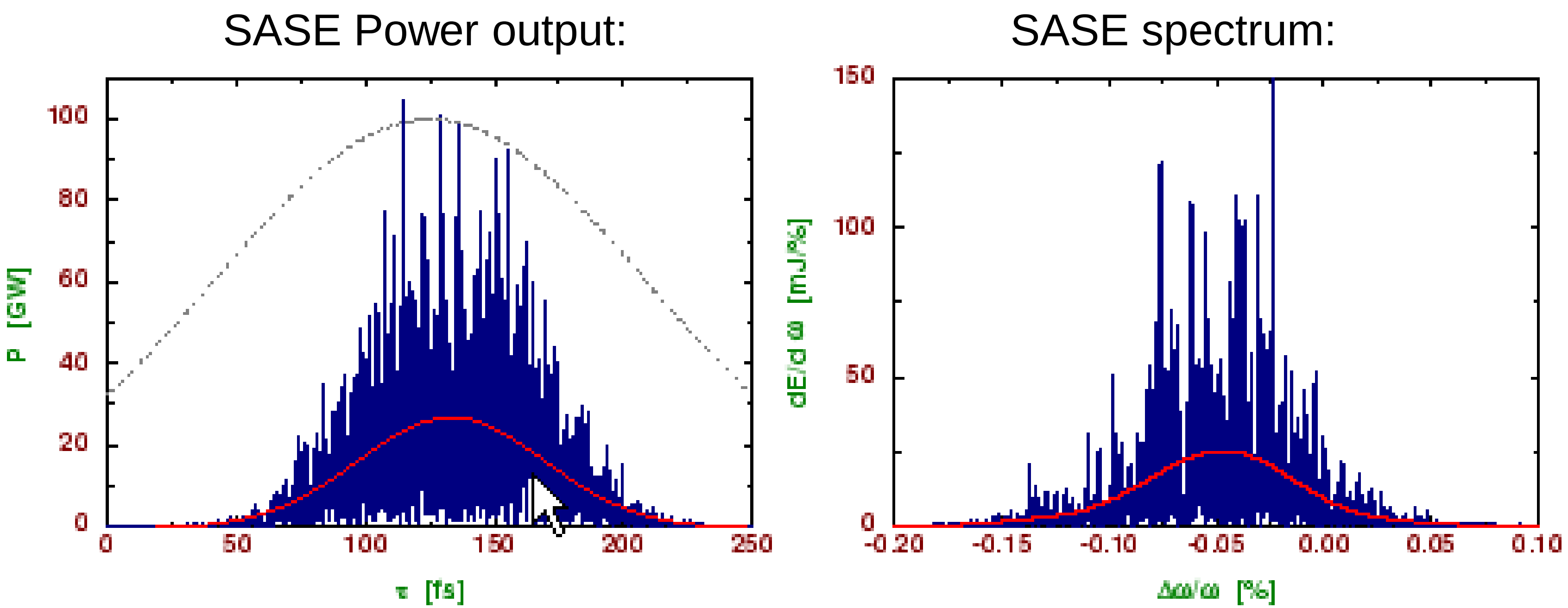


Figure 5: Typical temporal (left) and spectral (right) structure of the radiation pulse from a SASE XFEL at a wavelength of 1Å. The red lines correspond to averaged values. The dashed line represents the axial density profile of the electron bunch. Note that the growth rate in the electron bunch tail is reduced due to the reduced current. Therefore, the radiation pulse length of 100fs (FWHM) is about a factor of two shorter than the electron bunch.

Estimate of initial mean SASE power

The radiation power evolves like: $$P_{rad}(\bar{z}) \approx \frac{P_{rad}(0)}{9}\exp(\sqrt{3}\bar{z}) \quad (1)$$

And the saturation power: $$P_{sat} \approx \rho P_{beam} \quad (2)$$

From the analysis of *, the power in the linear regime is: $$P_{rad}(\bar{z}) = \frac{2\sqrt{\pi}}{3^{5/4}}\frac{\exp\left(\sqrt{3}\bar{z}\right)}{\sqrt{\bar{z}}N_\lambda}\rho^2 P_{beam} \quad (3)$$

N_λ is # e$^-$ in radn. wavelength.

