A 3-DIMENSIONAL THEORY OF FREE ELECTRON LASERS

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Abstract

In this paper, we present an analytical three-dimensional theory of free electron lasers. Under several assumptions, we arrive at an integral equation similar to earlier work carried out by Ching, Kim and Xie, but using a formulation better suited for the initial value problem of Coherent Electron Cooling. We use this model in later papers to obtain analytical results for gain guiding, as well as to develop a complete model of Coherent Electron Cooling.

INTRODUCTION

Existing work on the analytical three-dimensional theory of FELs ([1], [2] and citations therein) provide a number of useful results, and cover transverse modes and dispersion relations thoroughly. However, these approaches lack several features useful for applications to Coherent Electron Cooling (CeC) [3]. Specifically, existing theory for the initial signal for the kicker considers an infinite electron beam and provides an initial value problem for the FEL [4]. It is therefore desirable to develop a three-dimensional theory of the FEL process that is compatible with this work. With the existing formalism in [4] in mind, we develop here a three-dimensional theory of FELs which can be readily generalized to the case of the infinite beam, and which quickly reduces to the one-dimensional theory in [2].

We treat the transverse dynamics of the electron beam as a parameter whose dynamics are dictated by the Maxwell equations. The beam is assumed to have no transverse velocity spread, and the only magnetic field present is assumed to be an helical wiggler field. Operating in a transverse Fourier space, we obtain an integral equation in which the kernel is the Fourier transform of the transverse beam profile. A mode expansion obtains dispersion relations for each transverse mode as a function of their eigenvalue. For an infinite beam, the Fourier transform is a delta function, and results in an equation similar to the one-dimensional theory presented in [2]. Some specific applications of this theory are discussed in another conference proceeding.

EQUATIONS OF MOTION

To develop this model, we employ the Maxwell-Vlasov coupled equations to obtain a linearized equation of motion for the current density, which is directly related to the phase space distribution.

Vlasov Equation

The equations of motion as a function of z are given by

$$\frac{d\mathcal{H}}{dz} = \frac{1}{c}\left\{-\frac{1}{p_0}\left(\frac{e}{c}\right)^2 \vec{A}_w \cdot \vec{A}_\perp + \frac{e}{c}\frac{\partial A_z}{\partial t}\right\} \qquad \text{(1a)}$$

$$\frac{dt}{dz} = \frac{1}{c}\left\{1 + \frac{1}{2}\frac{1}{p_0^2}\left[\left(\frac{e}{c}\right)^2 (\vec{A}_w^2 + 2\vec{A}_w \cdot \vec{A}_\perp) + m^2c^2\right]\right\} \qquad \text{(1b)}$$

Defining $\mathcal{H} = \mathcal{E} + \mathcal{E}_0$ where $\mathcal{E}_0$ is the nominal energy of the electron beam, and linearizing the phase space density $f = f_0 + f_1$ where f_1 is small gives the linearized Vlasov equation:

$$\frac{\partial f_1}{\partial z} + \frac{1}{c}\left\{1 + \frac{1}{2}\frac{1}{\gamma_0^2}\left[K^2+1\right]\left(1 - 2\frac{\mathcal{E}}{\mathcal{E}_0}\right)\right\}\frac{\partial f_1}{\partial t} + \left\{\frac{1}{\mathcal{E}_0}\left(\frac{e}{c}\right)^2 \vec{A}_w \cdot \frac{\partial \vec{A}}{\partial t} + eE_z\right\}\frac{\partial f_0}{\partial \mathcal{E}} = 0 \qquad (2)$$

It is now necessary to solve for the vector potential and space charge fields to obtain the full equations of motion.

