THREE-DIMENSIONAL MODES OF A LAMELLAR GRATING FOR SMITH-PURCELL EXPERIMENTS

J. T. Donohue
Centre d'Etudes Nucléaires de Bordeaux-Gradignan, Université Bordeaux 1, CNRS/IN2P3, BP 120, 33175 Gradignan, France,
J. Gardelle
CEA, CESTA, F-33114 Le Barp, France

Abstract

Several years ago Andrews and Brau presented a two-dimensional theory for the production of coherent Smith-Purcell radiation by an initially continuous beam. An essential component of their analysis was the dispersion relation for a lamellar grating (i.e., rectangular profile) relating frequency and axial wave number *k*. Both simulations and an experiment performed at CESTA using a wide beam have confirmed the validity of their approach. However, all gratings are three-dimensional objects, and one may ask what modifications of the theory might be necessary. We present here our solution to the problem, which assumes a progressive wave in the direction of the grooves, with wave number *q*. A surprisingly simple modification of the Andrews and Brau 2-D dispersion relation is found. We have extensively tested our theory, both with simulations using the 3-D PIC code "MAGIC", and with measurements of the properties of the surface wave on the CESTA grating made using a network analyzer. Extremely good agreement is found, both with and without sidewalls on the grating.

INTRODUCTION

In 1953 Smith and Purcell (SP) [1] sent an electron beam of about 300 keV energy along the surface of a diffraction grating, and observed visible radiation which satisfied the condition they proposed,

$$\lambda = L\left(1/\beta - \cos\theta\right)/\left|n\right|,$$

where λ denotes the wavelength of the radiation produced at angle θ with respect to the beam, the grating period is *L*, and *n*, the order of diffraction, is a negative integer (for SP radiation). The quantity $\beta = v/c$, where *v* denotes the electron's velocity and *c* the speed of light. Although many contributions to the understanding of this radiation have been made over the past half-century, a renaissance of interest followed the publication of a theory for the production of intrabunch coherent SP radiation by Andrews and Brau [2]. In this work the authors solved the Maxwell equations in two dimensions (2-D) (the grooves were assumed infinitely long with no variation of the fields in their direction). They established the existence of an evanescent wave in the vicinity of the grating, and obtained the dispersion relation between frequency ω and axial wave number *k* as the condition that a determinant vanishes. They showed that the intersection of the dispersion relation with the beam line, $\omega = vk$, occurs at a frequency ω less than the minimum allowed SP frequency. The interaction between beam and wave leads to beam bunching at that frequency, and the evanescent wave will be radiated from the ends of the grating. If the bunching is strong enough, harmonics of the bunching frequency may appear in the current, and this could produce coherent monochromatic radiation in a small interval around the corresponding SP angle. It was pointed out by AB that for sufficiently low beam energy, the intersection would occur on that part of the dispersion relation where the slope $d\omega/dk$ is negative, as in a Backward Wave Oscillator. The theory of Andrews and Brau was subsequently described in greater detail by them and their collaborators [3]. Support for their view was provided by simulations [4,5] that used the 2-D electromagnetic code "MAGIC" [6]. Shortly thereafter Kumar and Kim [7] analyzed the problem using a different approach, but arrived at conclusions quite similar to those of AB. It is also true that the analysis of the evanescent wave on a lamellar grating has long been available in the literature, notably in the treatise of Collin [8].

Despite the consensus concerning the AB theory, until quite recently no experimental results supported their scenario. However, two recent experiments have reported evidence in its favor. The Vanderbilt-Vermont Photonics collaboration [9] observed the evanescent wave, but they used a grating equipped with sidewalls at the groove ends, which is not strictly 2-D. At CEA-CESTA a demonstration experiment in the microwave domain without sidewalls has found results which confirm the scenario of AB [10]. The width of both the grating and the beam, *w*, was 10 cm. Under these conditions the main results of 3-D "MAGIC" simulations were similar to those of the 2-D simulations presented by us in Ref. 4 [11].