From last 2 eqns.: $$\ln\left(\bar{z}_{sat}\right) - 2\sqrt{3}\bar{z}_{sat} - \ln\left(\frac{4\pi\rho^2}{3^{5/2}N_\lambda^2}\right) = 0$$

$$\Rightarrow \quad \bar{z}_{sat} \approx \ln\left(N_\lambda/\rho\right)/\sqrt{3}$$ for: $2\sqrt{3}\bar{z}_{sat} \gg \ln\left(\bar{z}_{sat}\right)$ (4)

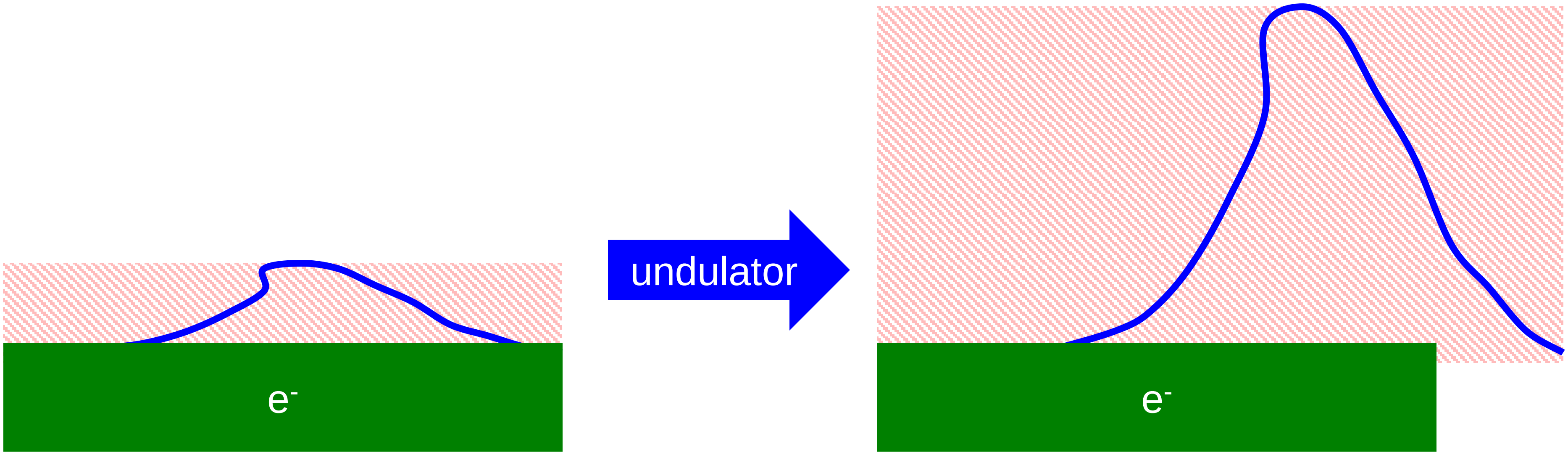
Equating (1) & (3): $$P_{rad}(0) \approx 2\frac{3^{3/4}\sqrt{\pi}}{\sqrt{\bar{z}_{sat}}N_\lambda}\rho^2 P_{beam}$$ and using (4):

$$P_{rad}(0) \approx \frac{6\sqrt{\pi}}{N_\lambda\sqrt{\ln\left(N_\lambda/\rho\right)}}\rho^2 P_{beam} \quad \Rightarrow \quad |A_0|^2 = \frac{6\sqrt{\pi}\rho}{N_\lambda\sqrt{\ln\left(N_\lambda/\rho\right)}}$$

* Kwang-Je Kim, Phys. Rev. Lett., **57**, 1871, (1986)

Seeded FEL – improves SASE

Region of seed with good longitudinal coherence :



