Stochastic Temporal Behavior of SASE FEL

Samuel Krinsky
BNL

FEL 2009, Liverpool, UK
August 24, 2009

Acknowledgements

I have greatly benefited from collaborations with:

Ilan Ben-Zvi
Robert Gluckstern
Zhirong Huang
Yuelin Li
Li-Hua Yu

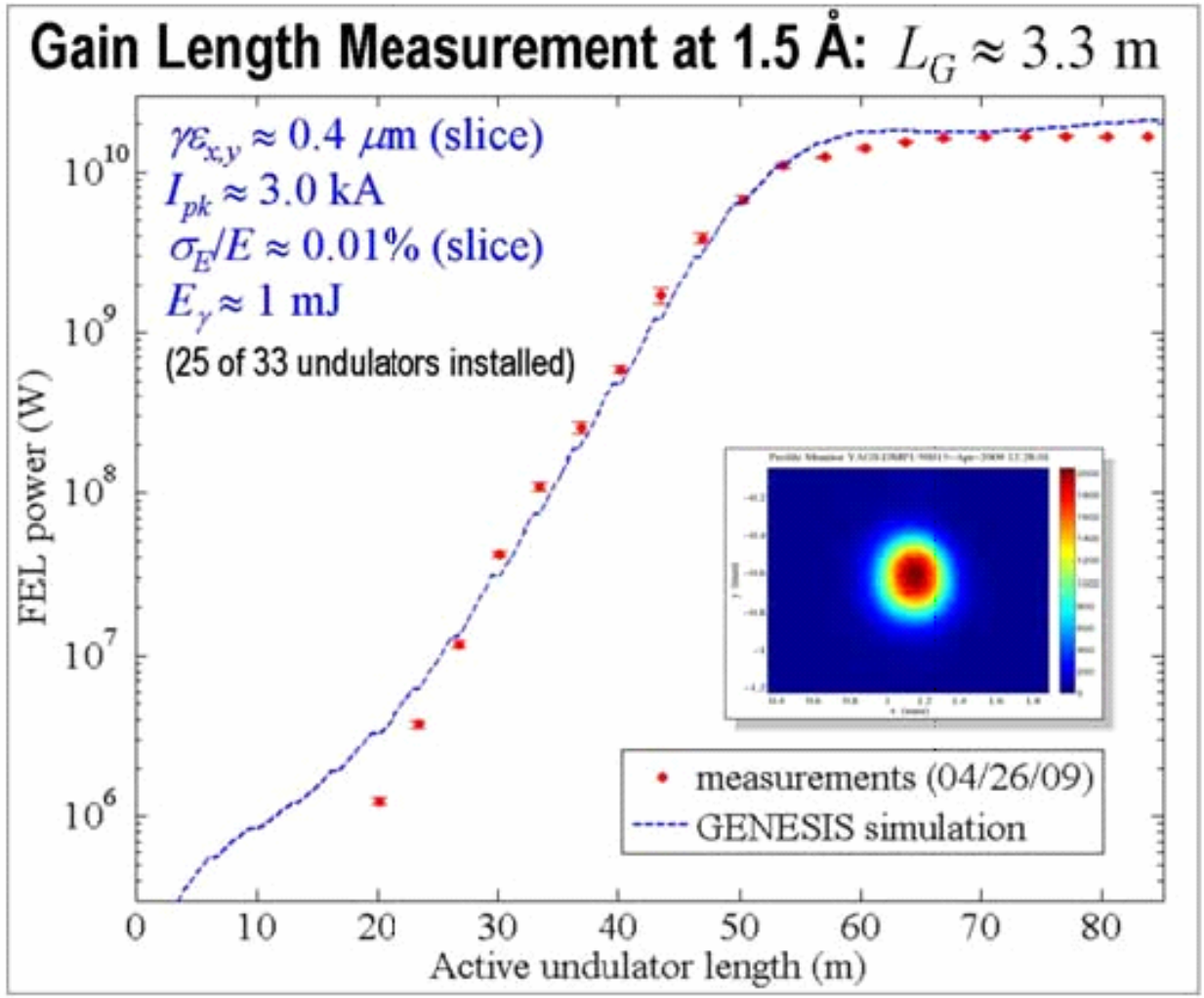
Early Work on SASE

C. Pellegrini and E. Saldin
Suggested importance of SASE

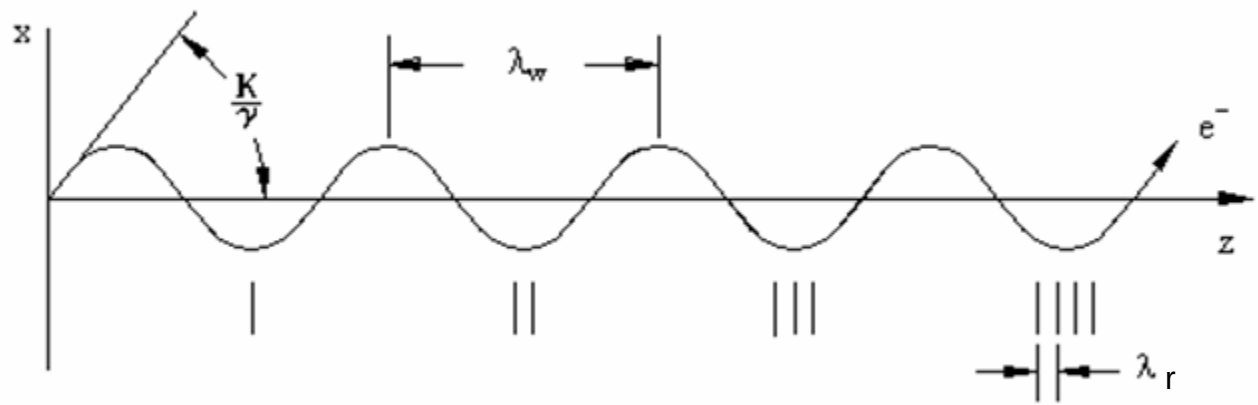
K.J. Kim and J.M Wang & L.H. Yu
1-D Theory of Start-up from Shot Noise

