NONLINEAR TRAVELING WAVES IN AN ELECTROMAGNETICALLY PUMPED FREE ELECTRON LASER

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Abstract

The relativistic cold fluid model is used to study the propagation of the nonlinear travelling wave in a free electron laser (FEL) with electromagnetic wiggler. It is convenient to transform the relevant equations to the frame of reference rotating with the wiggler. The travelling-wave ansatz is employed to obtain three coupled, nonlinear ordinary differential equations that describe the nonlinear propagation of the coupled wave. Saturation and solitary waves in FELs with electromagnetic wiggler may be investigated using these equations. In the small signal limit, the wave equations are linearized and the dispersion relation for the travelling wave is obtained. The numerical solution of the travelling-wave dispersion relation reveals the range of parameters for its unstable solutions. Instability curves with two peaks are found, for which the phase velocity is smaller and larger than the beam velocity.

INTRODUCTION

Electromagnetic wiggler has been proposed as an alternative to the conventional magnetostatic wiggler. There are investigations that have considered employing electromagnetic pumps in FELs [1-7]. One advantage of the electromagnetic wiggler is its favourable scaling law for the radiation wavelength compared to the magnetostatic wiggler [8]. For this and other reasons optical wiggler has been used for investigation of x-ray FEL [6,7].

The purpose of the present investigation is to use the relativistic cold fluid model to study the propagation of the nonlinear travelling-wave in a FEL with electromagnetic wiggler. The method of analysis and notations are similar to Ref. [9] where this problem for the magnetostatic wiggler was solved.

THEORETICAL MODEL AND NONLINEAR TRAVELING-WAVE EQUATIONS

Electromagnetic wiggler can be described as [10]

$$\mathbf{B}^0(\mathbf{x},t) = -B\left[\cos(k_0 z+\omega_0 t)\hat{e}_x + \sin(k_0 z+\omega_0 t)\hat{e}_y\right]$$

$$\mathbf{E}^0(\mathbf{x},t) = (\omega_0 B_0/ck_0)\left[\sin(k_0 z+\omega_0 t)\hat{e}_x - \cos(k_0 z+\omega_0 t)\hat{e}_y\right] \quad (1)$$

The macroscopic cold-fluid model which will be employed, consists of the continuity equation

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(nV_z) = 0, \quad (2)$$

and the z component of relativistic momentum equation

$$\frac{\partial p_z}{\partial t} = -mc^2\frac{\partial \gamma}{\partial z} + e\frac{\partial \varphi}{\partial z}, \quad (3)$$

where $\gamma = \left(1+\mathbf{p}^2/m^2c^2\right)^{1/2}$ is the relativistic factor, and p_z is the longitudinal momentum. The axial component of the Maxwell equation

$$\nabla\times\delta\mathbf{B} = \frac{4\pi}{c}\delta\mathbf{J} + \frac{1}{c}\frac{\partial}{\partial t}\delta\mathbf{E}, \quad (4)$$

can be written as

$$\frac{\partial^2\varphi}{\partial z\,\partial t} = -4\pi e\left(nV_z - n_0V_{0z}\right). \quad (5)$$

Using $V_z = p_z/\gamma m$ in Eq. (5) and differentiating Eq. (3) with respect to t, give

$$\frac{\partial^2 p_z}{\partial t^2} = -mc^2\frac{\partial^2\gamma}{\partial t\,\partial z} - \omega_{p0}^2\left[\frac{n}{n_0}\frac{p_z}{\gamma} - \frac{p_{0z}}{\gamma_0}\right]. \quad (6)$$

The transverse canonical momentum is a constant of motion. We use $\mathbf{p}_\perp = e\mathbf{A}/c$. The wave equation for perturbed vector potential in terms of momentum can be written as

$$\left[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\delta\mathbf{p}_\perp = \frac{\omega_{p0}^2}{c^2}\left[\frac{n}{n_0}\frac{\mathbf{p}_\perp}{\gamma} - \frac{\mathbf{p}_{0\perp}}{\gamma_0}\right], \quad (7)$$

where $\omega_{p0}^2 = 4\pi e^2 n_0/m$ and $\gamma_0 = \left(1+\mathbf{p}_0^2/m^2c^2\right)^{1/2}$ denote unperturbed values. In the wiggler coordinate system which defined by [10]

$$\hat{e}_1 = \cos(k_0 z+\omega_0 t)\hat{e}_x + \sin(k_0 z+\omega_0 t)\hat{e}_y,$$
$$\hat{e}_2 = -\sin(k_0 z+\omega_0 t)\hat{e}_x + \cos(k_0 z+\omega_0 t)\hat{e}_y,$$
$$\hat{e}_3 = \hat{e}_z, \quad (8)$$

