

DERIVATION OF BUNCHING FOR POISSON STATISTICS*

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Abstract

We derive the average and rms bunching for Poisson statistics. Unlike a bunch with a fixed number of independent particles, the shot noise is independent of frequency.

INTRODUCTION

For a collimated bunch of ultrarelativistic electrons, the radiation field emitted at a given wavelength is proportional to the Fourier transform of the electron line density—the so-called bunching. For a bunch with a fixed number of independent particles, the mean and rms values of the bunching describe coherent radiation from the bunch's density profile in addition to incoherent radiation from fluctuations in particle locations [1, 2].

We derive the mean and rms values of the bunching for a Poisson process, in which the number of particles in a bunch varies from bunch to bunch. Our results agree with a previous analysis [3]. Unlike a bunch with a fixed number of statistically independent particles, the variance and standard deviation of the bunching (i.e., the shot noise) is independent of frequency.

POISSON STATISTICS

Consider a bunch that obeys Poisson statistics [4], in which the total number of particles N varies while the mean \bar{N} is finite. Let the number of particles that have passed by an observer at time t be described by a statistical process $N(t)$ for which process increments are independent, i.e., the number of particles observed in non-overlapping time intervals $(t_0, t_1]$, $(t_1, t_2]$, $(t_2, t_3]$, etc. are independent.

For a Poisson distribution in which the mean number of particles is μ , the probability of observing k particles is $\mu^k e^{-\mu} / k!$. For a bunch with particle line density profile $v(t)$, the mean number of particles in the time interval $(s, s+t)$ is $\int_s^{s+t} v(t') dt'$. Describing such a bunch as a nonhomogeneous Poisson process [4], we have

$$\begin{aligned} \Pr[N(s+t) - N(s) = k] \\ = \left(\int_s^{s+t} v(t') dt' \right)^k \exp\left(-\int_s^{s+t} v(t') dt'\right) / k! \end{aligned} \quad (1)$$

Since $N(-\infty) = 0$, the probability that the j th particle is observed in the time interval $(t, t+dt)$ is given by the distribution

$$\begin{aligned} p_j(t)dt &= \Pr[N(t) = j-1, N(t+dt) \geq j] \\ &= \Pr[N(t) - N(-\infty) = j-1, N(t+dt) - N(t) \geq 1] \end{aligned} \quad (2)$$

For independent process increments,

$$\begin{aligned} p_j(t)dt &= \Pr[N(t) - N(-\infty) = j-1] \Pr[N(t+dt) - N(t) \geq 1] \\ &= \Pr[N(t) - N(-\infty) = j-1] \{1 - \Pr[N(t+dt) - N(t) = 0]\}. \end{aligned} \quad (3)$$

Using $\Pr[N(t+dt) - N(t) = 0] = \exp(-\int_t^{t+dt} v(t') dt')$
 $= \exp(-v(t)dt) = 1 - v(t)dt$, we have

$$p_j(t)dt = \frac{[V(t)]^{j-1}}{(j-1)!} e^{-V(t)} v(t)dt \quad (4)$$

where

$$V(t) \equiv \int_{-\infty}^t v(t') dt'. \quad (5)$$

The mean number of particles in the bunch is $\bar{N} = V(\infty)$, while the probability that the bunch has n particles is

$$\Pr[N = n] = \Pr[N(\infty) - N(-\infty) = n] = \bar{N}^n e^{-\bar{N}} / n! \quad (6)$$

Noting that

$$\begin{aligned} \int_{-\infty}^{\infty} p_j(t)dt &= \frac{1}{(j-1)!} \int_{-\infty}^{\infty} [V(t)]^{j-1} e^{-V(t)} v(t)dt \\ &= \frac{1}{(j-1)!} \int_0^{\bar{N}} V^{j-1} e^{-V} dV, \end{aligned} \quad (7)$$

we have [5]

$$\begin{aligned} \int_{-\infty}^{\infty} p_j(t)dt &= 1 - e^{-\bar{N}} \sum_{n=0}^{j-1} \frac{\bar{N}^n}{n!} = \sum_{n=j}^{\infty} \frac{\bar{N}^n}{n!} e^{-\bar{N}} \\ &= \sum_{n=j}^{\infty} \Pr[N = n] = \Pr[N \geq j] \end{aligned} \quad (8)$$

As expected, the integral of $p_j(t)$ over all time gives the probability that the entire bunch has j or more particles.

