# DISPERSION RELATIONS FOR 1D HIGH-GAIN FELS

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*Abstract*

We present analytical results for the one-dimensional dispersion relation for high-gain FELs. Using kappa-n distributions, we obtain analytical relations between the dispersion relations for various order kappa distributions. Since an exact solution exists for the kappa-1 (Lorentzian) distribution, this provides some insight into the number of modes on the way to the Gaussian distribution.

## INTRODUCTION

For the 1D FEL model in [1] analytical results exist for the roots of the dispersion relation in the case of a cold beam or a Lorentzian energy distribution. For the cold beam, the relevant integrals in the dispersion relation may be evaluated directly due to the delta function. For any thermal spread, however, the relevant integral must be evaluated by contour integration, a procedure which breaks down for a Gaussian energy distribution. An exact dispersion relation exists for a Gaussian energy distribution, obtained using other methods.

The dispersion relation is given by

$$s - \frac{\hat{D}}{1 - \imath\Lambda_p^2\hat{D}} = 0 \tag{1}$$

for any arbitrary energy spread function, where

$$\hat{D} = \int_{-\infty}^{\infty} d\hat{P} \frac{\hat{F}'(\hat{P})}{s + \imath(\hat{P} + \hat{C})}$$

and $\hat{F}(\hat{P})$ is the normalized energy distribution of the electron beam in the linear approximation.

For a cold beam, $\hat{F}(\hat{P}) = \delta(\hat{P})$, and the familiar dispersion relation for solutions of the form $\sim e^{s\hat{z}}$ is given by

$$s\left((s + \imath\hat{C})^2 + \hat{\Lambda}_p^2\right) = \imath \tag{2}$$

As this is a cubic equation, an exact analytical solution may be written down using the Cardano formula. For the case of a Lorentzian distribution, $N = 1$, another cubic equation is obtained by contour integration. For a Gaussian distribution, the dispersion relation is obtained $\hat{D}$ using integral tricks [1] to yield

$$\hat{D} = \frac{\imath}{\hat{\Lambda}_T^2} - \frac{\imath\sqrt{\pi/2}}{\hat{\Lambda}_T^3}\left(s + \imath\hat{C}\right)\exp\left[\frac{(s+\imath\hat{C})^2}{2\hat{\Lambda}_T^2}\right] \times \left\{1 - \operatorname{erf}\left(\frac{s + \imath\hat{C}}{\sqrt{2}\hat{\Lambda}_T}\right)\right\} \tag{3}$$

For applications to Coherent Electron Cooling (CeC) [2] it is necessary to understand the details of the phase space density evolution of the electron beam perturbation generated by the FEL instability. Work already exists for this goal [3] [4], and exact analytical solutions exist for the initial perturbation generated by the hadron beam [5]. However, the solutions in [5] only exist for $\kappa = 2$ energy distributions, and it is therefore desirable for consistency to obtain dispersion relations for the FEL process for such an energy distribution. A solution also exists in the FEL modulator, which is also obtained for the $\kappa - 2$ distribution [6].

In this paper I present formal relationships for arbitrary $\kappa$ distributions, obtaining results for the growing roots recursively. Formally this provides an avenue of studying the analytical properties of the roots of the dispersion relation in the limit approaching a Gaussian energy spread, while from a practical application this allows us to study the CeC process from start to finish using a consistent background electron beam in the equations of motion.

## $\kappa - N$ DISTRIBUTIONS

A properly normalized $\kappa = N$ distribution is given by

$$f_N(P) = \frac{\Gamma[N]}{\sqrt{\pi}q_N\Gamma[N-1/2]}\frac{1}{(1 + (P/q_N)^2)^\kappa} \tag{4}$$

where $q_N$ is some function of $N$ and $\Gamma[N]$ is the gamma function. The zeroth through $N^{\text{th}}$ moment in $P$ is well-defined, but $\langle P^{N+1}\rangle \to \infty$. This distribution is convenient for modelling because it vanishes at infinity when it is analytically continued to the complex plane, thereby allowing contours to be closed in the upper- and lower-half planes, which cannot be done with a Gaussian distribution.. This allows the use of the Cauchy residue theorem to evaluate integrals of the function.