Maxwell Equations

The transverse Maxwell equations in Fourier space are given by

$$\left(-k_\perp^2 + \partial_z^2 - \frac{1}{c^2}\partial_t^2\right)\vec{A}_\perp = \frac{4\pi}{c}\vec{j}_\perp \qquad (3)$$

The transverse current is related to the longitudinal current by assuming that the transverse velocity is given by $\vec{v}_\perp = K/\gamma_0(\cos k_w z\hat{e}_x - \sin k_w z\hat{e}_y)$ for all electrons. This allows the solution in Fourier space of the potential $\vec{A}_w \cdot \vec{A}_\perp$ to be

$$\tilde{A}_\perp - \tilde{A}_\perp\,|_{z=0} = -e^{\imath k_\perp^2 cz/2\nu\omega_r}\frac{\imath\pi K}{\nu\omega_r\gamma_0}\begin{pmatrix}1\\-\imath\end{pmatrix}\int_0^z \tilde{j}_z dz' \qquad (4)$$

where the Fourier transform on the current density is defined by

$$j_z = \frac{1}{\sqrt{2\pi}^3}\int d\nu\, d^2k_\perp e^{\imath\vec{k}_\perp\cdot\vec{r}_\perp}e^{\imath\nu\omega_r(z/c-t)}\times e^{\imath k_u z}e^{\imath k_\perp^2 cz/2\nu\omega_r}\tilde{j}_z \qquad (5)$$

Space charge is accounted for by the Maxwell equation

$$\partial_t E_z = -\frac{4\pi}{c}j_z \qquad (6)$$

Applying the identical Fourier transform on j_z to E_z gives the space charge equation

$$\tilde{E}_z = -\frac{4\pi\imath}{c\nu\omega_r}\tilde{j}_z \qquad (7)$$

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These two results may now be inserted into the full coupled Maxwell-Vlasov equation.

Maxwell-Vlasov Equation

At this point we are able to write down the full Maxwell-Vlasov equation under the assumption that $f_0 = n_0 F(\mathcal{E}) G(\vec{r}_\perp)$ where the normalization is such that integrating over energy and the transverse coordinates gives the longitudinal density n_0. Under this assumption, the Maxwell-Vlasov equation takes the form of an integral equation given by equation 8 where $\mathcal{U}_0$ is related to initial seeding, and

$$\phi_0 = \imath k_w(1-\nu) + \imath 2 \frac{\mathcal{E}}{\mathcal{E}_0} k_u \nu + \frac{\imath k_\perp^2 c}{2\nu\omega_r}$$

Introducing normalized coordinates simplifies the equation as

$$\begin{aligned} \tilde{j}_z = \int d\hat{\mathcal{E}} e^{\imath(\hat{C}+\hat{\mathcal{E}}+\hat{k}_\perp^2)\hat{z}} \tilde{f}_1 \mid_0 + \dots \\ \int d\hat{\mathcal{E}} \int_0^{\hat{z}} d\hat{z}' \; e^{\imath(\hat{C}+\hat{\mathcal{E}}+\hat{k}_\perp^2)(\hat{z}'-\hat{z})} \int d^2\hat{q}\; e^{\imath(\hat{q}^2-\hat{k}_\perp^2)\hat{z}'} \times \dots \\ \left\{ \imath\hat{\mathcal{U}}_0 + \int_0^{\hat{z}'} \tilde{j}_z d\hat{z}'' + \imath\hat{\Lambda}_p^2 \tilde{j}_z \right\} \frac{d\hat{F}}{d\hat{\mathcal{E}}} \tilde{G}(\hat{q}-\hat{k}_\perp) \end{aligned} \tag{9}$$

At this point, the problem is solving the integral equation in the transverse Fourier space. For a finite beam size, a mode expansion similar to the methods in [1] can be employed, while for an infinite beam direct solution by Laplace transform is possible, and is the method of choice in GANG'SPAPER. I highlight both results briefly below.

INFINITE BEAM LIMIT

The limit of an infinite, homogeneous beam is useful for applications to the theory of CeC, as analytical results exist in the modulator [4] and the kicker [5]. In the limit of an infinite, homogeneous beam, $\tilde{G}(\hat{k}_\perp - \hat{q}) = \delta(\hat{k}_\perp - \hat{q})$ and the integral equation becomes a simple differential equation in $\tilde{j}_z$. In this case, solution may be carried out by Laplace transform, as in [2], but with the identification that $\hat{C} \mapsto \hat{C} + \hat{k}_\perp^2$. We therefore see that, for an infinite beam, finite transverse size is directly equivalent to a detuning.