The head of a uniform beam is described when $v(t)$ is zero for negative times and constant for $t > 0$ [6], in which

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case the Poisson process is homogenous and \bar{N} is infinite.

AVERAGE BUNCHING

For a collimated bunch of ultrarelativistic electrons, the radiation emitted at angular frequency ω is proportional to the Fourier transform of the line charge density. For an n -particle bunch in which the j th particle is observed at time t_j , the radiation is therefore proportional to the Fourier transform of the particle line density, given by $T(\omega) = \int_{-\infty}^{\infty} \sum_{j=1}^n \delta(t-t_j) \exp(i\omega t) dt = \sum_{j=1}^n \exp(i\omega t_j)$ [1, 2].

For a Poisson process, in which the number of particles in a bunch varies, the mean value of $T(\omega)$ is

$$\langle T(\omega) \rangle = \sum_{n=1}^{\infty} \Pr[N = n] \sum_{j=1}^n \langle e^{i\omega t_j} \rangle \Big|_{N=n} \quad (9)$$

where $\langle \exp(i\omega t_j) \rangle \Big|_{N=n}$ is the mean value of $\exp(i\omega t_j)$ for a bunch with n particles. Changing the order of summation gives

$$\begin{aligned} \langle T(\omega) \rangle &= \sum_{j=1}^{\infty} \sum_{n=j}^{\infty} \Pr[N = n] \langle e^{i\omega t_j} \rangle \Big|_{N=n} \\ &= \sum_{j=1}^{\infty} \Pr[N \geq j] \langle e^{i\omega t_j} \rangle \Big|_{N \geq j} \end{aligned} \quad (10)$$

Here,

$$\langle e^{i\omega t_j} \rangle \Big|_{N \geq j} = \int_{-\infty}^{\infty} f_j(t) e^{i\omega t} dt, \quad (11)$$

where $f_j(t) = p_j(t) / \int_{-\infty}^{\infty} p_j(t) dt$ is the conditional probability distribution for the j th particle, given that the bunch has j or more particles. Thus, eq. (10) gives

$$\begin{aligned} \langle T(\omega) \rangle &= \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} p_j(t) \exp(i\omega t) dt \\ &= \int_{-\infty}^{\infty} \sum_{j=1}^{\infty} \frac{[V(t)]^{j-1}}{(j-1)!} \exp[-V(t)] v(t) \exp(i\omega t) dt \end{aligned} \quad (12)$$

Since the sum over j gives $\exp[V(t)]$, we have

$$\langle T(\omega) \rangle = \int_{-\infty}^{\infty} v(t) \exp(i\omega t) dt = \bar{N} F(\omega), \quad (13)$$

where $F(\omega)$ is the bunch form factor at angular frequency ω , equal to the Fourier transform of the normalized line density profile $v(t) / \bar{N}$.

RMS BUNCHING

Consider $|T(\omega)|^2$, whose value for an n -particle bunch is $\sum_{j=1}^n \sum_{k=1}^n \exp[i\omega(t_j - t_k)]$. To evaluate its average, consider the probability that the j th particle is in the interval $(t_1, t_1 + dt_1]$ while the k th particle is in the interval $(t_2, t_2 + dt_2]$. For $j < k$, the nonzero probability for $t_1 < t_2$ is

$$p_{jk}(t_1, t_2) dt_1 dt_2 = \Pr[N(t_1) = j-1, N(t_1 + dt_1) \geq j, N(t_2) = k-1, N(t_2 + dt_2) \geq k] \quad (14)$$