From the definition of the exponential function, a Gaussian distribution may be written as the limit of $\kappa - N$ distributions

$$\exp\left(-\frac{x^2}{2\sigma^2}\right) = \lim_{N\to\infty}\left(1 + \frac{x^2}{2\sigma^2 N}\right)^{-N} \tag{5}$$

The simplest appropriate choice for $q_N$ that converges to the Gaussian is then given by $q_N = \sqrt{2N}\sigma$. This gives the $N^{\text{th}}$ order $\kappa$ distribution that we study to be

$$f_N(P) = \frac{\Gamma[N]}{\sqrt{2\pi N\sigma^2}\Gamma[N-1/2]}\frac{1}{(1 + P^2/(2\sigma^2 N))^N} \tag{6}$$

It is clear that $f_{N\to\infty}(P) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\{-P^2/(2\sigma^2)\}$.

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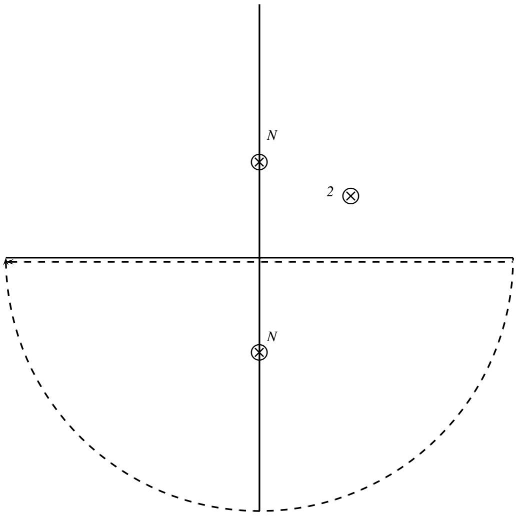


Figure 1: The pole structure for the growing roots. There are two poles of order $N$ on the imaginary axis, and another pole of order 2 located shifted off the axes.

## EVALUATING $\hat{D}$ FOR $F_N(P)$

We define

$$\hat{D}_N = \imath \int d\hat{P} \, \frac{1}{\left(s + \imath(\hat{C} + \hat{P})\right)^2} f_N(\hat{P}) \tag{7}$$

which is equivalent to $\hat{D}$ for the $N^{\text{th}}$ kappa distribution, and we have introduced normalized variables for direct comparison to [1]. For $\Re(s) > 0$, the pole structure is given by fig. 1.

Closing the contour on the lower half-plane and applying the Cauchy residue theorem gives two contributions, one from the pole on the positive imaginary axis, and the other from the pole due to the growing root. The imaginary axis contribution is given by

$$\begin{aligned}\phi_{\Im} = -\frac{2\imath\pi}{(N-1)!} \frac{d^{N-1}}{d\hat{P}^{N-1}} \Bigg[ \left(s + \imath(\hat{C} + \hat{P})\right)^{-2} \times \\ \left(1 - \imath\hat{P}/q_N\right)^{-N} (-\imath q_N)^N \Bigg]_{\hat{P} = -\imath q_N}\end{aligned} \tag{8}$$

which gives the resulting $\hat{D}_N$ in terms of the single pole

$$\hat{D}_N = \imath \frac{\Gamma[N]}{q_N \Gamma[N - 1/2]} \phi_{\Im} \tag{9}$$

Using the method due to Landau, $\hat{D}_N$ can be defined in such a way that the solution is independent of the sign of $\Re(s)$.