K.J. Kim and S. Krinsky & L.H. Yu
SASE Power in Guided Modes

SASE Saturation at LCLS



Spontaneous Undulator Radiation



Fundamental radiation wavelength

$$\lambda_r = \frac{\lambda_w}{2\gamma_0^2}(1+a_w^2)$$

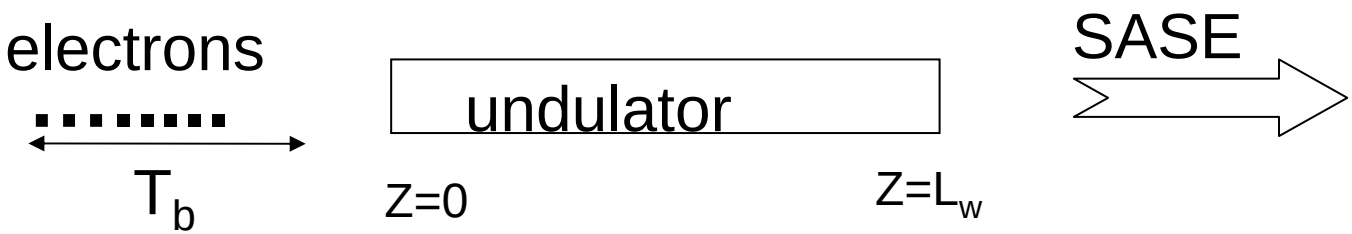
Radiation bandwidth

$$\frac{\Delta\lambda}{\lambda_r} = \frac{1}{N_w}$$

Slippage distance

$$\ell_s = N_w \lambda_r$$

Shot Noise

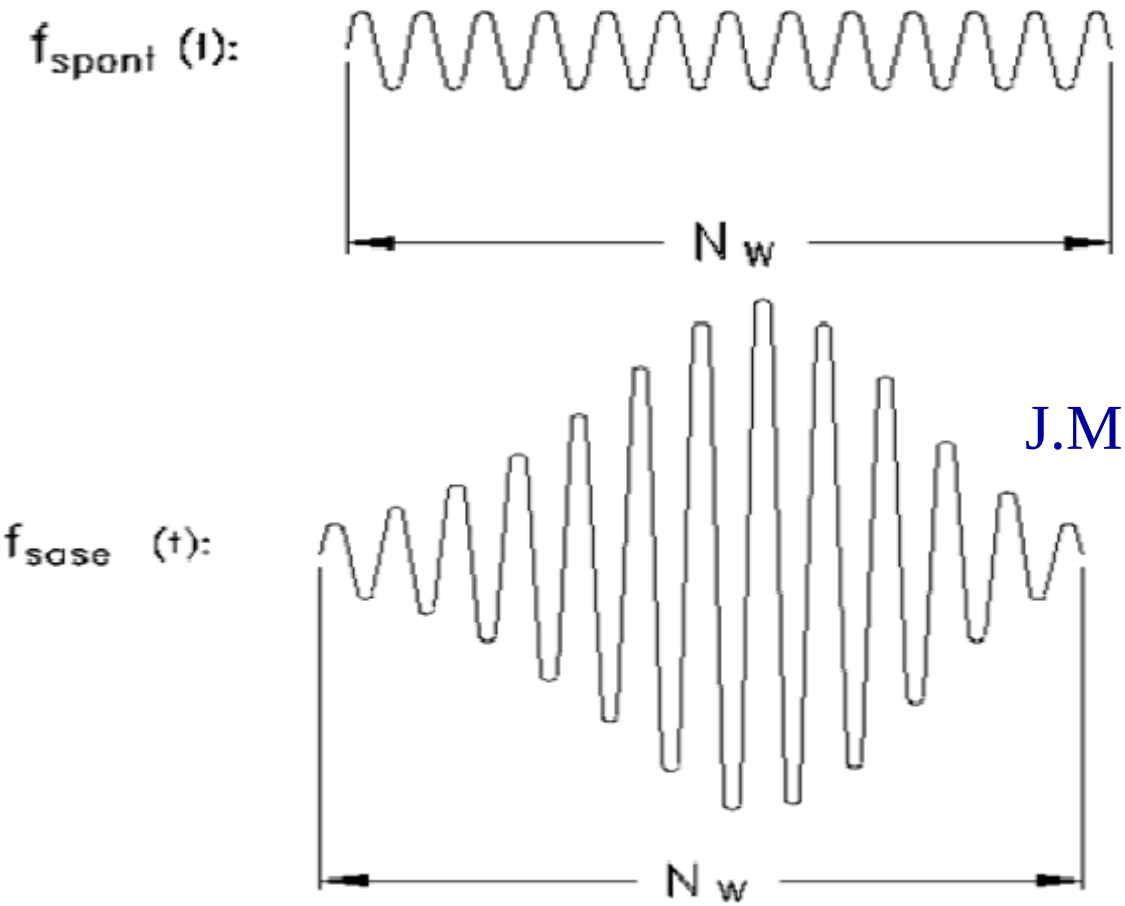


t_j : arrival time of jth electron at z=0

t_j stochastically distributed over $0< t_j <T_b$

Radiated Field in Linear Regime Before Saturation

$$E(t) = \sum_j f(t - t_j)$$



J.M. Wang & L.H. Yu (1986)

K.J. Kim (1986)

Distributions for Instantaneous Power and Pulse Energy

Central Limit Theorem implies instantaneous power follows the Negative Exponential Distribution:

$$p(P) = \exp(-P / P_{av})$$

The energy in a single SASE pulse is

$$W(z) \propto \int_0^{T_b} |E(z, t)|^2 \, dt$$

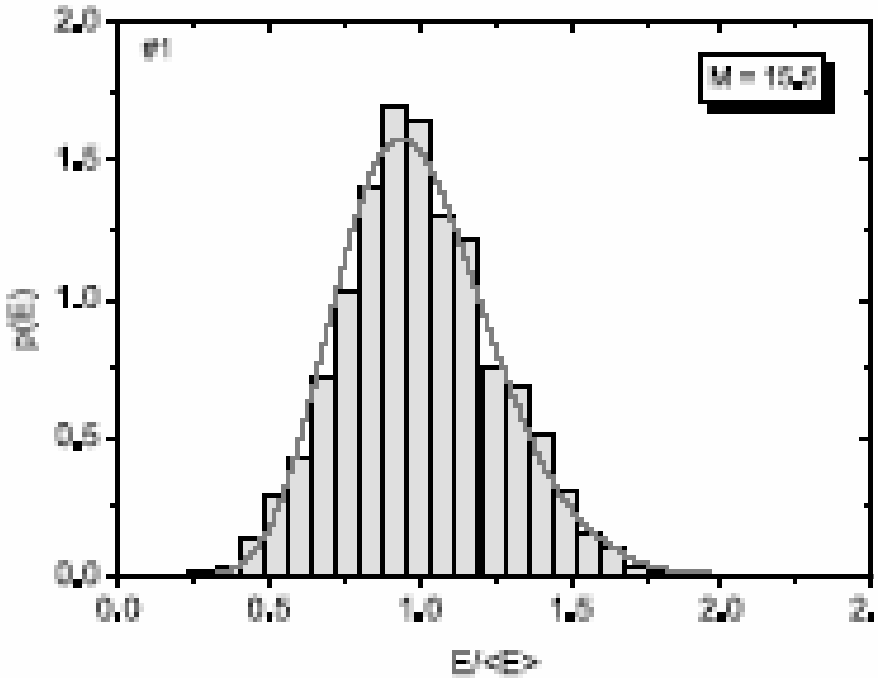
Pulse Energy fluctuation

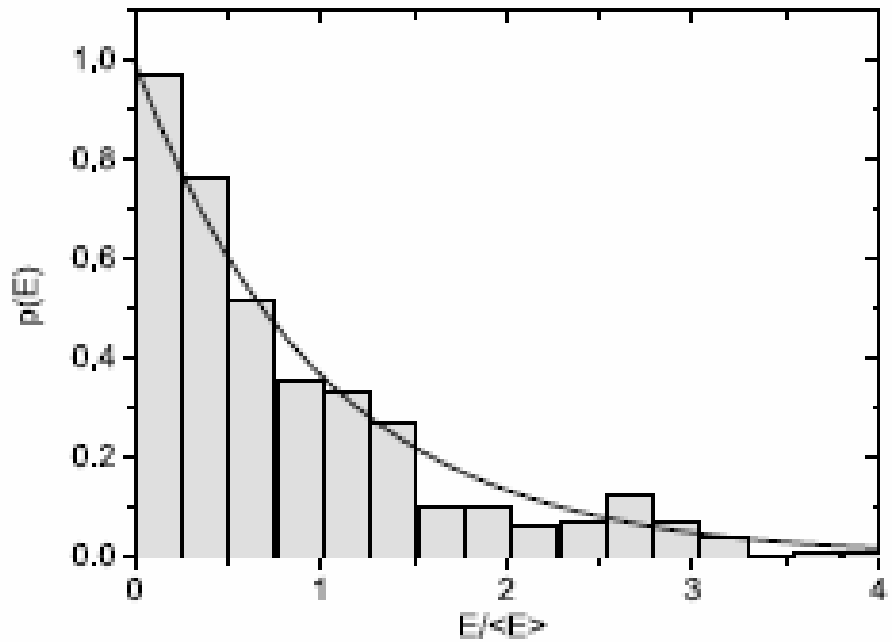
$$\frac{\sigma_W^2}{W^2} = \frac{\langle W - \langle W \rangle \rangle^2}{\langle W \rangle^2} = \frac{1}{M} \equiv \frac{T_{coh}}{T_b} \qquad T_{coh} = \sqrt{\pi} / \sigma_\omega$$

E. Saldin et al (1998)

L.H. Yu & S.Krinsky (1998)