Several attempts have been made to study coherent Smith-Purcell in three dimensions, since all gratings have finite width. Kim and Kumar [12] extended their previous analysis to 3-D and provided estimates for the beam parameters and minimum current for a set-up similar to the Dartmouth terahertz experiments [13]. Dazhi Li and co-workers performed 3-D simulations using "MAGIC" [14,15]. They showed that one could expect bunching for sufficiently large currents, and that if sidewalls were placed at the ends of the grooves, the start current would be reduced by a factor of two. They noted that with sidewalls the transverse profile of the axial component of the electric field has a cosine form, vanishing at the sidewalls, so that it couples well to a narrow beam in the middle of the grating. Andrews, Jarvis and Brau [16] presented a 3-D dispersion relation for a grating with sidewalls, assuming cosine profiles in the grooves. They examined in detail the dispersion relation for the grating used in the experiment described in reference 9, and showed that the predicted modes of operation for a beam of 30 keV were quite different according to whether the 2-D or their new 3-D theory applied. Experiment favored the 3-D prediction. Jarvis, Andrews and Brau [17] have recently provided a more detailed description of their theory. It differs from ours, in that they assume the axial magnetic field may be neglected, while we find a non-zero value for this field if the transverse wave number $q \neq 0$. We recognize that their theory furnishes a better description than ours does of the results presented in reference 9. However, it doesn't agree with the simulations and measurements we present here, especially when q is large.

We present here our 3-D dispersion relation for a mode that propagates with non-vanishing wave number q in the direction of the grooves. We find that the general solution, with an assumed $e^{i(qx-\omega t)}$ dependence, has zero electric field in the direction of the grooves, and that the dispersion relation is a simple generalization of the 2-D AB solution, in which for a given axial wave number k, the 3-D frequency $\omega_{3\text{-D}}(k,q)$ is given by

$$\omega_{3\text{-}D}(k,q) = \sqrt{\left(\omega_{2\text{-}D}(k)\right)^2 + (cq)^2},$$

where $\omega_{2\text{-D}}(k)$ is the 2-D AB frequency. In reference 12, Kim and Kumar anticipated this result, using their approach based on the reflection coefficient matrix for the grating. The case of sidewalls is then obtained by constraining the transverse wave number q to take on discrete values and adding the contribution with $-q$, to produce standing waves of the form $\cos(qx)$ or $\sin(qx)$. In support of this theory we show the results of 3-D "MAGIC" simulations as well as measurements of the transmission coefficient of the grating used in the experiment described in reference 10. Extremely good agreement is found among theory, simulation and measurements carried out at loops and nodes of the transverse waves.

THEORY

The details of our theory are given elsewhere [18], so we present here only a brief sketch of the main results. The geometry of our grating is shown in Fig. 1.

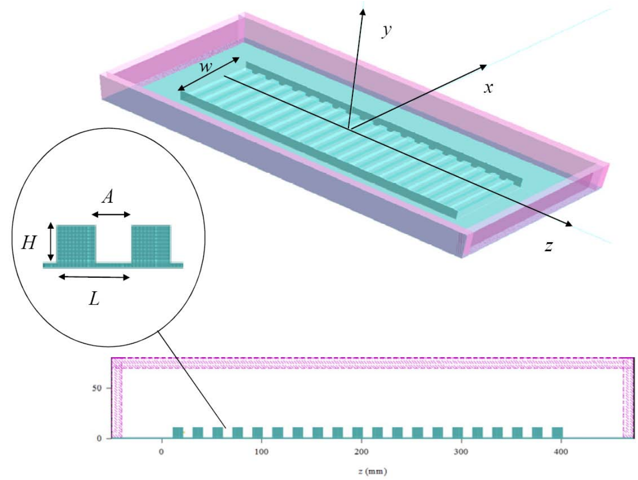

Figure 1: Schematic of the grating.

We assume the following form for the electric field

$$\vec{E}(\vec{x},t) = \nabla \times \left(\psi(y,z)\, e^{i(qx-\omega t)} \hat{e}_x\right),$$

where the complex function $\psi(y,z)$ will be chosen so as to satisfy the source-free Maxwell equations, ω is the frequency and q is the transverse wave number. The electric and magnetic fields may then be written as

$$\vec{E}(\vec{x},t) = \left(\frac{\partial\psi}{\partial z}\hat{e}_y - \frac{\partial\psi}{\partial y}\hat{e}_z\right) e^{i(qx-\omega t)},$$

$$\vec{B}(\vec{x},t) = \frac{1}{\omega}\left(q\left(\frac{\partial\psi}{\partial z}\hat{e}_z + \frac{\partial\psi}{\partial y}\hat{e}_y\right) - i\left(\frac{\omega^2}{c^2} - q^2\right)\psi\hat{e}_x\right) e^{i(qx-\omega t)},$$

where $\psi(y,z)$ satisfies the 2-D Helmholtz equation ,

$$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2} - q^2\right)\psi(y,z) = 0.$$