The explicit equation of motion for $\tilde{j}_z$, solved through Laplace transform, is given by

$$\mathcal{J} = \frac{\int d\hat{\mathcal{E}} \frac{\tilde{f}_1|_0}{s+\imath(\hat{\mathcal{E}}+\hat{C}_{3D})} + \hat{\mathcal{U}}_0}{s - \hat{D}(1-\imath s\hat{\Lambda}_p^2)} \tag{10}$$

where $\hat{D}$ is the familiar integral

$$\hat{D} = \int d\hat{\mathcal{E}} \frac{\hat{F}'}{s + \imath(\hat{C}_{3D} + \hat{\mathcal{E}})} \tag{11}$$

which is examined in greater detail in [6], where $\hat{C}_{3D} = \hat{C} + \hat{k}_\perp^2$. It is studies of this particular model that are required for the solutions presented in [7] and further discussion is left there.

FINITE BEAM

For the case of finite beam, solution can best be achieved by solving the eigenmode problem for the transverse beam profile, searching for a solution of the integral equation

$$\psi_n(k) = \omega_n \int d^2q\; e^{(k^2-q^2)z} \tilde{G}(k-q)\psi_n(q) \tag{12}$$

In this case, it is convenient to expand the solutions for the current equation as

$$\tilde{j}_z = \sum_n a_n e^{-\imath\hat{k}_\perp^2 \hat{z}} \phi_n(\hat{k}_\perp) \tag{13}$$

where ϕ_n is an eigenmode of the $\tilde{G}$ kernel. Here the a_n are in general a function of $\hat{z}$, $\hat{C}$, and $\hat{k}_\perp$. A differential equation is obtained for the coefficients of ϕ_n in terms of the eigenvalues of the mode, which arises from the integral equation 9

$$\begin{aligned} a'_\ell - \imath Q_{m,\ell} a_m = \int d^2\hat{k}_\perp\; e^{\imath(\hat{C}+\hat{\mathcal{E}}+\hat{k}_\perp^2)\hat{z}} \tilde{f}_0 \mid_0 \phi_\ell(\hat{k}_\perp) - \dots \\ \int d\hat{\mathcal{E}} \int_0^{\hat{z}} d\hat{z}'\; e^{\imath(\hat{C}+\hat{\mathcal{E}})(\hat{z}'-\hat{z})} \times \dots \\ \frac{1}{\omega_\ell}\Big\{ \int d^2\hat{k}_\perp \hat{\mathcal{U}}_0(\hat{k}_\perp)\phi_\ell(\hat{k}_\perp) + a_n \dots \\ + \imath\hat{\Lambda}_p^2 \left[a'_\ell - \imath Q_{m,\ell} a_m\right] \Big\} \end{aligned}$$

where $Q_{m,\ell} = \int d^2q\; q^2\; \phi_m(q)\; \phi_\ell(q)$. If the individual ϕ_n has an exponential or faster drop-off for large q, then these terms will generally be fairly small, as the integrand is close to zero for $q < 1$ and drops off exponentially outside that range. This lends itself to a perturbative expansion in the Q matrix to get at least the first order coupling between modes.

The integral equation can be solved by use of a Laplace transform. Applying the Laplace transform in $\hat{z}$ gives the equation for the Laplace transformed A_ℓ to be

$$\begin{aligned} \Big[\left(s - \hat{D}/\omega_m(1+\imath s\hat{\Lambda}_p^2)\right)\delta_{\ell,m} - \dots \\ (1+\imath\hat{\Lambda}_p^2/\omega_m)Q_{\ell,m}\Big] A_m = \\ \left(1+\imath\hat{\Lambda}_p^2/\omega_\ell\right) a_\ell(0) + \tilde{F}_1^\ell + \hat{U}_0^\ell \end{aligned} \tag{14}$$

where a quantity $\mathcal{G}^\ell$ is the initial condition quantity integrated with the ℓ^{th} eigenmode.