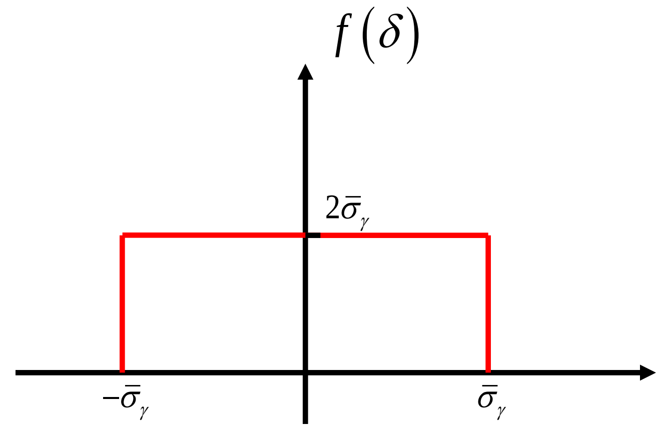
Longitudinal coherence of radiation pulse is inhereted from that of seed if $P_{seed} >> P_{noise}$

Spoiling effects

Energy spread

The effects of energy spread can be investigated by introducing a spread in the initial values of p_j :

$$\delta_j = p_j\left(\bar{z}=0\right) \equiv \frac{\gamma_r - \gamma_j\left(\bar{z}=0\right)}{\rho\gamma_r}$$

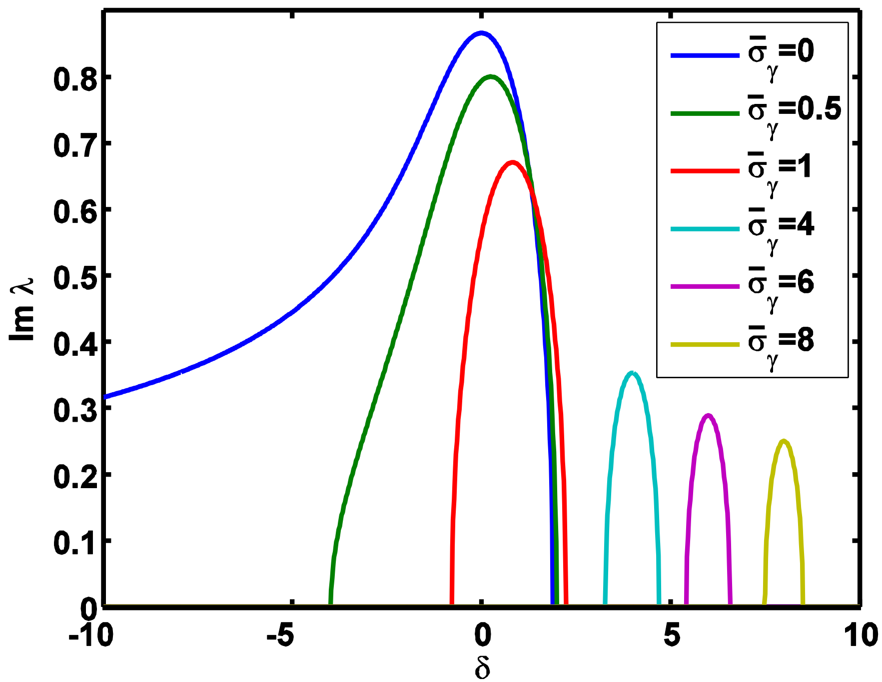


The steady state dispersion relation becomes:

$$\lambda - \int_{-\infty}^{\infty} \frac{\mathrm{d}\delta\, f(\delta)}{(\lambda-\delta)^2} = 0$$

with solutions for the imaginary part of lambda, determining the high gain case, shown opposite. Energy spread effects become less important when:

$$\bar{\sigma}_\gamma < 1 \quad \Rightarrow \quad \frac{\sigma_\gamma}{\gamma} < \rho$$



R. Bonifacio [1], L. De Salvo Souza and B.W.J. McNeil Optics Communications 93 (1992) 179–185

Emittance*

The beam emittance introduces two main effects:

1) The electron beam radius in a matched focussing channel** is determined by the emittance via:

$$r_b = \sqrt{\frac{\varepsilon_n \beta}{\gamma}} \Rightarrow \rho = \left(\frac{e}{16\pi\epsilon_0 mc^3} \frac{I_{pk} a_w^2 f_B^2}{\gamma_r^2 k_w^2 \epsilon_n \beta} \right)^{1/3}$$

β – betafunction of focussing lattice.