The function $\psi(y,z)$ is, to within a factor, the profile of B_x. Following AB, we write it as:

in groove: $0 \le z \le A$, $\quad -H \le y \le 0$,

$$\psi(y,z) = \sum_{n=0}^{\infty} \overline{\psi}_n \cos(n\pi z / A) \cosh\left(\kappa_n (y+H)\right) / \cosh(\kappa_n H),$$

above groove: $0 \le y \le \infty$, $\quad \psi(y,z) = \sum_{p=-\infty}^{p=\infty} \psi_p e^{i(k+pK)z - \alpha_p y}$,

$$\kappa_n = \sqrt{(n\pi / A)^2 + \omega^2 / c^2 - q^2}, \quad \alpha_p = \sqrt{(k+pK)^2 + \omega^2 / c^2 - q^2}.$$

where K=2π/L denotes the grating wave number. If one imposes at $y = 0$ continuity of ψ and $\partial\psi / \partial y$ across a groove and $\partial\psi / \partial y = 0$ atop a tooth, one obtains a homogeneous linear equation for the ψ_n,

$$\sum_{n=0}^{\infty}\left(\overline{R}_{mn} - \delta_{mn}\right)\overline{\psi}_n = 0.$$

The dispersion relation (DR) is just the condition that the matrix $\overline{R}$ has 1 as an eigenvalue. The matrix elements are given by

$$\overline{R}_{mn} = -2\kappa_n \tanh(\kappa_n H)\left(\sum_{p=-\infty}^{\infty} K_{pm}{}^{*} K_{pn} / \alpha_p\right) / (1+\delta_{m0}) AL,$$

where the K_{pn} (which depend only on k) are defined in AB. The key point is that $\bar{R}_{mn}$ depend only on k and $\sqrt{\omega^2 - c^2 q^2}$. If $\omega_{2\text{-D}}$ and k satisfy the DR for $q = 0$, then $\sqrt{\omega_{2-D}{}^2 + c^2 q^2}$ and k form a solution for arbitrary q. Thus the usual AB DR can be easily extended to any value of q. The profile function ψ depends only on k. If $q \to -q$, one obtains a solution with the same frequency and ψ, and one may form symmetric or anti-symmetric combinations that represent standing waves.

If the grooves are terminated with conducting sidewalls, B_x must vanish there, which imposes that

$$q = \frac{(2m+1)\pi}{w}, \quad m = 0,1, .., f_{3-D}(k) = \sqrt{f_{2-D}(k)^2 + \left(\frac{(m+\frac{1}{2})c}{w}\right)^2}$$

$$q = \frac{2n\pi}{w}, \quad n = 1, 2.., f_{3-D}(k) = \sqrt{f_{2-D}(k)^2 + \left(\frac{nc}{w}\right)^2},$$

for symmetric and anti-symmetric modes, respectively. Here w denotes the grating width and we express the frequency f in Hz. We display in Fig. 2 the DRs for the standard 2-D theory (green), four symmetric modes (black) and three anti-symmetric modes (red). We have used the grating parameters L = 2 cm, A = H =1 cm and w = 10 cm. The main difference between 2-D and 3-D DRs is that the former acts like a low-pass filter, while the latter acts like a pass-band filter with multiple bands. The blue curves are described in the next section.

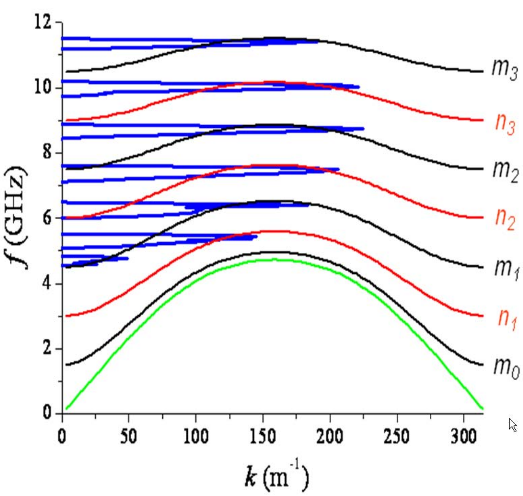

Figure 2: Dispersion relation for grating with sidewalls.

