THE SASE FEL TWO-TIME CORRELATION FUNCTION

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Abstract

The new approach for the SASE radiation properties calculation was proposed recently. It is based on the use of BBGKY chain of equations, adapted for FEL. In fact, it is the only known logically correct way to describe the SASE phenomenon. The two-time correlation function is necessary for calculation of averaged SASE spectrum. The solution of the correlation function equation for linear stage of SASE process is obtained.

INTRODUCTION

As it is well known the SASE FEL principle of operation is based on the amplification of the initial fluctuations which are present in the electron beam current density due to the discreteness of electrons. Therefore radiation of such FEL has stochastic nature and its parameters in a single shot can not be predicted. But averaged over many shots radiation properties obey some definite statistical laws which can be obtained by standard methods of statistical mechanics. It worth noting, that the averaging happens naturally in the experiment when one accumulates the data obtained in different shots.

The regular approach to the averaging procedure which has been developed recently [1-3] is based on the BBGKY chain of equations. It allows for calculating of the beam current correlation function at the given moment of specially chosen time variable. To find many important radiation properties like averaged spectral density one needs to know the two-time correlation function. Similar to the BBGKY chain the two-time correlation function equation can be obtained by averaging of the continuity equation for the microscopic density distribution [4].

In this paper we find the explicit solution of the two-time correlation function equation at linear stage for the 1-D case.

Single Shot Radiation Field

Let us first consider the FEL radiation field which is observed in a single shot. Vector potential of the field in paraxial approximation can be determined from the following expression:

$$\vec{A}(\vec{r},z,\xi)=\int_0^z\!\!\int\!\!\int \frac{\vec{j}_\xi\left(\vec{r}',z',\xi-\dfrac{1}{2}\dfrac{(\vec{r}-\vec{r}')^2}{(z-z')}\right)}{z-z'}dx'dy'dz' \tag{1}$$

where $\xi=t-z$ is the new time variable [1] and $\vec{j}_\xi(\vec{r},z,\xi)$ is the beam current density written as a function of this new time:

$$\vec{j}_\xi(\vec{r},z,\xi)=\vec{j}(\vec{r},z,z+\xi) \tag{2}$$

We assume that velocity of light $c=1$.

The single shot beam current density (2) can be expressed trough the microscopic density distribution (Klimontovich function) in the single-particle phase plane (z,Δ) [4]:

$$\mathrm{N}(z,\Delta,X;\theta)=\sum_k\delta\left(z-z^{(k)}(\theta)\right)\delta\left(\Delta-\Delta^{(k)}(\theta)\right)\delta\left(X-X^{(k)}\right) \tag{3}$$

where $\theta=2\gamma_{//}^2\xi$ is scaled time variable, $\Delta=\delta\gamma/\gamma_0$ is energy deviation, and $X=\left(\vec{r}_0,\dot{\vec{r}}_0\right)$ is a 4-*D* vector of initial transverse coordinates and angles. We assume that transverse motion is not influenced by longitudinal one and particles move along prescribed trajectories. One can consider (3) as continuous set of distributions marked with continuous index X. So, that each transverse trajectory $\vec{r}=\vec{r}(X,z)$ has its own distribution. There also exists inverse mapping $X=X\left(\vec{r},\dot{\vec{r}},z\right)$ with Jacobian $\dfrac{\partial\left(\vec{r}_0,\dot{\vec{r}}_0\right)}{\partial\left(\vec{r},\dot{\vec{r}}\right)}=1$

For the electron beam moving in undulator the transverse current density can be written in the following way

$$j_\perp(\vec{r},z,\theta)\approx\int\frac{V_\perp}{1-V_z}\mathrm{N}\left(z,\Delta,X\left(\vec{r},\dot{\vec{r}},z\right);\theta\right)d\Delta d\dot{\vec{r}} \tag{4}$$

where $V_\perp=K\gamma^{-1}\sin(k_w z)$ is transverse velocity of electron wiggling motion (K - undulator deflection parameter, k_w - undulator wave number) and $(1-V_z)^{-1}\approx 2\gamma_\parallel^2$ ($\gamma_\parallel$ - relativistic factor for the electron longitudinal velocity). The longitudinal distance between two particles for the given θ is $(1-V_z)^{-1}$ times more, that this distance for given *t*. Therefore this factor appears in Eq. (4).