two component of Eq. (7) can be transformed to the following expressions,

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$$\frac{\partial^2 \delta p_1}{\partial z^2} - 2k_0 \frac{\partial \delta p_2}{\partial z} - k_0^2 \delta p_1 - \frac{1}{c^2}\left[\frac{\partial^2 \delta p_1}{\partial t^2} - 2\omega_0 \frac{\partial \delta p_2}{\partial t} - \omega_0^2 \delta p_1\right] = \frac{\omega_{p0}^2}{c^2}\left[\frac{n}{n_0}\frac{p_1}{\gamma} - \frac{p_{10}}{\gamma_0}\right], \tag{9}$$

$$\frac{\partial^2 \delta p_2}{\partial z^2} + 2k_0 \frac{\partial \delta p_1}{\partial z} - k_0^2 \delta p_2 - \frac{1}{c^2}\left[\frac{\partial^2 \delta p_2}{\partial t^2} + 2\omega_0 \frac{\partial \delta p_1}{\partial t} - \omega_0^2 \delta p_2\right] = \frac{\omega_{p0}^2}{c^2}\frac{n}{n_0}\frac{p_2}{\gamma} \tag{10}$$

By making the travelling-wave ansatz that the dependencies of all of the quantities δp_1, δp_2, p_3, and n on z and t are only through the combination $\xi = z - ut$, where $u = const$ is the speed of the wave, and introducing dimensionless quantities

$$\rho_\alpha = \frac{p_\alpha}{mc}, \quad \delta\rho_\alpha = \frac{\delta p_\alpha}{mc}, \quad \beta = \frac{u}{c}, \tag{11}$$

where $\alpha = 1,2,3$, Eqs. (2), (6), (9), and (10), may then be reduced to

$$\left(1-\beta^2\right)\frac{d^2\delta\rho_1}{d\xi^2} - 2\left(k_0 + \beta\omega_0\right)\frac{d\,\delta\rho_2}{d\xi} - \left(k_0^2 - \frac{\omega_0^2}{c^2}\right)\delta\rho_1 = \frac{\omega_{p0}^2}{c^2}\left[\frac{n}{n_0}\frac{\rho_1}{\gamma} - \frac{\rho_{01}}{\gamma_0}\right] \tag{12}$$

$$\left(1-\beta^2\right)\frac{d^2\delta\rho_2}{d\xi^2} + 2\left(k_0 + \beta\omega_0\right)\frac{d\,\delta\rho_1}{d\xi} - \left(k_0^2 - \frac{\omega_0^2}{c^2}\right)\delta\rho_2 = \frac{\omega_{p0}^2}{c^2}\frac{n}{n_0}\frac{\delta\rho_2}{\gamma} \tag{13}$$

$$\beta\frac{d^2}{d\xi^2}\left(\beta\rho_3 - \gamma\right) = -\frac{\omega_{p0}^2}{c^2}\left[\frac{n}{n_0}\frac{\rho_3}{\gamma} - \frac{\rho_{03}}{\gamma_0}\right] \tag{14}$$

$$\frac{d}{d\xi}\left[n\left(\beta - \frac{\rho_3}{\gamma}\right)\right] = 0, \tag{15}$$

where $\beta = u/c$. Equation (15) can be integrated to give

$$\frac{n}{n_0} = \frac{\left(\beta - \beta_b\right)\gamma}{\gamma\beta - \rho_3}, \tag{16}$$

where n_0 and $\beta_b = \rho_{03}/\gamma_0$ are constants. With the introductions of the dependent variable $Z = \beta\rho_3 - \gamma$, dimensionless traveling-wave variable $\zeta = k_0\xi$, and the use of Eq. (16) in Eqs. (12)-(14), and performing some algebraic manipulation, we obtain

$$\left(1-\beta^2\right)\ddot{\rho}_1 - 2\left(1+\beta\hat{v}_{ph}\right)\dot{\rho}_2 - \rho_1\left[1-\hat{v}_{ph}^2 + \frac{\hat{\omega}_p^2\gamma_0\left|\beta-\beta_b\right|}{\left[\left(\beta^2-1\right)\left(1+\rho_1^2+\rho_2^2\right)+Z^2\right]^{1/2}}\right] = -\hat{\omega}_c\gamma_0\left(1+\hat{\omega}_p^2-\hat{v}_{ph}^2\right), \tag{17}$$