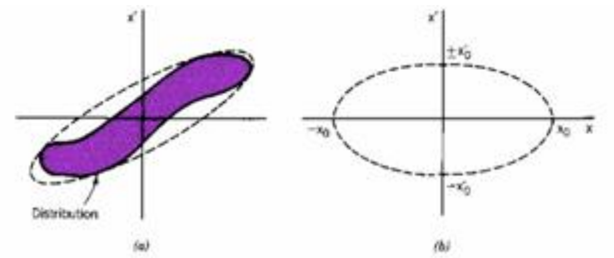
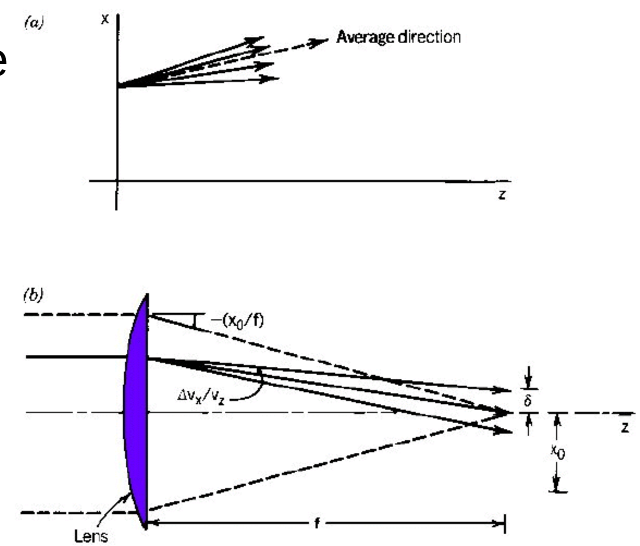


Figure 3.8. Definition of emittance. (*a*) Uniform orbit-vector distribution inside a boundary, surrounded by a minimum-area ellipse. (*b*) Upright trace-space ellipse — the enclosed emittance equals $x_0 x_0'$ π-m-rad.

2) The emittance introduces an energy spread in the resonant electron energy***. This can be added in quadrature with the real energy spread to estimate emittance effects in a 1D model:

$$\sigma_\epsilon = \frac{\epsilon_n a_w^2 k_w^2 \beta}{4\gamma_r (1 + a_w^2)} \Rightarrow \sigma_{eff} = \sqrt{\sigma_\epsilon^2 + \sigma_\gamma^2}.$$