Substituting (4) in (1) one can obtain explicit expression of single-shot radiation field trough the microscopic density distribution (3)

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Radiation Properties Averaged over Shots

Many of the important radiation properties like power and averaged spectral density can be determined from the field correlation function

$$\langle A(\vec{r}_1,z_1,t_1)A(\vec{r}_2,z_2,t_2)\rangle = \langle A(\vec{r}_1,z_1,\xi_1)A(\vec{r}_2,z_2,\xi_2)\rangle \tag{5}$$

For the coasting beam this function depends only on the time difference $t_2 - t_1$ or equivalently on $\xi_2 - \xi_1$:

$$\langle A(\vec{r}_1,z_1,\xi_1)A(\vec{r}_2,z_2,\xi_2)\rangle = C(\vec{r}_1,z_1,\vec{r}_2,z_2,\xi_2-\xi_1) \tag{6}$$

In this case the radiation spectral density at given observation point $(\vec{r},z)$ can be obtained by applying Fourier transformation to the correlation function (6) by the time difference.

After substitution of the explicit expression for the radiation field (1) with the beam current density (4) in the correlator (6) it becomes evident that calculation of correlator requires calculation of the averaged product of microscopic density distributions (3) $\langle \mathrm{N}(1,\theta_1)\mathrm{N}(2,\theta_2)\rangle$ where both multipliers are taken at different moments of time.

This product can be expressed as a sum of one-particle two-time and two-particle two-time distribution functions [4]

$$\langle \mathrm{N}(1,\theta_1)\mathrm{N}(2,\theta_2)\rangle = NF_2(1,\theta_1;2,\theta_2) + N(N-1)\times \times f_2^{(2)}(1,\theta_1;2,\theta_2) \tag{7}$$

where N is the total number of electrons (or for the coasting beam - the number of electrons per unit length).

Both functions obey similar equations but have different initial conditions

$$\begin{aligned} F_2(1,\theta_1;2,\theta_2)\big|_{\theta_1=\theta_2} &= F(1,\theta_1)\delta(1-2) \\ f_2^{(2)}(1,\theta_1;2,\theta_2)\big|_{\theta_1=\theta_2} &= f^{(2)}(1,2;\theta_1) \end{aligned} \tag{8}$$

where $F(1,\theta_1)$ is one-particle one-time and $f^{(2)}(1,2;\theta_1)$ is two-particle one-time distribution functions.

In turn the two-particle two-time distribution function can be separated in two parts

$$f_2^{(2)}(1,\theta_1;2,\theta_2) = F(1,\theta_1)F(2,\theta_2) + G_2(1,\theta_1;2,\theta_2) \tag{9}$$

where $G_2(1,\theta_1;2,\theta_2)$ is so called two-particle two-time correlation function.

For the coasting beam $F(1,\theta_1)$ does not depend on time, and the first term of Eq. (9) does not give any contribution to the correlator (6). All the required information about FEL coherent radiation can be obtained from the second term. It worth noting that contribution of the one-particle two-time distribution function (first term of Eq. 7) corresponds to spontaneous emission.

TWO-TIME CORRELATION FUNCTION

Correlation Function Equation

The two-time correlation function obeys the following equation [4]

$$\left(\frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial z_1}v_1\right)G_2(1,\theta_1;2,\theta_2) = -N\frac{\partial F(1)}{\partial\Delta_1}\int \Phi(1,3)G_2(3,\theta_1;2,\theta_2)d\{3\} \tag{10}$$

where $v(z,\Delta,X) = [1+2\Delta-2\gamma_{//}^2\Delta\beta(z,X)]$ is longitudinal velocity and $\Phi(1,2)$ is the two-particle interaction force [1]. The unit length for longitudinal coordinates is $\lambda_w/2\pi = 1/k_w$.

This equation has to be solved with the initial condition

$$G_2(1,\theta_1;2,\theta_2)|_{\theta_1=\theta_2} = G(1,2,\theta_1) \tag{11}$$

where $G(1,2,\theta_1)$ is one-time correlation function.

For simplicity we restrict our further consideration to the coasting beam and 1-D case. It can be applied for both the model of charged sheets and the thin beam model. In this case Eq. (10) can be written in the following form

$$\begin{aligned} &\left(\frac{\partial}{\partial\tau} + (1+2\Delta_1)\frac{\partial}{\partial z_1}\right)G_2(z_1,\Delta_1,z_2,\Delta_2;\tau) = \\ &= -N\frac{\partial}{\partial\Delta_1}F(z_1,\Delta_1)\int_0^{z_1}\int_{-\infty}^{\infty}\Phi(z_1-z_3)G_2(z_3,\Delta_3,z_2,\Delta_2;\tau)d\Delta_3 dz_3 \end{aligned} \tag{12}$$

where $\tau = \theta_1 - \theta_2$.