Using $\Pr[N(t+dt) - N(t) \geq 1] = 1 - \Pr[N(t+dt) - N(t) = 0] = 1 - \exp[-v(t)dt] = v(t)dt = \Pr[N(t+dt) - N(t) = 1]$ gives

$$\begin{aligned} p_{jk}(t_1, t_2) dt_1 dt_2 &= \Pr[N(t_1) - N(-\infty) = j-1, \\ N(t_1 + dt_1) - N(t_1) &= 1, N(t_2) - N(t_1 + dt_1) = k-j-1, \\ N(t_2 + dt_2) - N(t_2) &= 1] \end{aligned} \quad (15)$$

The assumption of independent process increments yields

$$\begin{aligned} p_{jk}(t_1, t_2) dt_1 dt_2 &= \Pr[N(t_1) - N(-\infty) = j-1] \\ &\times \Pr[N(t_1 + dt_1) - N(t_1) = 1] \\ &\times \Pr[N(t_2) - N(t_1 + dt_1) = k-j-1] \\ &\times \Pr[N(t_2 + dt_2) - N(t_2) = 1] \end{aligned} \quad (16)$$

so that, for $j < k$

$$\begin{aligned} p_{jk}(t_1, t_2) &= H(t_2 - t_1) \frac{V(t_1)^{j-1}}{(j-1)!} \exp[-V(t_1)] v(t_1) \\ &\times \frac{[V(t_2) - V(t_1)]^{k-j-1}}{(k-j-1)!} \exp\{-[V(t_2) - V(t_1)]\} v(t_2) \end{aligned} \quad (17)$$

where $H(t)$ is the Heaviside function equaling 1 for positive t and zero for negative t .

For $j > k$, $p_{jk}(t_1, t_2)$ is given by exchanging t_1 and t_2 , and exchanging j and k , on the RHS of eq. (17).

For the case $j = k$,

$$p_{jj}(t_1, t_2) = \delta(t_1 - t_2) p_j(t_1), \quad (18)$$

where $p_j(t)$ is given by eq. (4).

In all cases ($j < k$, $j = k$, and $j > k$), we have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{jk}(t_1, t_2) dt_1 dt_2 = \Pr[N \geq \max(j, k)]. \quad (19)$$

The average of $|T(\omega)|^2$ is

$$\langle |T(\omega)|^2 \rangle = \sum_{n=1}^{\infty} \Pr[N = n] \sum_{j=1}^n \sum_{k=1}^n \langle e^{i\omega(t_j - t_k)} \rangle \Big|_{N=n} \quad (20)$$

where $\langle \exp[i\omega(t_j - t_k)] \rangle \Big|_{N=n}$ is the average for a bunch with n particles. The contribution to $\langle |T(\omega)|^2 \rangle$ from the terms with $j = k$ is

$$\sum_{n=1}^{\infty} n \Pr[N = n] = \bar{N} \quad (21)$$

The contribution from the remaining terms is, after changing the order of summation

$$\begin{aligned} & \sum_{j=1}^{\infty} \sum_{k \neq j} \sum_{n=\max(j,k)}^{\infty} \Pr[N = n] \langle e^{i\omega(t_j - t_k)} \rangle \Big|_{N=n} \\ &= \sum_{j=1}^{\infty} \sum_{k \neq j} \Pr[N \geq \max(j,k)] \langle e^{i\omega(t_j - t_k)} \rangle \Big|_{N \geq \max(j,k)} \end{aligned} \quad (22)$$

Here,

$$\langle e^{i\omega(t_j - t_k)} \rangle \Big|_{N \geq \max(j,k)} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{jk}(t_1, t_2) e^{i\omega(t_1 - t_2)} dt_1 dt_2, \quad (23)$$

where $f_{jk}(t_1, t_2) = p_{jk}(t_1, t_2) / \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{jk}(t_1, t_2) dt_1 dt_2$ is the conditional probability distribution, given that the number of particles in the bunch is at least $\max(j, k)$. Thus, the contribution from terms with $j \neq k$ is given by

$$\sum_{j=1}^{\infty} \sum_{k \neq j} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[i\omega(t_1 - t_2)] p_{jk}(t_1, t_2) dt_1 dt_2. \quad (24)$$