$$\left(1-\beta^2\right)\ddot{\rho}_2 + 2\left(1+\beta\hat{v}_{ph}\right)\dot{\rho}_1 - \rho_2\left[1-\hat{v}_{ph}^2 + \frac{\hat{\omega}_p^2\gamma_0\left|\beta-\beta_b\right|}{\left[\left(\beta^2-1\right)\left(1+\rho_1^2+\rho_2^2\right)+Z^2\right]^{1/2}}\right] = 0, \tag{18}$$

$$\ddot{Z} + \frac{\hat{\omega}_p^2\gamma_0\left|\beta-\beta_b\right|Z}{\left(\beta^2-1\right)\left[\left(\beta^2-1\right)\left(1+\rho_1^2+\rho_2^2\right)+Z^2\right]^{1/2}} + \frac{\hat{\omega}_p^2\gamma_0}{\left(\beta^2-1\right)}\left(1-\beta\beta_b\right) = 0, \tag{19}$$

where the overdot denotes $d/d\zeta$, and

$$\hat{\omega}_c = \omega_c/ck_0,\ \omega_c = eB_0/\gamma_0 mc,\ \hat{\omega}_p^2 = \omega_p^2/c^2k_0^2,\ \omega_p^2 = \omega_{p0}^2/\gamma_0,\ \omega_{p0}^2 = 4\pi n_0 e^2/m,\ \hat{v}_{ph} = \omega_0/ck_0,\ \beta_b = V_0/c = \left[1-\hat{\omega}_c^2 - 1/\gamma_0^2\right]^{1/2}. \tag{20}$$

By multiplying $\dot{\rho}_1$, $\dot{\rho}_2$, and $\dot{Z}$ by Eqs. (17)-(19), respectively, we find an exact integral of motion,

$$Q = \frac{1}{2}\left[\left(\beta^2-1\right)\left(\dot{\rho}_1^2+\dot{\rho}_2^2\right)+\dot{Z}^2\right] + \frac{1}{2}\left(1-\hat{v}_{ph}^2\right)\left(\rho_1^2+\rho_2^2\right) + \frac{\hat{\omega}_p^2\gamma_0\left|\beta-\beta_b\right|}{\beta^2-1}\left[\left(\beta^2-1\right)\left(1+\rho_1^2+\rho_2^2\right)+Z^2\right]^{1/2} + \frac{\hat{\omega}_p^2\gamma_0\left(1-\beta\beta_b\right)}{\beta^2-1}Z - \hat{\omega}_c\gamma_0\rho_1\left(1+\hat{\omega}_p^2-\hat{v}_{ph}^2\right) = \text{const.}. \tag{21}$$

The stationary solutions to Eqs. (17)-(19), denoted by ρ_{01}, ρ_{02}, and Z_0, can be obtained by setting

$d/d\zeta = 0$. Using $\dot{\rho}_1 = \dot{\rho}_2 = \dot{Z} = 0$ gives $\rho_{01} = \hat{\omega}_c\gamma_0$, $\rho_{02} = 0$, and $Z_0 = \gamma_0\left(\beta\beta_b - 1\right)$.

SMALL-SIGNAL EQUATIONS

We will now investigate Eqs. (17)-(19) in the small-signal limit. By using $\rho_1 = \rho_{01} + \delta\rho_1$, $\rho_2 = \delta\rho_2$, and $Z = Z_0 + \delta Z$, and by linearizing for small $\delta\rho_1$, $\delta\rho_2$, and δZ, the following set of equations will be obtained

$$\left(1-\beta^2\right)\delta\ddot{\rho}_1 - 2\left(1+\beta\hat{v}_{ph}\right)\delta\dot{\rho}_2 - \left[\left(1-\hat{v}_{ph}^2+\hat{\omega}_p^2\right) + \frac{\hat{\omega}_c^2\hat{\omega}_p^2\left(1-\beta^2\right)}{\left(\beta-\beta_b\right)^2}\right]\delta\rho_1 - \frac{\hat{\omega}_c\hat{\omega}_p^2\left(1-\beta\beta_b\right)}{\left(\beta-\beta_b\right)^2}\delta Z = 0, \quad (22)$$

$$\left(1-\beta^2\right)\delta\ddot{\rho}_2 + 2\left(1+\beta\hat{v}_{ph}\right)\delta\dot{\rho}_1 - \left(1+\hat{\omega}_p^2-\hat{v}_{ph}^2\right)\delta\rho_2 = 0, \quad (23)$$

$$\delta\ddot{Z} + \frac{\hat{\omega}_p^2\left(1-\beta_b^2\right)}{\left(\beta-\beta_b\right)^2}\delta Z + \frac{\hat{\omega}_c\hat{\omega}_p^2\left(1-\beta\beta_b\right)}{\left(\beta-\beta_b\right)^2}\delta\rho_1 = 0. \quad (24)$$