To zeroth order in $Q_{m,\ell}$ the equations of motion for each mode behaves like a one-dimensional growth with the root equation being modified by appropriate factors of ω_ℓ. The

$$\tilde{f}_1 = e^{\imath\phi_0 z}\tilde{f}_1\mid_0 + \int_0^z dz'\, e^{\imath\phi_0(z'-z)} \int d^2q\, e^{\imath\frac{(q^2-k_\perp^2)c}{2\nu\omega_r}z'} \times \left[\imath\nu\omega_r\left(\frac{e}{c}\right)^2 \frac{1}{\mathcal{E}_0}\left(\mathcal{U}_0 - \frac{\imath\pi K}{\nu\omega_r\gamma_0}\int_0^{z'} \tilde{j}_z dz''\right) + \frac{4\pi e\imath}{c\nu\omega_r}\tilde{j}_z\right] n_0\frac{dF}{d\mathcal{E}}\tilde{G}(k_\perp - q) \tag{8}$$

real part of the roots are a monotonically decreasing function of ω_ℓ, so that the minimal value of ω_ℓ dominates.

The first order inverse in powers of $Q_{m,\ell}$ is given by

$$A_m \approx \left[\left(s - \hat{D}/\omega_m(1+\imath s\hat{\Lambda}_p^2)\right)^{-1}\delta_{\ell,m} - \ldots \frac{1+\imath\hat{\Lambda}_p^2/\omega_m}{\left(s-\hat{D}/\omega_m(1+\imath s\hat{\Lambda}_p^2)\right)^2}Q_{\ell,m}\right]\times \left[\left(1+\imath\hat{\Lambda}_p^2/\omega_\ell\right)a_\ell(0) + \tilde{F}_1^\ell + \hat{U}_0^\ell\right] \tag{15}$$

THE EIGENVALUE PROBLEM

The first obvious problem is to consider calculating the eigenvectors and eigenvalues of the transverse beam size kernel, $\tilde{G}(\hat{k}-\hat{q})$. A convenient method for doing this would be to consider a countable set of orthonormal functions in k-space and then write down an equation for the series expansion of the eigenfunctions in terms of orthonormal functions

Suppose the eigenfunction can be written as

$$\psi_\ell = \sum_{m,n} a_{\ell,m,n}F_m(\hat{k})\cos(n\theta) \tag{16}$$

while a Gaussian beam profile, for example, can be written as a series of Bessel functions

$$\exp\left(-L^2(\vec{k}_\perp^2 - \vec{q}^2)\right) = \exp\left(-L^2(\hat{k}^2+\hat{q}^2)\right)\sum_\ell J_\ell(L^2\hat{k}\hat{q})\cos(\ell(\theta-\phi)) \tag{17}$$

where θ and ϕ are the angular components of $k_\perp$ and q respectively, and J_ℓ is the ℓ^{th} Bessel function. Due to the properties of the angular integrals, each angular mode is independent, and the series expansion boils down to calculating the solutions to the eigenvalue equation

$$\int d\hat{k}\, \hat{k}F_m(\hat{k})J_m(L^2\hat{k}\hat{q})e^{-L^2(\hat{k}^2+\hat{q}^2)} = \lambda_m F_m(\hat{q}) \tag{18}$$

If one can safely assume that F_m can be expanded as a series of orthonormal functions, then the series may be expanded and each individual term in the series is given by

$$\lambda_m b_n = \int d\hat{q}\, d\hat{k}\, \hat{k}\psi_n(\hat{k})\psi_n(\hat{q})J_n(L^2\hat{k}\hat{q})e^{-L^2(\hat{k}^2+\hat{q}^2)} \tag{19}$$

where ψ_n is some orthonormal function, and it is assumed that

$$F_m = \sum_n b_n\psi_n$$

and

$$\int d\hat{k}\psi_a(\hat{k})\psi_b(\hat{k}) = \delta_{a,b}$$

Once the individual components are solved, then it is relatively straightforward to calculate the corresponding eigenvalue by evaluating

$$\lambda_m \approx \frac{\int d\hat{q}\, d\hat{k}\, \hat{k}\, F_m(\hat{k})F_m(\hat{q})J_m(L^2\hat{k}\hat{q})e^{-L^2(\hat{k}^2+\hat{q}^2)}}{\int d\hat{q}\, d\hat{k}\, \hat{k}F_m(\hat{k})F_m(\hat{q})} \tag{20}$$