Combining the terms for $j < k$ and $j > k$, we write the contribution from terms with $j \neq k$ as

$$\begin{aligned} & 2 \sum_{j=1}^{\infty} \sum_{k=j+1}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(t_2 - t_1) \cos[\omega(t_1 - t_2)] \\ & \times \frac{[V(t_1)]^{j-1}}{(j-1)!} \exp[-V(t_1)] v(t_1) \\ & \times \frac{[V(t_2) - V(t_1)]^{k-j-1}}{(k-j-1)!} \exp\{-[V(t_2) - V(t_1)]\} v(t_2) dt_1 dt_2 \end{aligned} \quad (25)$$

Performing the summations yields

$$\begin{aligned} & 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(t_2 - t_1) \cos[\omega(t_1 - t_2)] v(t_1) v(t_2) dt_1 dt_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos[\omega(t_1 - t_2)] v(t_1) v(t_2) dt_1 dt_2 \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [e^{i\omega(t_1 - t_2)} + e^{-i\omega(t_1 - t_2)}] v(t_1) v(t_2) dt_1 dt_2 \\ &= \left| \int_{-\infty}^{\infty} e^{i\omega t} v(t) dt \right|^2 = \bar{N}^2 |F(\omega)|^2 \end{aligned} \quad (26)$$

Combining eqs. (21) and (26) gives the result

$$\langle |T(\omega)|^2 \rangle = \bar{N} + \bar{N}^2 |F(\omega)|^2. \quad (27)$$

The variance of the bunching is

$$\langle |T(\omega) - \langle T(\omega) \rangle|^2 \rangle = \bar{N}. \quad (28)$$

PREVIOUS RESULTS

For a bunch containing a fixed number n of independent particles, with joint probability distribution $p(t_1, t_2, \dots, t_n) = \prod_{j=1}^n f(t_j)$, the Fourier transform of the particle line density has average value

$$\begin{aligned} \langle T_n(\omega) \rangle &= \sum_{j=1}^n \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} p(t_1 \dots t_n) e^{i\omega t_j} dt_1 \dots dt_n \\ &= \sum_{j=1}^n \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt = nF(\omega). \end{aligned} \quad (29)$$

The square of its rms value is well known

$$\begin{aligned} \langle |T_n(\omega)|^2 \rangle &= \sum_{j=1}^n \sum_{k=1}^n \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} p(t_1 \dots t_n) e^{i\omega(t_j - t_k)} dt_1 \dots dt_n \\ &= \sum_{j=1}^n 1 + \sum_{j=1}^n \sum_{k \neq j} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \\ &= n + n(n-1) |F(\omega)|^2, \end{aligned} \quad (30)$$

where $F(\omega)$ is the Fourier transform of $f(t)$ [1, 2].

For a process in which the number of particles in a bunch varies while the particles in an n -particle bunch are independent, $\langle T(\omega) \rangle = \sum_{n=1}^{\infty} \Pr[N = n] nF(\omega)$ and $\langle |T(\omega)|^2 \rangle = \sum_{n=1}^{\infty} \Pr[N = n] [n + n(n-1) |F(\omega)|^2]$. For a Poisson distribution of bunch populations, the average of N is \bar{N} and the average of $N(N-1)$ is \bar{N}^2 [4], so that these formulas give eqs. (13) and (27), as noted in Ref. [3].

Thus, these bunching formulas for Poisson statistics may be rigorously obtained from previous results by showing that randomly chosen particles in an n -particle bunch are independent with probability distribution $v(t) / \bar{N}$.