By assuming ζ dependency of the form $\exp\left(i\hat{k}\zeta\right)$, where $\hat{k} = k/k_0$ and $\hat{k}\zeta = k\left(z-\beta ct\right)$, plane wave solutions to Eqs. (40)-(42) are considered. In this case $\delta\rho_1 = \delta\hat{\rho}_1\exp\left(i\hat{k}\zeta\right)$, $\delta\rho_2 = \delta\hat{\rho}_2\exp\left(i\hat{k}\zeta\right)$, and $\delta Z = \delta\hat{Z}\exp\left(i\hat{k}\zeta\right)$, are substituted in Eqs. (22)-(24) to obtain matrix equations relating the complex amplitudes $\delta\hat{\rho}_1$, $\delta\hat{\rho}_2$, and $\delta\hat{Z}$. For a nontrivial solution of Eq. (43) the determinant of the matrix should vanish. This, after some straightforward algebraic manipulation, will lead to the traveling-wave full dispersion relation,

$$\left[\left(\beta-\beta_b\right)^2\hat{k}^2 - \hat{\omega}_p^2/\gamma_b^2\right]\left[\left(\beta\hat{k}-\hat{v}_{ph}\right)^2 - \left(\hat{k}+1\right)^2 - \hat{\omega}_p^2\right]\left[\left(\beta\hat{k}+\hat{v}_{ph}\right)^2 - \left(\hat{k}-1\right)^2 - \hat{\omega}_p^2\right] = -\hat{\omega}_c^2\hat{\omega}_p^2\left[\left(\beta^2-1\right)\hat{k}^2 - \left(1+\hat{\omega}_p^2-\hat{v}_{ph}^2\right)\right]\left[\left(\beta^2-1\right)\hat{k}^2 - \hat{\omega}_p^2\right], \quad (25)$$

where $\gamma_b^2 = \left(1-\beta_b^2\right)^{-1}$. For $\hat{v}_{ph} = 0$, this dispersion relation reduces to that of magnetostatic wiggler FEL [9].

A formal correspondence between the traveling-wave dispersion relations and normal-mode dispersion relations, which are functions of normalized frequency $\hat{\omega} = \omega/ck_0$ and wave number $\hat{k}$, is established by $\hat{\omega} = \beta\hat{k}$ substitution. For the equation (25), this gives

$$\left[\left(\hat{\omega}-\hat{k}\beta_b\right)^2 - \hat{\omega}_p^2/\gamma_b^2\right]\left[\left(\hat{\omega}-\hat{v}_{ph}\right)^2 - \left(\hat{k}+1\right)^2 - \hat{\omega}_p^2\right]\left[\left(\hat{\omega}+\hat{v}_{ph}\right)^2 - \left(\hat{k}-1\right)^2 - \hat{\omega}_p^2\right] = -\hat{\omega}_c^2\hat{\omega}_p^2\left[\hat{\omega}^2 - \hat{k}^2 - \left(1+\hat{\omega}_p^2-\hat{v}_{ph}^2\right)\right]\left(\hat{\omega}^2 - \hat{k}^2 - \hat{\omega}_p^2\right). \quad (26)$$

This dispersion relation is similar in form to the corresponding normal-mode dispersion relation in Ref. [11].

NUMERICAL ANALYSIS OF STABILITY PROPERTIES

The six-degree polynomial dispersion equation (25) is solved numerically, to find the instability of the small signal in a FEL with electromagnetic wiggler. For $\hat{v}_{ph} = 0.5$, $\hat{\omega}_c = 0.05$ and $\gamma_b = 1.96$, Fig. 1 shows three-dimensional plots of $\mathrm{Im}\,\hat{k}/\mathrm{Re}\,\hat{k}$ as functions of β/β_b and $\hat{\omega}_p$. In this figure there are two unstable curves. For one curve $\beta < \beta_b$ which corresponds to the FEL resonance and is due to the unstable coupling between the slow space-charge wave and the electromagnetic radiation with wave number $\hat{k}-1$. Whereas, for the other curve $\beta > \beta_b$ and the unstable coupling is between the fast space-charge wave with the same $\hat{k}-1$ electromagnetic wave, which is not relevant to the FEL resonance. Figure 1 also shows that β and β_b differ widely for the two unstable curves at large $\hat{\omega}_p$ and their difference becomes less as $\hat{\omega}_p$ is lowered, which is the characteristics of the Raman regime. However, for small values of $\hat{\omega}_p$, which corresponds to the Compton-regime, $\beta \approx \beta_b$ as it should be according to the theory.