TTF/DESY





Distribution of Pulse Energy: Gamma Distribution

Distribution of Energy after Monochromator: Exponential Distribution

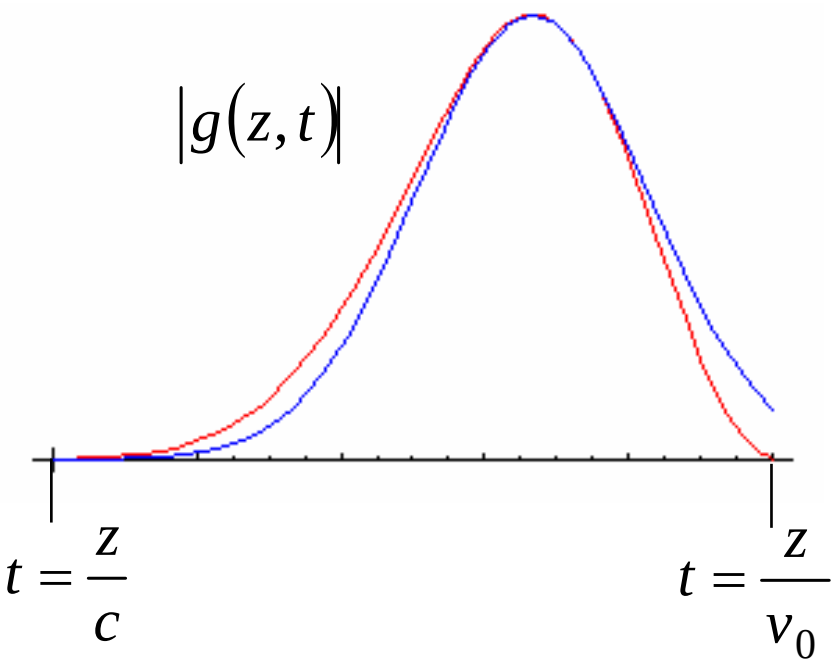
E. Saldin et al (1998)

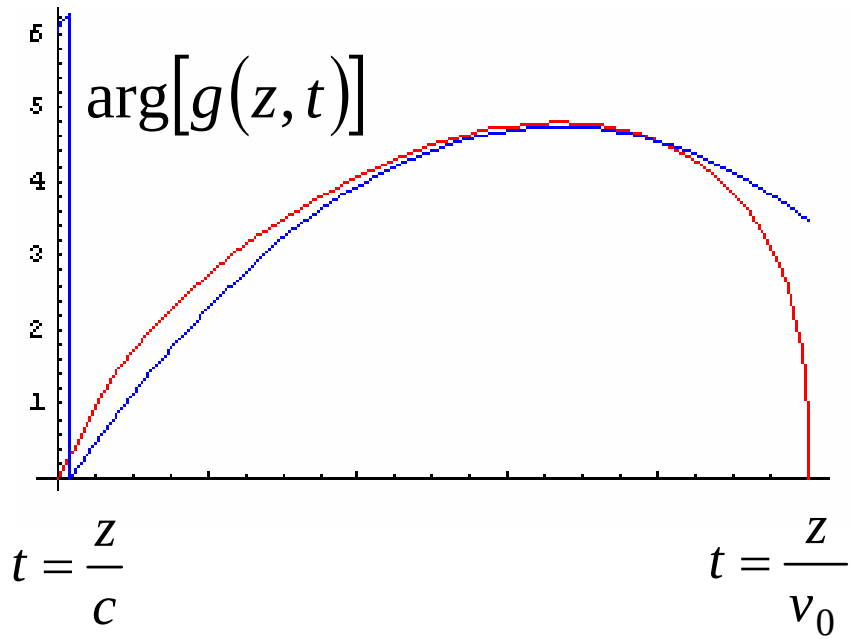
S.O. Rice (1945)

Green's Function

$$E(z,t) = A(z,t)\exp(ik_r z - i\omega_r t)$$

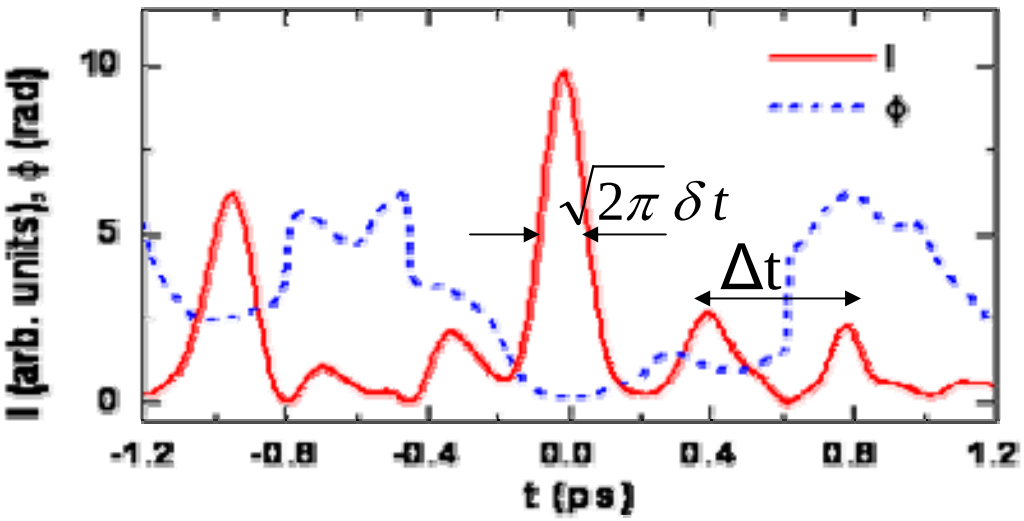
$$A(t,z) = \sum_{j=i}^{N_e} g\left(z, t - t_j\right)\exp\left(i\,\omega_r\, t_j\right)$$





In exponential regime, the Green's function (red) is well approximated by a Gaussian (blue).

Simulation of Intensity and Phase Evolution



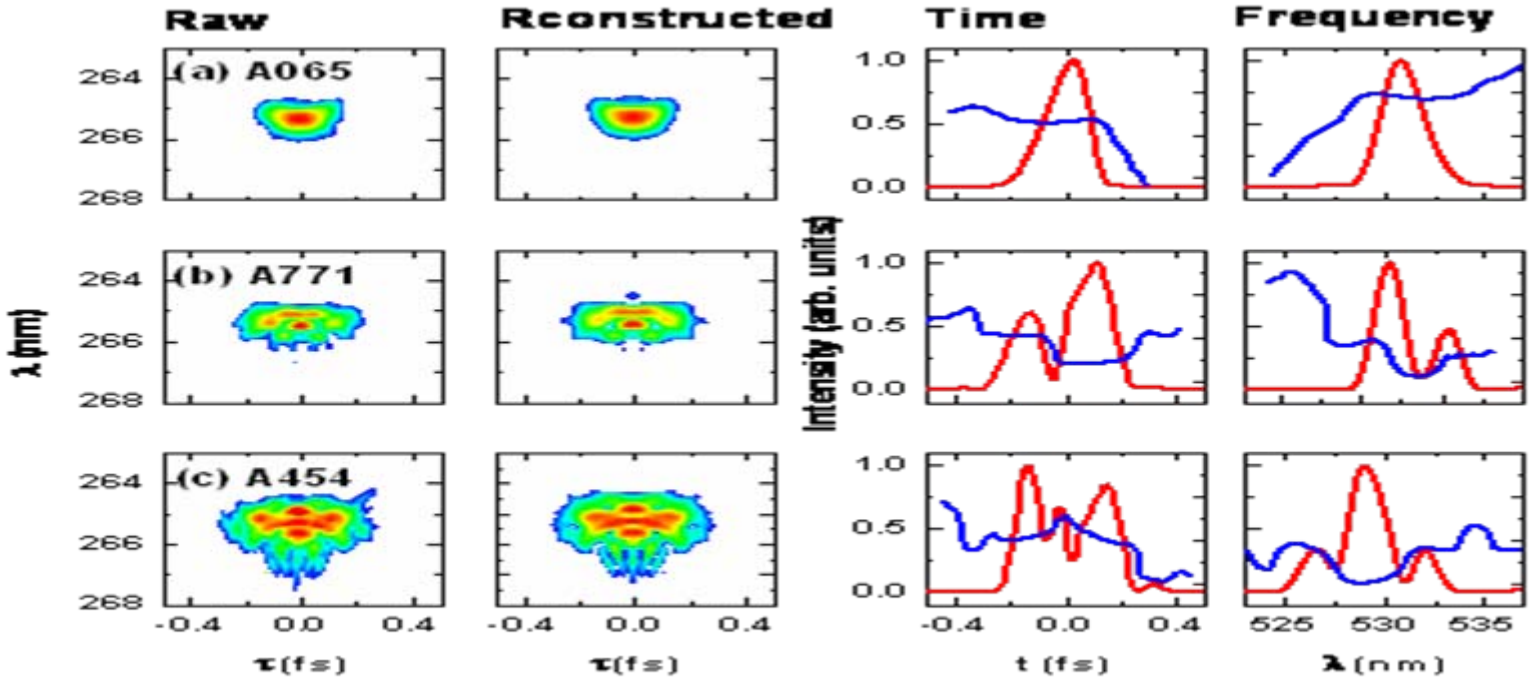
$$E(z,t) = A(z,t)\exp(ik_r z - i\omega_r t)$$