For an n -particle bunch with observation times t_1, t_2, \dots, t_n , let t'_1, t'_2, \dots, t'_n be the ordered observation times. There are $n!$ bunches with ordered times t'_1, t'_2, \dots, t'_n . The joint probability distribution for an n -particle bunch is $p(t_1, t_2, \dots, t_n) = (1/n!) f_{12..n}(t'_1, t'_2, \dots, t'_n)$, where $f_{12..n}(t'_1, t'_2, \dots, t'_n)$ is the conditional probability distribution given that the bunch has n particles. For a Poisson process,

$$\begin{aligned}
 f_{12\dots n}(t'_1, \dots, t'_n) dt'_1 \dots dt'_n &= \Pr[N(t'_1 = 0), N(t'_1 + dt'_1) = 1, \\
 N(t'_2 = 1), N(t'_2 + dt'_2) = 2, \dots, N(t'_n + dt'_n) = n]_{N=n} \\
 &= \Pr[N(t'_1) = 0, N(t'_1 + dt'_1) = 1, N(t'_2) = 1, N(t'_2 + dt'_2) = 2, \\
 \dots, N(t'_n + dt'_n) = n, N(\infty) = n] / \Pr[N = n] \quad (31) \\
 &= \Pr[N(t'_1) - N(-\infty) = 0, N(t'_1 + dt'_1) - N(t'_1) = 1, \\
 N(t'_2) - N(t'_1 + dt'_1) = 0, \dots, N(t'_n + dt'_n) - N(t'_n) = 1, \\
 N(\infty) - N(t'_n + dt'_n) = 0] / (\bar{N}^n e^{-\bar{N}} / n!)
 \end{aligned}$$

The assumption of independent process increments yields

$$\begin{aligned}
 f_{12\dots n}(t'_1, \dots, t'_n) dt'_1 \dots dt'_n &= e^{-V(t'_1)} v(t'_1) dt'_1 e^{-[V(t'_2) - V(t'_1)]} \\
 \times v(t'_2) dt'_2 \dots v(t'_n) dt'_n e^{-[N - V(t'_n)]} / (\bar{N}^n e^{-\bar{N}} / n!) \quad (32) \\
 &= n! \prod_{j=1}^n \left[\frac{v(t'_j) dt'_j}{\bar{N}} \right] = n! \prod_{j=1}^n \left[\frac{v(t_j) dt_j}{\bar{N}} \right],
 \end{aligned}$$

Therefore,

$$p(t_1, t_2, \dots, t_n) = \prod_{j=1}^n [v(t_j) / \bar{N}], \quad (33)$$

which justifies eqs. (13) and (27).

SUMMARY

For a collimated bunch of electrons obeying Poisson statistics with average bunch population \bar{N} , we have derived the average and rms bunching. Our results agree with Ref. [3].

The bunching, given by the Fourier transform of the particle line density, has average value

$$\langle T(\omega) \rangle = \bar{N} F(\omega). \quad (34)$$

The mean of its square is

$$\langle |T(\omega)|^2 \rangle = \bar{N} + \bar{N}^2 |F(\omega)|^2. \quad (35)$$

This equals the sum of an incoherent contribution \bar{N} and a coherent contribution $\bar{N}^2 |F(\omega)|^2$. Here, $F(\omega)$ is the bunch form factor, equal to the Fourier transform of the normalized line density profile.

The variance of the bunching is independent of frequency

$$\langle |T(\omega) - \langle T(\omega) \rangle|^2 \rangle = \bar{N}, \quad (36)$$

and its standard deviation is $\sqrt{\bar{N}}$.

In contrast, the variance of the bunching for a fixed number n of independent particles depends upon the frequency as $\langle |T_n(\omega) - \langle T_n(\omega) \rangle|^2 \rangle = n(1 - |F(\omega)|^2)$. Thus, for frequencies where $|F(\omega)| \approx 1$, the rms shot noise of a fixed number n of independent particles is much smaller than the frequency-independent shot noise of a Poisson process with average bunch population $\bar{N} = n$.

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