It can be shown that taking $M$ derivatives of a product of two functions behaves as a binomial expansion:

$$\frac{d^M}{dx^M}(f(x)g(x)) = \sum_{m=0}^{M} \left( \begin{array}{c} M \\ m-1 \end{array} \right) f^{(m)}(x) g^{(M-m)}(x)$$

This allows $\phi_{\Im}$ to be solved as an expansion in the derivatives of the two components. The resulting series solution for $\hat{D}_N$ is given by

$$\begin{aligned}\hat{D}_N = \imath \frac{\Gamma[N]}{q_N \Gamma[N - 1/2]} \times \ldots \\ \frac{2\pi}{(N-1)!} \frac{1}{2^{2N-1}} \sum_{m=0}^{N-1} \left( \begin{array}{c} N-1 \\ m \end{array} \right) \times \ldots \\ \left\{ \frac{2^m m!}{(s + q_N + \imath\hat{C})^{2+m}} q_N^{N-1-m} \frac{(2N-1-m)!}{(N-1)!} \right\}\end{aligned} \tag{10}$$

Solution of the dispersion equation (1) can then be obtained by whatever means are best.

The decaying roots ($\Re(s) < 0$) have the $\hat{D}_N^-$ given by mapping $q \mapsto -q$ and taking the negative of the result, and the oscillating roots ($\Re(s) = 0$) can be obtained by adding to $\hat{D}_N^+$ a term

$$2N\pi\imath \frac{(\imath s - \hat{C})/q_N^2}{\left(1 + (\imath s - \hat{C})^2/q_N^2\right)^{N+1}}$$

which arises from bumping the contour integral over the pole on the real axis that occurs with the oscillating root, and closing the contour in the upper-half plane.

It is interesting to note that, because $\hat{D}$ is some rational function, there will exist nonanalyticities in the roots where the real part may cross zero, sometimes abruptly, for a sufficiently large energy spread. However, for a given $N$, the next order typically appears to be well-behaved in that regime of the energy spread, and otherwise matches the previous order quite well. Therefore, one must be careful when using these dispersion relations with some finite energy spread.

## TWO LIMITS

We now examine two limits of these series: the $\kappa - 2$ distribution and the $N \to \infty$ limit. The first is of use to CeC modelling, while the second provides some insight into the poles for a Gaussian energy spread that might be more intuitive than the analytic continuation of the error function to the complex plane.

### $\kappa - 2$

For the case of a $\kappa - 2$ distribution we take $q_N = q_2 = q$ and obtain

$$\hat{D}_2 = \imath \frac{s + \imath\hat{C} + 3q}{(s + q + \imath\hat{C})^3} \tag{11}$$

This yields a fourth order equation in the dispersion relation, with the added condition that all roots must satisfy $\Re(s) > 0$ to obtain the growing roots. An analytical formula exists for the quartic, obtained by Ferarri's method, however the results are analytically complicated, and we present here only plots of the results. Supposing that $\sigma = .1$ for the definition of the $q_N$, this implies that $q_2 = .2$, while the result for the Lorentzian distribution has $q_1 = .141$. For comparison purposes, we present on the same plot the results for both.

It is worth noting for completeness that, for the decaying root, the dispersion relation can be obtained by mapping $q \mapsto -q$ in $\hat{D}_2$ and taking the negative of the result [1], while obtaining the oscillating roots requires taking attention to the pole on the real axis, which gives in addition to the decaying root $\hat{D}_2^-$ a term given by

$$4\pi\imath \frac{(\imath s - \hat{C})/q_2^2}{\left(1 + (\imath s - \hat{C})^2/q_2^2\right)^3}$$