Solution at Linear Stage

Eq. (12) can be solved at linear stage when the distribution function $F(z,\Delta)$ does not depend on z. Applying Laplace transformation by τ, z_1 and z_2 and taking into account initial condition (11) one obtains the following equation:

$$(p+(1+2\Delta_1)s_1)G_2(s_1,\Delta_1,s_2,\Delta_2;p)-G(s_1,\Delta_1,s_2,\Delta_2)= \\ = \quad -N\Phi(s_1)\frac{\partial}{\partial\Delta_1}F(\Delta_1)\int G_2(s_1,\Delta_3,s_2,\Delta_2;p)d\Delta_3 \tag{13}$$

The further simplification can be done by introducing of the correlation function moments

$$g_{n,m}(s_1,s_2)=\int \Delta_1^n\Delta_2^m G(s_1,\Delta_1;s_2,\Delta_2)d\Delta_1 d\Delta_2$$

$$G_2^{n,m}(s_1,s_2,p)=\int \Delta_1^n\Delta_2^m G_2(s_1,\Delta_1,s_2;p)d\Delta_1 d\Delta_2$$

These moments obey the chain of equations which can be obtained by integrating of Eq. (13) by energy. For the cold beam when $F(\Delta)=\delta(\Delta)$ this chain is reduced to the closed system of two equations

$$(p+s_1)G_2^{0,0}(s_1,s_2;p)+2s_1G_2^{1,0}(s_1,s_2;p)=g_{0,0}(s_1,s_2)$$

$$(p+s_1)G_2^{1,0}(s_1,s_2;p)=g_{1,0}(s_1,s_2)+N\Phi(s_1)G_2^{0,0}(s_1,s_2;p)$$

This system can be easily solved

$$G_2^{0,0}(s_1,s_2;p)\left(1+\frac{2Ns_1\Phi(s_1)}{(p+s_1)^2}\right)= \\ =\frac{1}{(p+s_1)}g_{0,0}(s_1,s_2)-\frac{2s_1}{(p+s_1)^2}g_{1,0}(s_1,s_2) \tag{14}$$

where $g_{0,0}(s_1,s_2)$ and $g_{1,0}(s_1,s_2)$ can be found by the similar approach applied to the one-time correlation function equation [2]

$$g_{0,0}(s_1,s_2)=\frac{1}{N(s_1+s_2)}\left(\frac{1}{D}\left(1+2N\frac{s_1\Phi(s_1)+s_2\Phi(s_2)}{(s_1+s_2)^2}\right)-1\right)$$

$$g_{1,0}(s_1,s_2)=\frac{1}{D}\frac{\Phi(s_1)}{(s_1+s_2)^2}\left(1+2N\frac{s_1\Phi(s_1)-s_2\Phi(s_2)}{(s_1+s_2)^2}\right)$$

where

$$D(s_1,s_2)=\left(1+2N\frac{s_1\Phi(s_1)+s_2\Phi(s_2)}{(s_1+s_2)^2}\right)^2-16\frac{N^2s_1s_2\Phi(s_1)\Phi(s_2)}{(s_1+s_2)^4}$$

Introducing the poles of Laplace image (14)

$$p_{1,2}=-s_1\pm i\sqrt{2Ns_1\Phi(s_1)}=-s_1\pm i\mu(s_1)$$

and making inverse Laplace transformation by p one obtains

$$G_2^{0,0}(s_1,s_2;\tau)=\frac{1}{2\pi i}\oint G_2^{0,0}(s_1,s_2;p)e^{p\tau}dp= \\ e^{-s_1\tau}\left(\cos(\mu(s_1)\tau)g_{0,0}(s_1,s_2)-2s_1\frac{\sin(\mu(s_1)\tau)}{\mu(s_1)}g_{1,0}(s_1,s_2)\right) \tag{15}$$

To make inverse Laplace transformation over s_1 and s_2 one needs to know the poles of Eq. (15) which can be found from the dispersion equation

$$D(s_1,s_2)=0$$

It can be shown that this equation is equivalent to the set of two equtions

$$1+\frac{2Ns_1\Phi(s_1)}{(-i\omega+s_1)^2}=0\wedge 1+\frac{2Ns_2\Phi(s_2)}{(i\omega+s_2)^2}=0 \tag{16}$$

where ω is some arbitrary complex number.