$$A = \sqrt{I}\, e^{i\phi}$$

$$A(t,z) \cong A_0(z)\sum_{j=i}^{N_e}\exp\left[i\omega_r t_j - \frac{(t-t_j - z/v_g)^2}{4\sigma^2}\left(1+\frac{i}{\sqrt{3}}\right)\right]$$

$$\sigma = \frac{1}{\sqrt{3}\,\sigma_\omega} \qquad \sigma_\omega = \omega_r\sqrt{3\sqrt{3}\rho / k_w z}$$

LEUTL: FROG Measurements



FROG Traces Red: Intensity; Blue: Phase

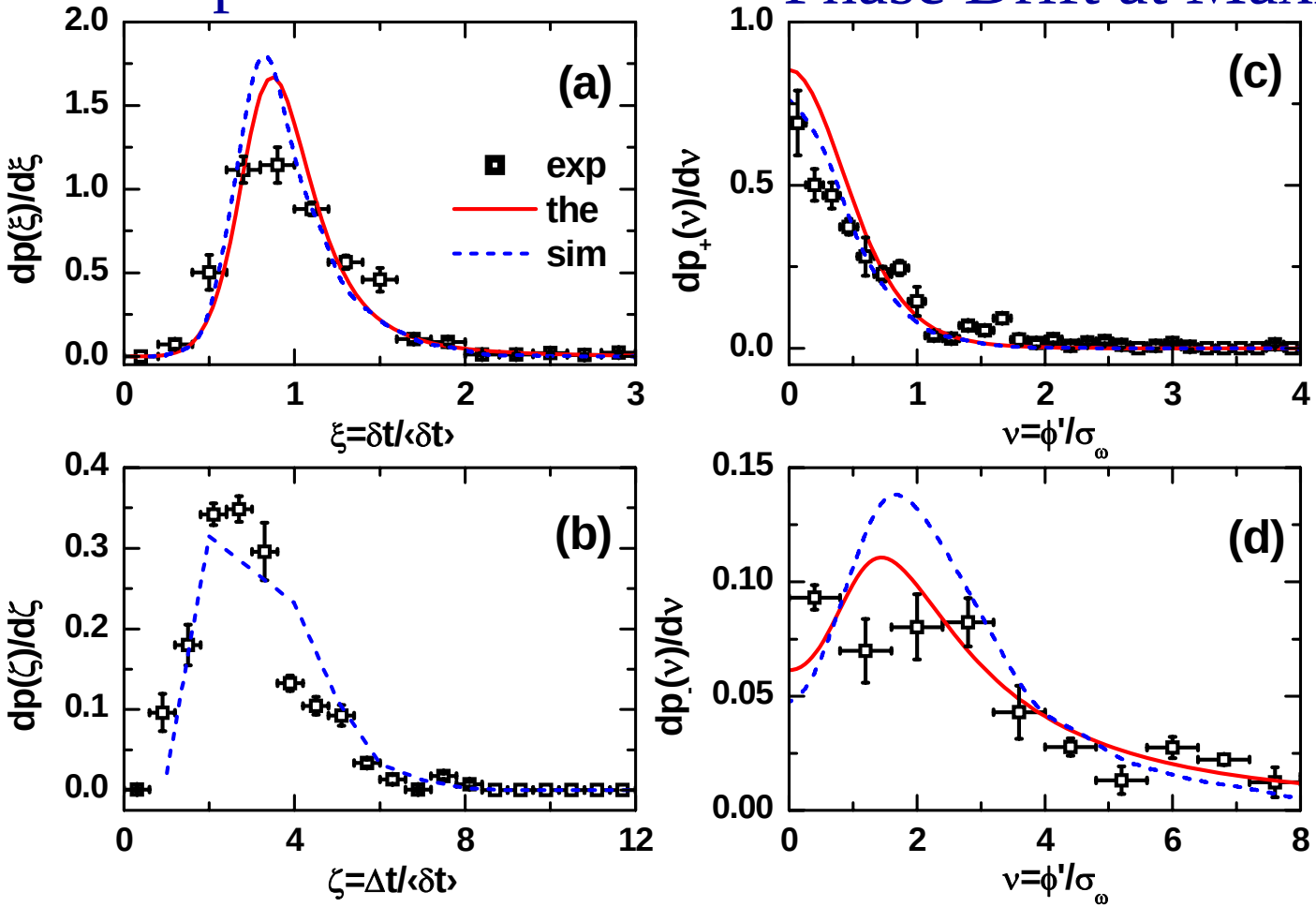
$$I_{FROG}(\omega,\tau) \propto \left| \int_{-\infty}^{\infty} E(t)E(t-\tau)\exp(-i\omega t)dt \right|^2$$

Y. Li et al (2003)

Distributions Characterizing Intensity Spikes

Spike Width

Phase Drift at Maximum



S.O. Rice (1945)

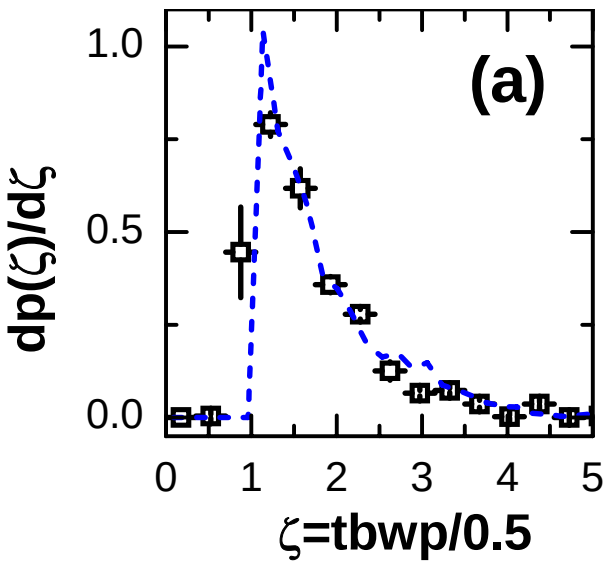
Y. Li et al (2003)