Further elaboration would require a particular choice of expansion functions. It should be clear that $\lambda_m = 1/\omega_m \leq 1$, so that the finite transverse size suppresses the growth of the individual modes.

FEL GREEN FUNCTION

From these results, a Green function for an initial phase space distribution can be calculated, given an initial perturbation of the form

$$\tilde{f}_1\mid_0 = \frac{1}{\sqrt{2\pi}^3}\frac{2\nu}{\rho\mathcal{E}_0}e^{-\imath\hat{k}_\perp\cdot\hat{r}_{\perp 0}}e^{\imath(1-\rho\hat{C})\omega_r t_0}\delta(\hat{P}-\hat{P}_0)$$

The current Green function for the infinite transverse beam limit is given by

$$\mathcal{G}(\hat{\vec{r}}_{\perp 0}, t_0, \hat{P}_0) = -ec\frac{1}{\sqrt{2\pi}^3}\sum_\jmath \frac{s_\jmath e^{s_\jmath\hat{z}}}{1-\hat{D}'_\jmath + \imath\hat{\Lambda}_p^2\left(\hat{D}_\jmath + s_\jmath\hat{D}'_\jmath\right)}\times \frac{1}{s_\jmath + \imath(\hat{C}+\hat{k}_\perp^2+\hat{P}_0+\hat{k}_\perp^2)}\left(\frac{1}{\sqrt{2\pi}^3}e^{-\imath\vec{\hat{k}}_\perp\cdot\vec{\hat{r}}_{\perp 0}}e^{\imath(1-\rho\hat{C})\omega_r t_0}\right) \tag{21}$$

while the phase space Green function is obtained from the equation of motion as

$$\mathcal{G}_{\text{FEL}} = e^{\imath(\hat{C}+\hat{P}+\hat{k}_\perp^2)\hat{z}} + \sum_\jmath \frac{1}{1-\hat{D}'_\jmath + \imath\hat{\Lambda}_p^2\left(\hat{D}_\jmath + s_\jmath \hat{D}'_\jmath\right)} \left(\frac{1}{s_\jmath + \imath(\hat{C}+\hat{P}_0+\hat{k}_\perp^2)}\right)^2 \times \left\{(1+\imath\hat{\Lambda}_p^2 s_\jmath)\left(e^{s_\jmath\hat{z}} - e^{-\imath(\hat{C}+\hat{P}_0+\hat{k}_\perp^2)\hat{z}}\right) - \left(1-e^{-\imath(\hat{C}+\hat{P}_0+\hat{k}_\perp^2)\hat{z}}\right)\right\}\frac{d\hat{F}}{d\hat{P}_0} \times e^{-\imath\hat{k}_\perp\cdot\hat{r}_{\perp 0}} e^{\imath(1-\rho\hat{C})\omega_r t_0}\delta(\hat{P}-\hat{P}_0) \quad (22)$$

Here the $s_\jmath$ refer to the roots of the dispersion relation. These Green functions neglect the oscillating term that arises in the FEL process due to the pole at $s = \imath(\hat{P}+\hat{C})$ which becomes a decaying root for a thermal spread in $\tilde{f}_1$.

CONCLUSION

We have presented a model for the three-dimensional dynamics of a high gain free-electron laser which generalizes to the infinite beam limit readily, and which provides a relatively simple set of equations for the dynamics of the individual modes of a finite transverse beam profile. An avenue is presented for approximating the transverse eigenmodes, and a preliminary though incomplete study of this method suggests that the finite beam model includes optical guiding.

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