It is convenient to introduce the following variables $s=s_1+s_2$ and $ik_z=(s_1-s_2)/2$. Then the inverse Laplace transformation of (15) over s gives the dependence of $G_2^{0,0}$ on $z=(z_1+z_2)/2$ and the inverse Fourier transformation over k_z gives its dependence on $x=z_1-z_2$.

Using inverse Laplace transformation of Eq. (15) and assuming that the gain length $L_g>>1$ one can determine asymptotic behaviour of the correlation function for large $z>>L_g$

$$G_2^{0,0}(z,k_z;\tau)\approx\frac{1}{4N}e^{\tilde{s}(k_z)z-i\omega(k_z)\tau} \tag{17}$$

where $\tilde{s}(k_z)$ and $\omega(k_z)$ are the solutions of Eq. (16). It can be easily shown that $\omega(-k_z)=-\omega(k_z)$ and $\tilde{s}(-k_z)=\tilde{s}(k_z)$.

To obtain the final expression for the correlation function one needs to calculate the following integral

$$G_2^{0,0}(z,x;\tau)\approx\frac{1}{4N}\frac{1}{2\pi}\int e^{\tilde{s}(k_z)z-i\omega(k_z)\tau+ik_zx}dk_z$$

As the exponent argument contains large factor z it can be done by saddle-point technique. For this purpose one needs to find extremal points of the exponent argument $\varphi(k_z) = \tilde{s}(k_z)z - i\omega(k_z)\tau + ik_z x$ at which $\varphi'_{k_z}(k_0) = 0$. Applying this technique one gets the following asymptotic expression for the correlation function

$$G_2^{0,0}(z,x;\tau) \approx \frac{1}{4N}\frac{e^{\varphi(k_0)}}{2\pi}\sqrt{\frac{2\pi}{\varphi''_{k_z}(k_0)}} + c.c. \qquad (18)$$

The Charged Sheets Model

As an example let us consider commonly used 1-D FEL model of charged sheets [6]. In the frame of this model one can write the following expression for the interaction force

$$N\Phi(s) = -8\rho^3 \frac{s}{1+s^2}$$

where $\rho << 1$ is the Pierce parameter.

Solution of dispersion equations (16) for this interaction force can be obtained by perturbation technique. Most simply it can be written in implicit form:

$$\begin{aligned} &\omega(k_z) = 1 + \delta \\ &\tilde{s}(k_z) = 2\rho\sqrt{3} - \frac{\sqrt{3}}{18\rho}\delta^2 \\ &k_z = 1 + \frac{2}{3}\delta + \frac{1}{32\rho}\delta^2 \end{aligned} \qquad (19)$$

where δ is some parameter which is assumed to be small.

For simplicity we shell find correlation function only for $x = 0$. There are no significant difficulties to find it for general case but the resulting expression come out cumbersome.

It can be easily shown that at the extremal point $\delta_0 = -i\frac{9\rho}{\sqrt{3}}\frac{\tau}{z}$. Taking this into account and substituting (19) in (18) one obtains

$$G_2^{0,0}(z,0;\tau) \approx \frac{e^{\frac{z}{L_g}}}{2N}\frac{e^{-\frac{3}{4}\frac{\tau^2}{zL_g}}}{\sqrt{3\pi}}\sqrt{\frac{1}{zL_g}}\left(\cos(\tau) - \frac{27}{32\sqrt{3}}\frac{\tau}{z}\sin(\tau)\right) \qquad (20)$$

where $L_g = 1/2\sqrt{3}\rho$ is the gain length. The Fourier transformation of (20) by τ is proportional to current spectral density at given point z in undulator

$$\left\langle j_\omega^2(z)\right\rangle \sim \frac{e^{\frac{z}{L_g}}}{18N}\left(e^{-\frac{(\omega+1)^2}{2\sigma_\omega^2}} + e^{-\frac{(\omega-1)^2}{2\sigma_\omega^2}}\right)$$

where the spectral width $\sigma_\omega = \sqrt{3/2zL_g}$ is the same as in conventional 1-D FEL theory [6].

CONCLUSION

We have shown that calculation of the averaged spectral density of the SASE FEL radiation requires calculation of the two-particle two-time correlation function. We also obtained explicit solution for this function in a simple 1-D FEL model.

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