SIMULATIONS AND MEASUREMENTS

In order to support our theory, we have performed simulations using the 3-D version of "MAGIC", and we have measured the transmission coefficient of the surface wave as a function of frequency with a network analyzer. In order to get a rough idea of the transmission properties, a simulated "ping" or short burst of localized electromagnetic field was excited in an upstream groove. The fields then propagated down the grating, and time signals of the magnetic field component B_x were observed in a downstream groove. From the Fast Fourier Transform (FFT) of this time signal, the spectrum of frequencies that can propagate is obtained at once. The blue curves in Fig. 2 show this FFT. It is clear that the maxima of the pass-bands coincide with those of the DRs. However, a more sophisticated test can be performed by placing the simulated source and detector at strategic transverse positions, i.e., nodes and loops of the sine and cosine functions. Similarly, the emitting and receiving antennas of the network analyzer were placed at these same points. For the grating with sidewalls, a selection of these results is shown in Fig. 3.

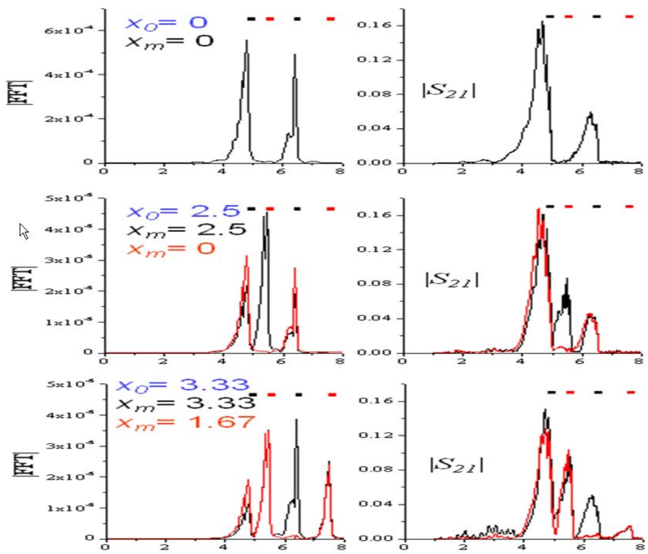

Figure 3: Ping FFTs and S21 *vs.* f (GHz).

The left side of Fig. 3 shows the ping FFTs, where x_0 is the source, and x_m the detector position. The right side shows the measured transmission coefficient $|S_{21}|$. The red and black squares indicate the theoretical position of the band-heads for anti-symmetric and symmetric modes, respectively. If either x_0 or x_m = 0, no anti-symmetric modes occur. For x_m = 2.5 cm, the second anti-symmetric mode is absent, while for x_m = 1.67 cm, the second symmetric mode is not seen. The overall consistency among predictions, pings, and $|S_{21}|$ measurements supports our theory for a grating with sidewalls.

We also simulated pings for a grating with no sidewalls, such as was used in ref. 10. We found, surprisingly, that sharp pass-band maxima occurred at 4.74, 4.9, 5.5, 6.3 and 7.2 GHz, while 2-D simulations indicate a maximum of 4.7 GHz. The second and fourth bands appeared only with asymmetric excitation. Empirically, these results are consistent with a list of discrete q values (in cm^{-1}) of {0.14, 0.31, 0.58, 0.86, 1.15}. For these values, the transverse B_x profile has a loop near the open ends of the grating, in contrast to the nodes for closed ends.

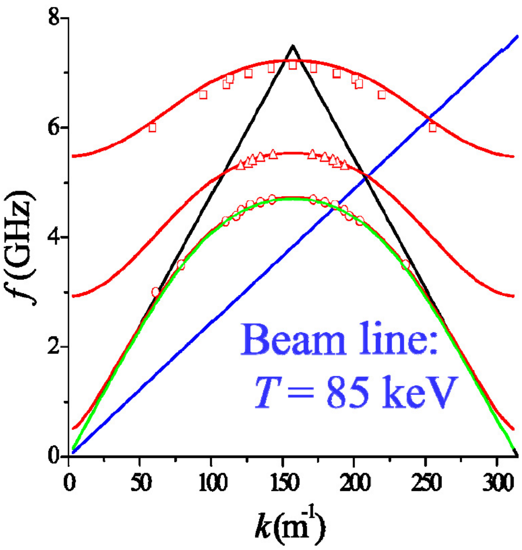

Figure 4: DR for grating without sidewalls.