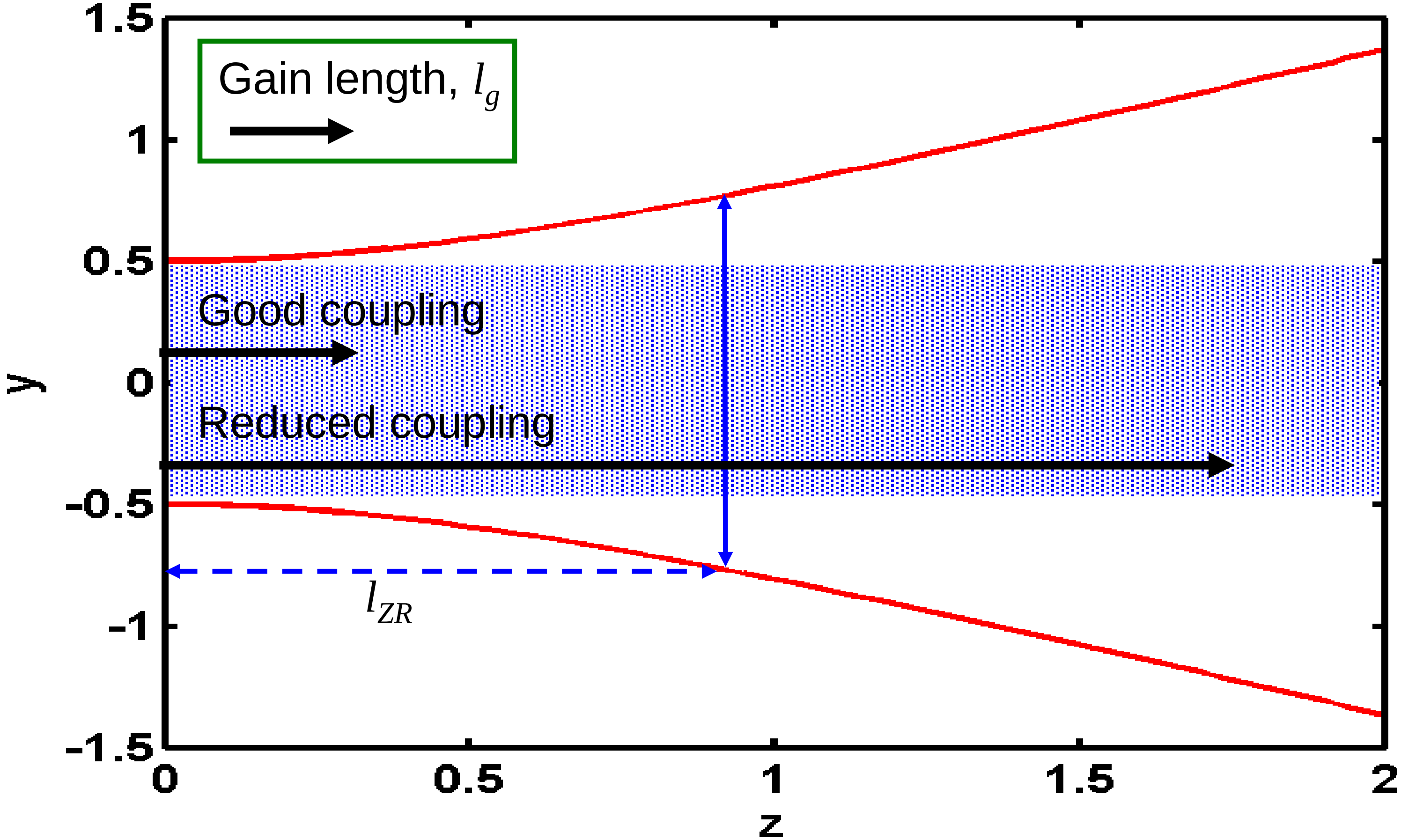
$$\Rightarrow \frac{\sigma_{eff}}{\gamma} < \rho$$



***Stanley Humphries,Charged Particle Beams:** http://www.fieldp.com/cpb.html

***R. Bonifacio [1], L. De Salvo Souza and B.W.J. McNeil Optics Communications 93 (1992) 179–185

Diffraction



The Rayleigh length l_{ZR} is that in which a beam diffracts to twice its transverse mode area. In an FEL amplifier, if the gain length of the FEL interaction is much greater than the Rayleigh length then diffraction can cause reduced coupling and longer saturation lengths.

8.1.3.3 Summary of Criteria for Optimum FEL Performance*

The one-dimensional theory describes the best-case limit for FEL operation. The following summarises the limits required for the one-dimensional equations to be a valid approximation for a high gain FEL interaction that achieves saturation:

- $L_u \gg L_g$ - The undulator is significantly longer than the interaction gain length
- $L_g \lesssim l_{ZR}$ - The gain length is $\lesssim$ the Rayleigh range
- $L_g \approx \beta$ - The gain length is $\approx$ the betatron function
- $\delta_r \ll r_b$ - Electron beam wander off-axis is much less than the beam radius
- $r_b \approx$ constant - The electron beam radius is approximately a constant
- $\sigma_\gamma \lesssim \rho$ - Homogeneous relative energy spread is less than the FEL coupling parameter
- $\sigma_\varepsilon \lesssim \rho$ - Resonant relative energy spread due to emittance is less than the FEL coupling parameter

Conditions 2 & 3 yield the ‘Kim-Pellegrini’ condition on the emittance: $\varepsilon_n \lesssim \gamma\lambda_1/4\pi$

Thank You!

Real FEL designs as taken from the 4GLS design*

*4GLS Conceptual Design Report, Chapter 8: http://www.4gls.ac.uk/documents.htm

4GLS CDR – April 2006