Setting $\hat{v}_{ph} = 0$ reduces the problem to the magnetostatic wiggler case, for which the unstable curves are shown in Fig. 2 for $\gamma_b = 1.96$ and $\hat{\omega}_c = 0.05$. Similar to the electromagnetic wiggler case, there are two unstable curves and instability exists for both $\beta < \beta_b$ and $\beta > \beta_b$ in the magnetostatic wiggler. The curve for $\beta > \beta_b$ in Fig. 2, which is due to the coupling of the fast space-charge wave with the electromagnetic radiation with $\hat{k}-1$ wave number, can be helpful in the numerical study of the nonlinear traveling-wave equations in the FEL with magnetostatic wiggler. The analytical investigation of magnetostatic wiggler in Ref. [9] was performed for $\beta < \beta_b$ case that is relevant to the FEL resonance. The reason that they only found $\beta < \beta_b$ for the small signal instability is that they only considered the coupling between the slow space-charge wave with the radiation.

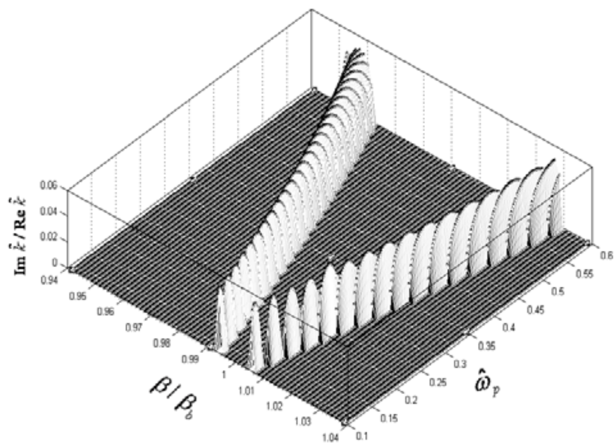

Figure 1: Three-dimensional plot of $\mathrm{Im}\,\hat{k}\,/\,\mathrm{Re}\,\hat{k}$ as a function of β/β_b and $\hat{\omega}_p$ for $\hat{v}_{ph}=0.5$, $\gamma_b=1.96$, $\hat{\omega}_c=0.05$.

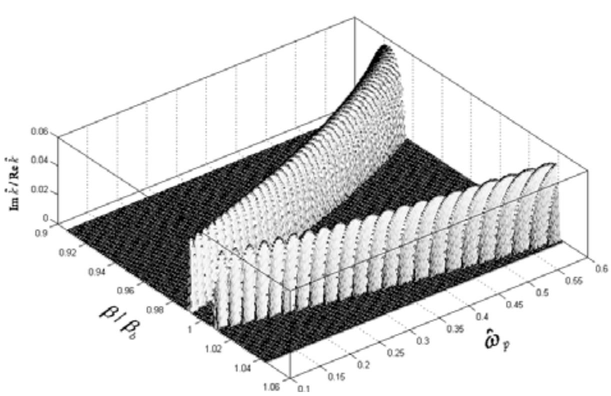

Figure 2: Three-dimensional plot of $\mathrm{Im}\,\hat{k}\,/\,\mathrm{Re}\,\hat{k}$ as a function of β/β_b and $\hat{\omega}_p$ for $\hat{v}_{ph}=0$, $\gamma_b=1.96$, $\hat{\omega}_c=0.05$.

CONCLUSION

The relativistic fluid theory is used to find three coupled and nonlinear differential equations [Eqs. (17)-(19)] that describe the nonlinear traveling-wave propagation in a FEL with electromagnetic wiggler. By linearizing these equations the small-signal analysis yields the traveling wave dispersion relation. Since the saturated states of linear instabilities are often associated with nonlinear wave equation, the traveling-wave dispersion relation is studied numerically to find appropriate range of parameters.

For both electromagnetic and magnetostatic wigglers, instability curves have two peaks. One with $\beta < \beta_b$ for the FEL resonance and the other with $\beta > \beta_b$ that corresponds to the coupling of the fast space-charge wave with the $\hat{k}-1$ electromagnetic wave.

The numerical investigation of the nonlinear traveling wave is expected to give new insights into the solitary solutions for the FELs.

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