Spike Separation

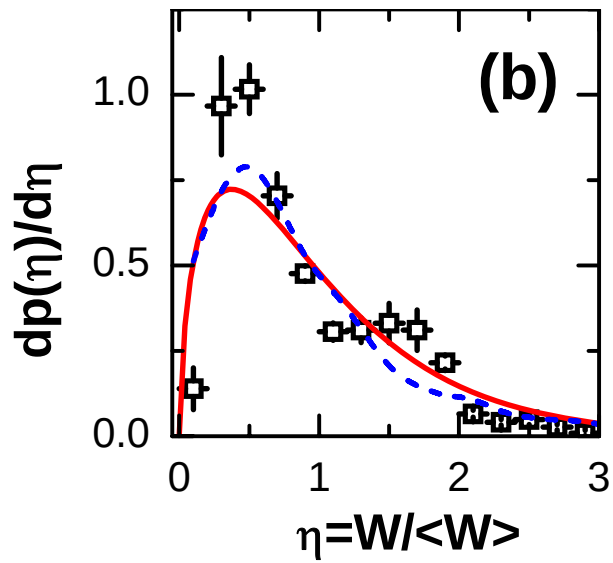
Phase Drift at Minimum

LEUTL: FROG Measurements

Pulse Time-Bandwidth Product



Pulse Energy

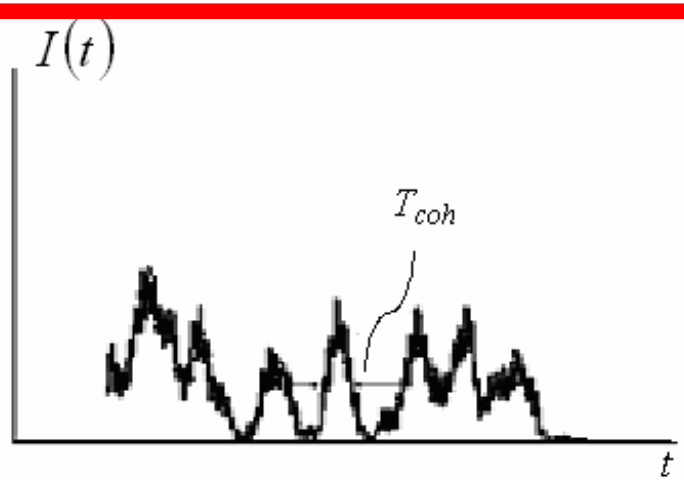


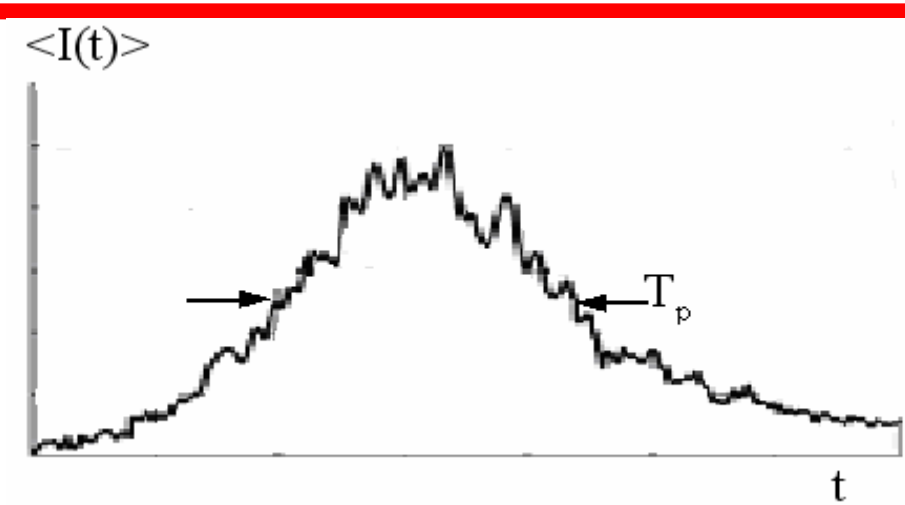
Y. Li et al (2005)

$$M \cong 2\langle tbwp \rangle \cong 1.8$$

Pulses with the highest energy had lowest time-bandwidth product (tbwp).

SASE Statistics: Time Domain





$$M \cong 2\,\sigma_t\,\sigma_\omega$$ Number of Modes in Pulse

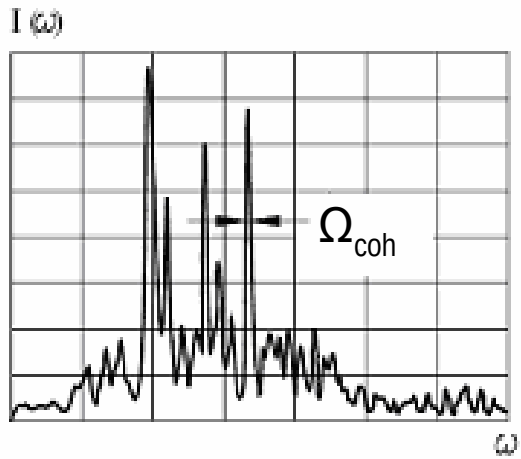
$$T_p \equiv 2\sqrt{\pi}\,\sigma_t$$ Average Pulse Duration

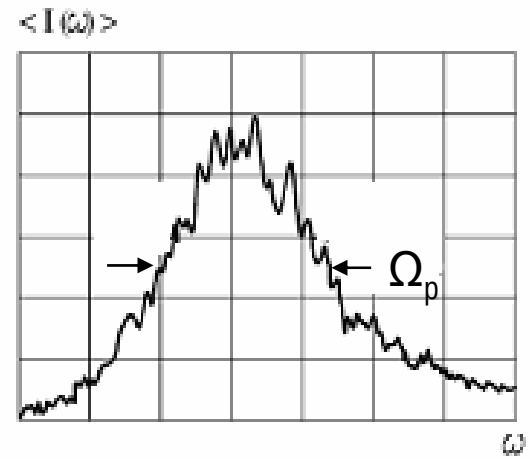
$$T_{coh} \equiv T_p / M \cong \sqrt{\pi} / \sigma_\omega$$ Coherence Time

$$\langle \delta t \rangle \cong T_{coh} / \sqrt{2\pi}$$ RMS Spike Width

$$\langle \Delta t \rangle \cong \sqrt{2\pi} / \sigma_\omega$$ Average Spike Separation

SASE Statistics: Frequency Domain





$$M \cong 2\,\sigma_t\,\sigma_\omega$$ Number of Modes in Pulse

$$\Omega_p \equiv 2\sqrt{\pi}\,\sigma_\omega$$ Average Spectral Width

$$\Omega_{coh} \equiv \Omega_p / M \cong \sqrt{\pi} / \sigma_t$$ Range of Spectral Coherence

$$\langle \delta\omega \rangle \cong \Omega_{coh} / \sqrt{2\pi}$$ RMS Spike Width

$$\langle \Delta\omega \rangle \cong \sqrt{2\pi} / \sigma_t$$ RMS Spike Separation

Mathematical Formalism

Wigner Function has Properties of Phase Space Density

$$W(t,\omega) = \int d\tau \left\langle E\left(t - \frac{\tau}{2}\right) E^*\left(t + \frac{\tau}{2}\right)\right\rangle \exp(-i\omega\tau)$$ Wigner Function

$$\left\langle |E(t)|^2 \right\rangle = \int \frac{d\omega}{2\pi} W(t,\omega)$$ Average Intensity

$$\left\langle \left|\tilde{E}(\omega)\right|^2 \right\rangle = \int dt\, W(t,\omega)$$ Average Spectral Intensity

$$W = \int \frac{dt\,d\omega}{2\pi} W(t,\omega) = \int dt \left\langle |E(t)|^2 \right\rangle$$ Integrated Intensity

Time Domain

Define Number of Modes M to be ratio of phase space area occupied by pulse divided by minimum phase space area

$$\frac{1}{M} \equiv \frac{\int \frac{dt d\omega}{2\pi} W^2(t,\omega)}{\left(\int \frac{dt d\omega}{2\pi} W(t,\omega)\right)^2} = \frac{\int dt_1 dt_2 \left|\left\langle E(t_1) E^*(t_2)\right\rangle\right|^2}{\left(\int dt \left\langle \left|E(t)\right|\right\rangle^2\right)^2} \qquad (M \geq 1)$$

$$\frac{1}{T_p} = \frac{\int dt \left\langle \left|E(t)\right|^2 \right\rangle^2}{\left(\int dt \left\langle \left|E(t)\right|^2 \right\rangle\right)^2}$$ Pulse Width

$$T_{coh} = \frac{T_p}{M} = \frac{\int dt_1 dt_2 \left|\left\langle E(t_1) E^*(t_2)\right\rangle\right|^2}{\int dt \left\langle \left|E(t)\right|^2 \right\rangle^2}$$ Coherence Time