To illustrate the DR for the open-ended grating, we show in Fig. 4 a comparison between our theory (red curves, lowest symmetric modes) and simulations (red symbols). The latter are determined by introducing a monochromatic current source in a groove, and using the "MAGIC" range feature to determine the corresponding wave number k. The green curve shows the 2-D DR of AB, the black light-lines are indicated, and a beam line of energy 85 keV is shown. It is clear that the lowest 3-D mode and the 2-D DR are almost identical, while the higher modes extend outside the limits of the light lines. These offer the interesting possibility of observing coherent SP radiation without the need for having harmonics in the bunching.

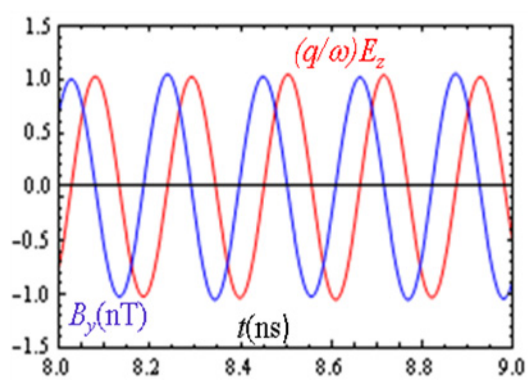

Figure 5: Test of $B_y(t) = \left(q\tan\left(qx\right)/\omega\right)E_z\left(t+T/4\right)$.

Our theory predicts that $B_y(t) = (q\tan(qx)/\omega)E_z(t+\pi/2\omega)$ for symmetric standing waves. A simulated current source of 4.7 GHz was used to excite the grating with sidewalls, and the result is shown in Fig. 5. The observation point was such that $qx = \pi/4$. It is clea that the prediction is verified, since E_z lags behind B_y by a quarter-period.

CONCLUSIONS

The agreement of our theory with both simulations and measured transmission coefficients provides convincing evidence of its validity.

REFERENCES

[1] S. J. Smith and E. M. Purcell, Phys. Rev. **92**, 1069 (1953).

[2] H. L. Andrews and C. A. Brau, Phys. Rev. ST Accel. Beams **7**, 070701 (2004).

[3] H. L. Andrews, C. H Boulware, C. A. Brau and J. D. Jarvis, Phys. Rev. ST Accel. Beams **8**, 050703 (2005); **8**, 110702 (2005),

[4] J. T. Donohue and J. Gardelle, Phys. Rev. ST Accel. Beams **8**, 060702 (2005).

[5] D. Li, Z. Yang, K. Imasaki and Gun-sik Park, Phys. Rev. ST Accel. Beams **9**, 040701 (2006).

[6] B. Goplen, L. Ludeking, D. Smithe, and G. Warren, Computer Physics Communications **87,** 54 (1995).

[7] Vinit Kumar and Kwang-Je Kim, Phys. Rev. **E 73**, 026501 (2006).

[8] R. E. Collin, *Foundations for Microwave Engineering,* Second Edition, McGraw-Hill, 1992, ch. 8.

[9] H. L. Andrews, C. A. Brau, J. D. Jarvis, C. F. Guertin, A. O'Donnell, B. Durant, T. H. Lowell and M. R. Mross, Phys. Rev. ST Accel. Beams **12**, 080703 (2009).

[10] J. Gardelle, L. Courtois, P. Modin and J. T. Donohue, Phys. Rev. ST Accel. Beams **12**, 110701 (2009).

[11] J. Gardelle and J. T. Donohue, IEEE Trans. Electron Devices, **56**, 769 (2009).

[12] Kwang-Je Kim and Vinit Kumar, Phys. Rev. ST Accel. Beams **10**, 080702 (2007).

[13] J. Urata M. Goldstein, M. F. Kimmit, A. Naumov, C. Platt and J. E. Walsh, Phys. Rev. Lett. **80**, 516 (1998).

[14] D. Li, K. Imasaki, Z. Yang, G-S. Park, S. Miyamoto, S. Amano, and T. Mochizuk, Jpn. J. Appl. Phys. **46**, 601 (2007).

[15] D. Li, K. Imasaki, X. Gao, Z. Yang and Gun-Sik Park, Appl. Phys. Lett. 91, 221506 (2007)

[16] H. L. Andrews, J. D. Jarvis and C. A. Brau, J. Appl. Phys. 105, 024904 (2009).

[17] J. D. Jarvis, H. L. Andrews, and C. A. Brau, Phys. Rev. ST Accel. Beams 13, 020701 (2010).

[18] J. T. Donohue and J Gardelle, submitted to Phys. Rev. STAB.