### *Large $N$ Limit*

It would be convenient to determine the relative importance of each term for large $N$ distributions, where $q_N \propto \sqrt{N}$ indicates that this term will eventually become much bigger than 1 for sufficiently large $N$. In this case it is safe to assume that $|s + \hat{q} + \imath\hat{C}| \approx \hat{q} + \mathcal{O}(1/\sqrt{N})$ near $\hat{C} = 0$. It is then convenient to look at the order of magnitude estimated coefficients

$$c_m = 2^m \frac{(2N-1-m)!}{(N-1-m)!} q_N^{N-1-m} q_N^{-2-m}$$

where the first coefficient of $q_N$ arises explicitly in the expansion, and the second comes from considering $(s + q_N + \imath\hat{C}) \sim q_n$ for the purposes of this approximation. The critical value for $m$, taken by maximizing $\ln(c_m)$ with respect to m, is given by

$$m \approx N - 1 - \frac{eN}{\frac{2}{v_0} - e} \tag{12}$$

where $v_0 = e^{2q_N}$, subject to the condition that $m \in [0, N-1]$. For some range of energy spreads the largest term in the series defining $\hat{D}$ will appear somewhere in the middle of the series, whereas below this range the largest term is for $m = N-1$ and below this range $m = 0$, specifically for small $q$ the smaller $m$ terms are more important, while for larger $q$ higher order terms are necessary to accurately calculate $\hat{D}$. It is important to note that the second derivative at this local maximum is on the order of magnitude of

$$\frac{d^2(\ln c_m)}{dm^2}\Big|_{m=N-1-\frac{eN}{2/v_0=e}} = \frac{1}{N + \frac{eN}{2/v_0 - e}} - \frac{2/v_0 - e}{N} \tag{13}$$

which is $\mathcal{O}(1/N)$ in smallness, and it is therefore possible to assume that truncating the series at the quadratic order gives some good approximation of the terms. It is important to note for approximation purposes that it is precarious to series expand near $m = 0$ since the Stirling approximation was used to obtain this result, and does not start to converge until $m \sim 40$.

As an example of obtaining an approximate dispersion relation, consider a case where the largest term is the first term, and that the logarithm is approximately linear from that point on[2]. Taking the linear expansion around $m = N/2$, we obtain as a result

$$\ln c_m \approx -N + \ln(\alpha(s)) + \ln(\beta(s))(m - N/2) \tag{14}$$

where

$$\alpha(s) = \frac{2^N \left(\frac{3N}{2} - 1\right)^{3N/2-1} q_N^{N/2-1}}{\left(\frac{N}{2} - 1\right)^{N/2-1} \left(s + q_N + \imath\hat{C}\right)^{2+N/2}}$$

and

$$\beta(s) = \frac{2}{\left(\frac{3N}{2} - 1\right) q_N \left(s + q_N + \imath\hat{C}\right)}$$

Summing the approximate series from $m = 0 \ldots N-1$ gives as a result that

$$\hat{D}_N \approx e^{-N} \alpha(s) \beta(s)^{-N/2} \frac{1 - \beta(s)^{N-1}}{1 - \beta(s)} \tag{15}$$

The achievement here is that the transcendental equation on $s$ that arises from Gaussian distributions has been converted to an algebraic equation. Since $\beta(s) \sim N^{-3/2}$ for large $N$, it may be fair to drop the $\beta(s)^{N-1}$ term as small.

## CONCLUSION

We have shown here a method of generating the dispersion relation for the $1D$ self-consistent model of high-gain FELs for any natural number choice of $\kappa - N$ distribution. We have then applied this method to obtain an exact result for the $\kappa - 2$ distribution, of interest for a theoretical model of coherent electron cooling, as well as providing some insight in obtaining the approximate dispersion relation for large $N$, where the dispersion relation is still a rational function, rather than the formal Gaussian limit, which is given in terms of transcendental functions.

## REFERENCES

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[3] S.D. Webb and V.N. Litvinenko, "Evolution of electron beam phase space distribution in a high-gain FEL", *Proceedings of FEL '09* MOPC83.

[1] This is achieved by flipping the contour from the lower-half to the upper-half plane, and then reversing its orientation to give the correct integral.

[2] Where we only consider solutions near $\hat{C} = 0$, so as to avoid the case at far detuning when $|s| \sim \hat{C}$ which could become on the order of $q_N$. This holds in the area of growing roots, which are the dominant terms in the FEL process.

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