Definition of Pulse Width

$$\frac{1}{T_p} = \frac{\int dt \left\langle \left| E(t) \right|^2 \right\rangle^2}{\left(\int dt \left\langle \left| E(t) \right|^2 \right\rangle \right)^2}$$

Top Hat Distribution: $\left\langle \left| E(t) \right|^2 \right\rangle = \begin{Bmatrix} E_0^2 & 0 \le t \le T \\ 0 & otherwise \end{Bmatrix}$

$$T_p = T$$

Gaussian Distribution: $\left\langle \left| E(t) \right|^2 \right\rangle = E_0^2 \exp\left(-\frac{t^2}{2\sigma_t^2} \right)$

$$T_p = 2\sqrt{\pi}\,\sigma_t$$

Frequency Domain

$$\frac{1}{M} = \frac{\int \frac{d\omega_1}{2\pi}\frac{d\omega_2}{2\pi}\left|\left\langle \tilde{E}(\omega_1)\tilde{E}(\omega_2)\right\rangle\right|^2}{W^2}$$

$$\frac{1}{\Omega_p} = \frac{\int d\omega \left\langle \left|\tilde{E}(\omega)\right|^2 \right\rangle^2}{\left(\int d\omega \left\langle \left|\tilde{E}(\omega)\right|^2 \right\rangle\right)^2}$$

Spectral Width

$$\Omega_{coh} = \frac{\Omega_p}{M} = \frac{\int d\omega_1 d\omega_2 \left|\left\langle \tilde{E}(\omega_1)\tilde{E}^*(\omega_2)\right\rangle\right|^2}{\int d\omega \left\langle \left|\tilde{E}(\omega)\right|^2 \right\rangle^2}$$

Range of Spectral Coherence

Energy Fluctuation

$$\sigma_W^2 = \int dt_1 dt_2 \left[\left\langle |E(t_1)|^2 |E(t_2)|^2 \right\rangle - \left\langle |E(t_1)|^2 \right\rangle \left\langle |E(t_2)|^2 \right\rangle \right]$$

For Gaussian Process with Zero Mean

(SASE Before Saturation)

$$\left| \left\langle E(t_1) E^*(t_2) \right\rangle \right|^2 = \left\langle |E(t_1)|^2 |E(t_2)|^2 \right\rangle - \left\langle |E(t_1)|^2 \right\rangle \left\langle |E(t_2)|^2 \right\rangle$$

$$\boxed{\frac{1}{M} = \frac{\int dt_1 dt_2 \left| \left\langle E(t_1) E^*(t_2) \right\rangle \right|^2}{W^2} = \frac{\sigma_W^2}{W^2}}$$

Simplified Model of SASE Pulse

Gaussian Electron Bunch

$$w_b\left(t_j\right) = \frac{1}{\sqrt{2\pi}\ \sigma_b} \exp\left(-t_j^2 / 2\sigma_b^2\right)$$

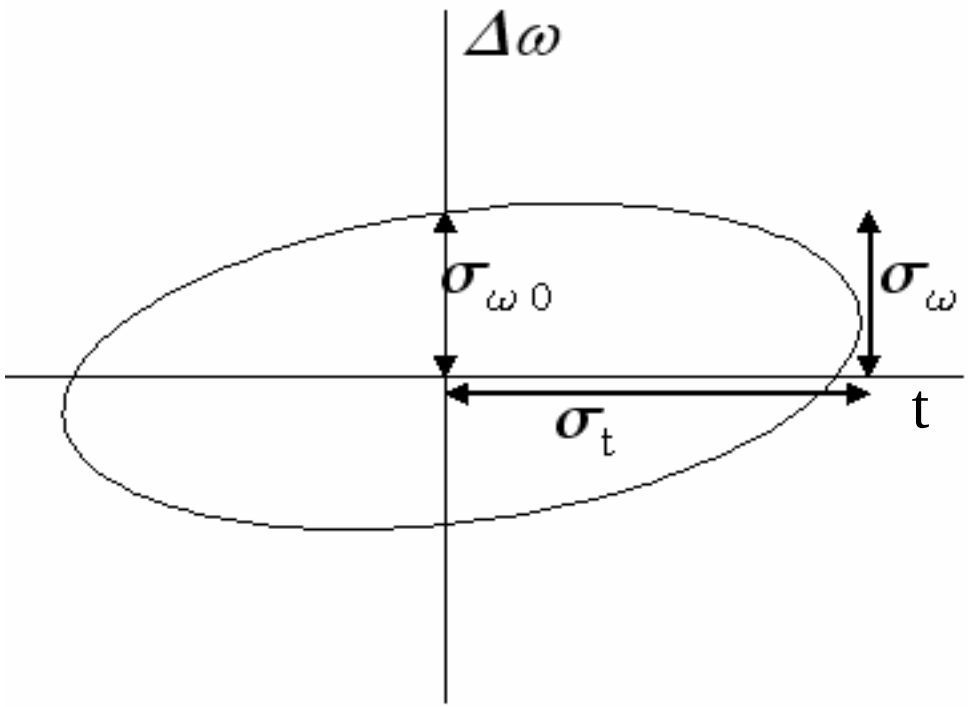
SASE Field Amplitude

$$A(t) \cong A_0 \sum_{j=1}^{N_e} \exp\left[i\,\omega_r\, t_j - \frac{\left(t - t_j\right)^2}{4\sigma^2}\left(1 + \frac{i}{\sqrt{3}}\right)\right]$$

$$\omega_r \sigma_b >> 1 \quad and \quad \omega_r \sigma >> 1 \quad \Longrightarrow \quad \langle A(t) \rangle \approx 0$$

Model Wigner Function

$$W(t,\omega) = \frac{N_e \sigma A_0^2 \sqrt{2\pi}}{\sigma_t \sigma_{\omega 0}} \exp\left(-\frac{t^2}{2\sigma_t^2} - \frac{(\omega - \omega_r - \mu t)^2}{2\sigma_{\omega 0}^2} \right)$$



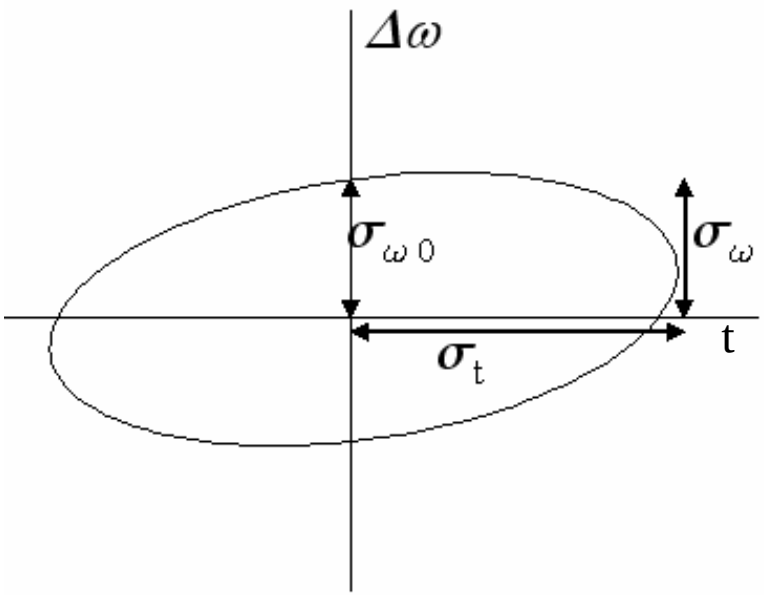
$$\sigma_t^2 = \sigma_b^2 + \sigma^2$$

$$\sigma_\omega^2 = \frac{1}{3\sigma^2}$$

$$\sigma_{\omega 0}^2 = \sigma_\omega^2 - \frac{1}{12\sigma_t^2}$$

$$\mu = \frac{1}{2\sqrt{3}\,\sigma_t^2}$$

Model Statistical Properties



$$M = 2\,\sigma_t\,\sigma_{\omega 0} = \sqrt{4\sigma_b^2\,\sigma_\omega^2 + 1}$$

$$\frac{\sigma_W}{W} = \frac{1}{\sqrt{M}}$$

$$T_p = 2\sqrt{\pi}\,\sigma_t \qquad \Omega_p = 2\sqrt{\pi}\,\sigma_\omega$$

$$T_{coh} = \frac{T_p}{M} = \frac{\sqrt{\pi}}{\sigma_{\omega 0}} \qquad \Omega_{coh} = \frac{\Omega_p}{M} = \sqrt{\frac{\pi}{\sigma_b^2 + \frac{1}{4\sigma_\omega^2}}}$$

Frequency Chirped SASE

Energy chirped electron beam $\frac{\gamma_j - \gamma_0}{\gamma_0} = \alpha \frac{t_j}{T_b}$

Leads to frequency chirped SASE

$$\omega_j = \omega_0 + u t_j \qquad u = 2\alpha\omega_0 / T_b$$

C. Schroeder et al (2002)

$$E(z,t) \propto \sum_{j=1}^{N_e} e^{ik_j z - i\omega_j (t - t_j)} g(z, t - t_j; u)$$

$$g(z, t - t_j; u) \cong e^{\rho(\sqrt{3}+i)k_w z} e^{-b\left(t - t_j - \frac{z}{v_g}\right)^2} e^{-\frac{iu}{2}\left(t - t_j - \frac{z}{v_0}\right)\left(t - t_j - \frac{z}{c}\right)}$$

$$b = \frac{3}{4}\left(1 + \frac{i}{\sqrt{3}}\right)\sigma_\omega^2$$

S. Krinsky & Z. Huang (2003)

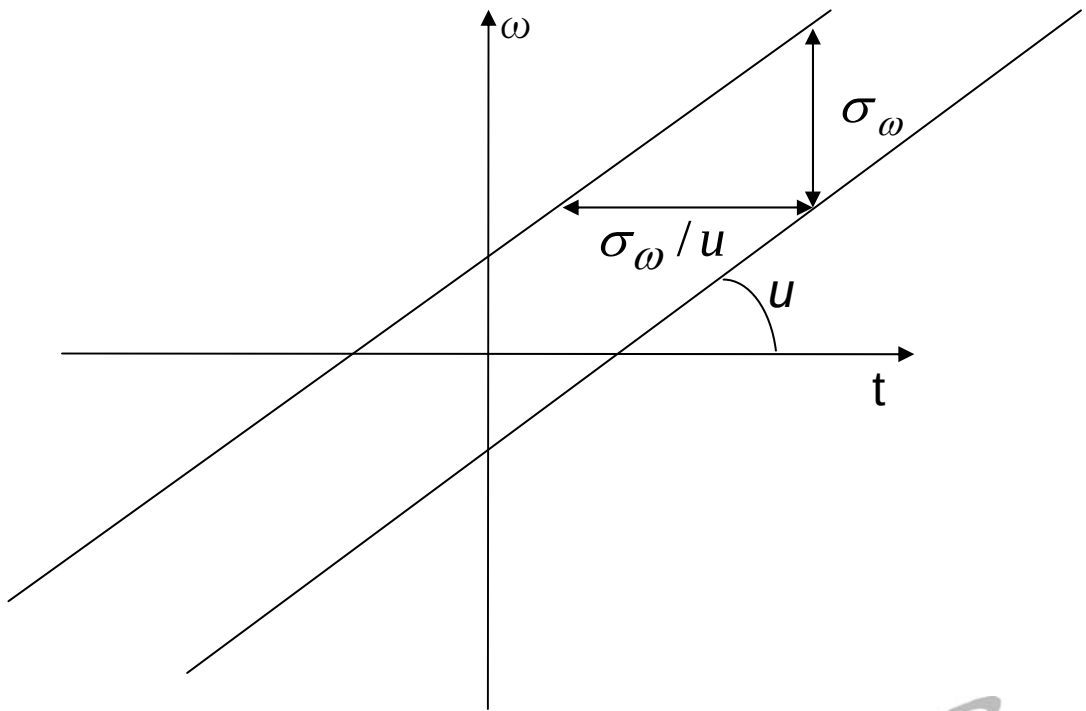
Wigner Function for Frequency Chirped SASE

$$W(z;t,\omega) \propto e^{2\rho\sqrt{3}\,k_w z} \exp\left(-\frac{1}{2\sigma_\omega^2}\left[\omega-\omega_0-u\left(t-\frac{1}{2}\left(\frac{z}{v_0}+\frac{z}{c}\right)\right)\right]^2\right)$$

$$M = \frac{\sigma_\omega T_b}{\sqrt{\pi}}$$

$$T_{coh} = \frac{T_b}{M} = \frac{\sqrt{\pi}}{\sigma_\omega}$$

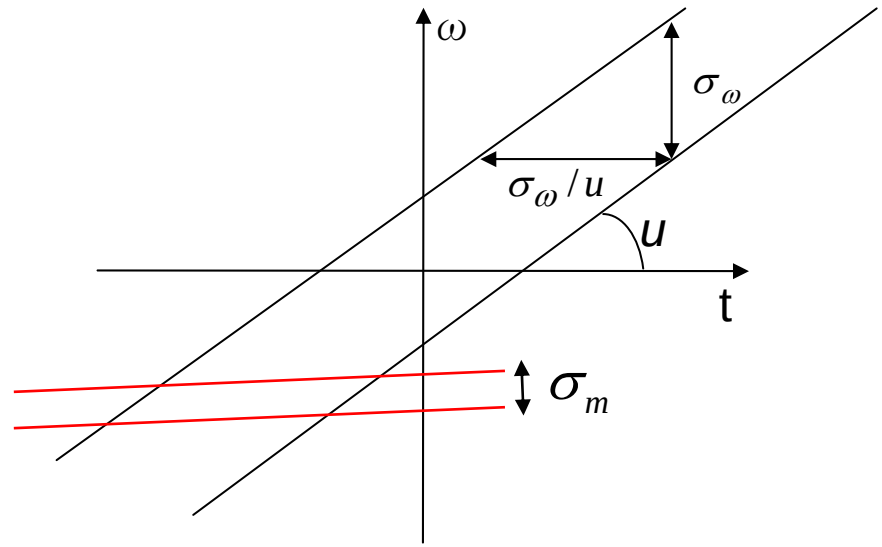
$$\Omega_{coh} = \frac{|u|\,T_b}{M} = \frac{\sqrt{\pi}}{\sigma_\omega / |u|}$$



Pulse Slicing Using Monochromator

$$\tilde{E}_F(z,\omega) = \tilde{E}(z,\omega)\exp\left[-\frac{(\omega-\omega_m)^2}{4\sigma_m^2}\right]$$

$$\left\langle \left|E_F(z,t)\right|^2 \right\rangle \propto \exp\left(-\frac{(t-t_m(z))^2}{2\sigma_t^2}\right)$$

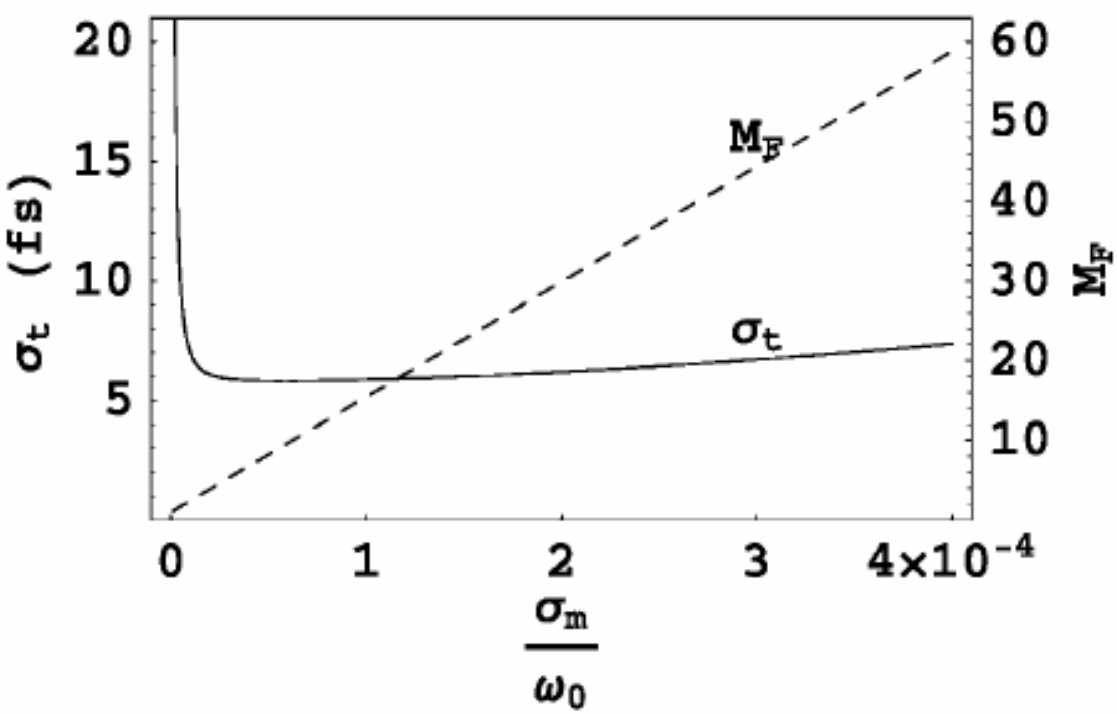


$$\sigma_t^2 = \frac{\sigma_\omega^2 + \sigma_m^2}{u^2} + \frac{1}{4\sigma_m^2}$$

$$M_F = \sqrt{\frac{4\sigma_m^2\sigma_\omega^2}{u^2} + 1} \qquad \frac{\sigma_W}{W} = \frac{1}{\sqrt{M_F}}$$

$$T_{coh} = \frac{2\sqrt{\pi}\,\sigma_t}{M_F} \qquad \Omega_{coh} = \frac{2\sqrt{\pi}\,\sigma_m}{M_F}$$

Region Near Pulse Length Minimum



Optimum

$$\sigma_m \approx \sqrt{|u|/2} << \sigma_\omega$$

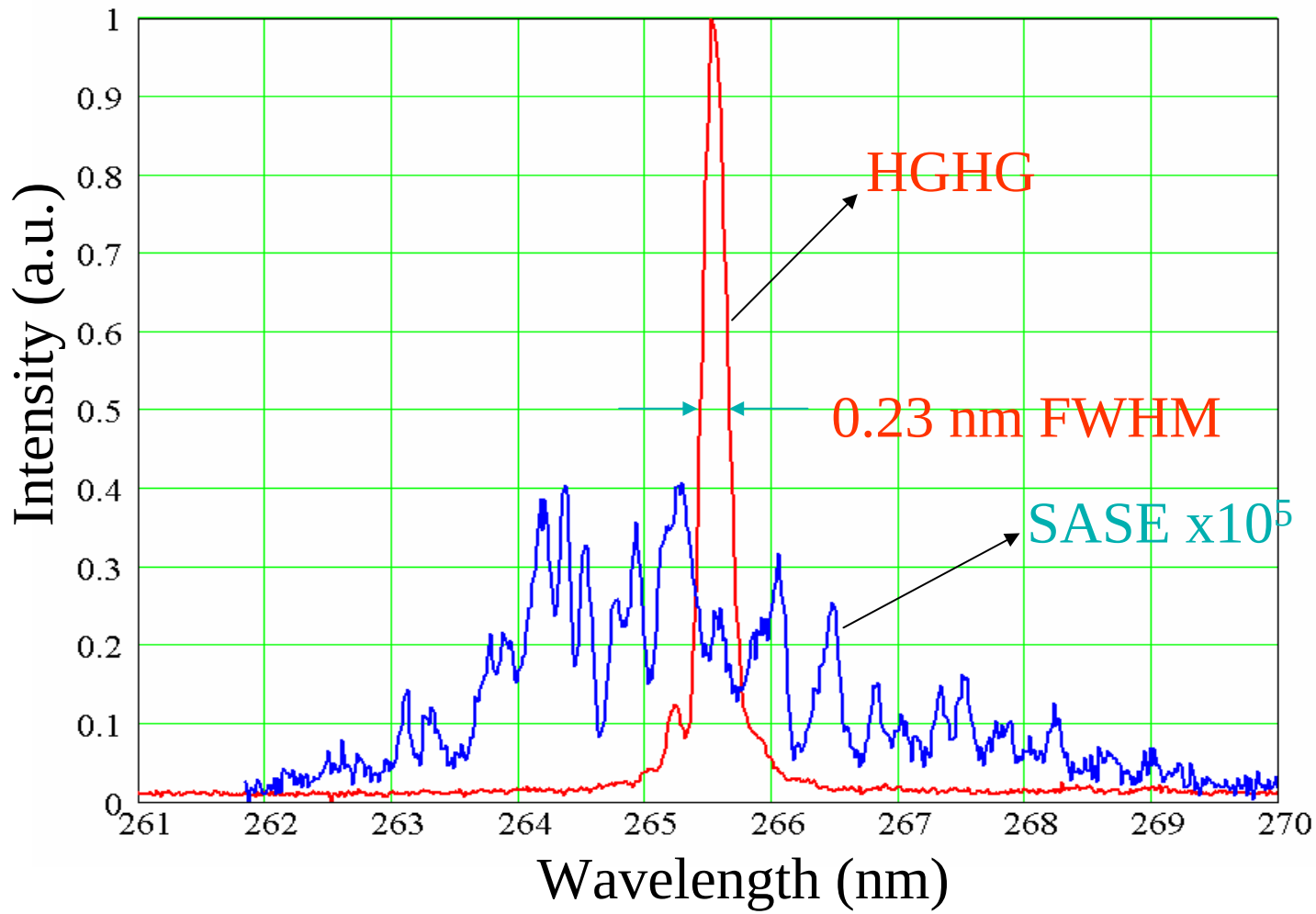
$$\sigma_t \approx \sigma_\omega / |u|$$

$$M_F \approx 2\sigma_m \sigma_\omega / |u|$$

Region Near Pulse Length Minimum: $\sigma_t \approx \sigma_\omega / |u|$

	Unchirped	Chirped	Filtered
M	$\sigma_\omega T_b / \sqrt{\pi}$	$\sigma_\omega T_b / \sqrt{\pi}$	$2\sigma_m \sigma_t$
T_p	T_b	T_b	$2\sqrt{\pi}\sigma_t$
T_{coh}	$\sqrt{\pi} / \sigma_\omega$	$\sqrt{\pi} / \sigma_\omega$	$\sqrt{\pi} / \sigma_m$
Ω_p	$2\sqrt{\pi}\sigma_\omega$	$\lvert u \rvert T_b$	$2\sqrt{\pi}\sigma_m$
Ω_{coh}	$2\pi / T_b$	$\sqrt{\pi}\lvert u \rvert / \sigma_\omega$	$\sqrt{\pi}\lvert u \rvert / \sigma_\omega$

Spectrum of HGHG at SDL/BNL and unsaturated SASE at 266 nm under the same electron beam condition



L.H. Yu et al (2003)