TWO EXAMPLES OF INTEGRABLE SYSTEMS WITH ROUND COLLIDING BEAMS

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Abstract

The work is devoted to analytical study of application of integrable systems to round colliding beams, aiming at enhancement of the beam-limit. Two examples of “integrable” beam-beam forces are presented, relevant to round counter beams with special density distributions. In such systems all the resonances will vanish, hence the beam-beam effects can be suppressed, and the intensities of colliding beams may be strongly increased, at least in the “weak-strong” case.

1 INTRODUCTION

The concept of “Round Colliding Beams” (RCB) is considered as a possibility to reach higher luminosity and to improve beam stability in colliders ([1] and references therein). The essential conditions of the RCB are: equal horizontal and vertical emittances \( \varepsilon_x = \varepsilon_y = \varepsilon \); equal horizontal and vertical beta-functions at the Interaction Point (IP) \( \beta_x^* = \beta_y^* = \beta^* \); equal horizontal and vertical tunes \( \nu_x = \nu_y = \nu \). The rotational symmetry of the kick from the round opposite beam, complemented with the \( X-Y \) symmetry of the betatron transfer matrix between the collisions, result in an additional integral of motion \( M = xy' - yx' \), i.e. the longitudinal component of particle’s angular momentum.

Thus, the transverse motion becomes equivalent to a one-dimensional (1D) motion [2]. Resulting elimination of all betatron coupling resonances is of crucial importance, since they are believed to cause the beam lifetime degradation and blow-up. Reduction to 1D motion makes impossible the diffusion through invariant circles. Although this 1D motion has more “regularity” in comparison with a general 2D motion, with the time-dependent Hamiltonian it is still stochastic in general. What we need here to make the motion regular, is to construct one more integral of motion, valid for any value of the angular momentum \( M \). At first glance, it is not evident, that we can find the needed forces (among those physically feasible), especially when we deal with the fields of the counter beam. But solutions exist [3], and we present two interesting examples, which may be already useful for practice.

2 EXAMPLE 1: INTEGRABLE BEAM-BEAM KICK

Let us take a drift space with the unity length (for simplicity) followed by an axially symmetric thin lens, as a representation of the angular-momentum-preserving linear optics in between the IPs, and the radial beam-beam kick.

The 2D map for particle trajectory displacements \( x, y \) and slopes \( x', y' \) through such a period is:

\[
\begin{align*}
\mathcal{X} &= x + x' \\
\mathcal{Y} &= y + y' \\
\mathcal{X}' &= x' + \mathcal{K}_x \\
\mathcal{Y}' &= y' + \mathcal{K}_y,
\end{align*}
\]

where \( k_x = \frac{x}{r}k(r), k_y = \frac{y}{r}k(r), \) and \( r = \sqrt{x^2 + y^2} \).

Due to conservation of the angular momentum, the motion is reducible to 1D. Previously we reported on existence of invariants of this map in the particular case of 1D motion ([4] and references therein). This corresponds in (1) to \( x' = 0 \), or \( y' = 0 \), or generally, to any meridional trajectory with \( M = xy' - yx' = 0 \). With \( x, x' \) lying in the plane of such a trajectory, the desired integrals of motion may be sought among these invariants:

\[
\mathcal{I}(x, x') = (a_2x^2 + a_1x + a_0)(x' + x)^2 + (a_1x^2 + b_1x + b_0)(x' + x) + a_0x^2 + b_0x,
\]

and the kick function \( k \) must have the form:

\[
k(x) = -2x - \frac{a_1x^2 + b_1x + b_0}{a_2x^2 + a_1x + a_0}.
\]

Here the 5 coefficients are arbitrary parameters of the kick force.

Turning back to the general case \( M = M \neq 0 \), we can use the generic form of 1D invariant (2) for construction of an axially symmetric invariant involving only \( r, r' \) as dynamic variables. The kick function (3) is now understood as a radial kick \( k(r) \), and we observe that only the case \( b_0 = a_1 = 0 \) is practically interesting, otherwise \( k(r) \) would have singularities at \( r = 0 \). The Courant–Snyder terms with \( a_0, b_1 \) in (2) give a clue to the form of the axially-symmetric invariant, to be tried for any value \( M \) of the angular momentum \( M = xy' - yx' \) (certainly valid at \( M = 0 \)):

\[
\mathcal{I}(r, r') = (a_2r^2 + a_1r + a_0)(r' + r)^2 + \frac{M^2}{r^2} + (a_1r^2 + b_1r + b_0)(r' + r) + a_0r^2 + b_0r.
\]

The variables here are changed to \( r, r' \), use has been made of the following relations: \( r' = (xx' + yy')/r, x'^2 + y'^2 = ((rr')^2 + (xy' - yx')^2)/r^2 = r'^2 + M^2 / r^2 \).

Rewriting accordingly the map (1) in terms of \( (r, r') \):

\[
\begin{align*}
\mathcal{T} &= \sqrt{r^2 + 2rr' + r'^2 + M^2 / r^2}, \\
\mathcal{T}' &= (r'(r' + r) + M^2 / r^2) \frac{1}{\mathcal{T}} + k(\mathcal{T}),
\end{align*}
\]
we apply this transformation to (4). The invariance relation 
\[ I_M (r, r') = I_M (r', r) \] then yields: \( a_1 = b_0 = 0 \). Thus we 
find the desired integral of motion which holds at any 
constant value \( M \) of \( M \):

\[ I_M (r, r') = (a_2 r^2 + a_0)(r' + r)^2 + b_1 r (r' + r) + a_0 \left( r^2 + \frac{M^2}{r^2} \right). \] (6)

The corresponding radial kick function

\[ k(r) = -2r - \frac{b_1 r}{a_0 + a_2 r^2}. \] (7)

has only 3 free parameters, just in accord with our 
assumption that the integrable systems for RCB form a subset of 
all 1D integrable systems.

In the present context we interpret the 2nd term in (7) as 
the beam-beam kick, while the 1st term together with the 
drift length form the linear optics in between the IPs. The 
optics appears to be a \( 90^\circ \) scheme requires short colliding bunches with radial distri-
butions close to (9), a linear optics with equal transfer ma-
trices in \( x \) and \( y \) planes, and with equal betatron phase ad-
vances of \( 90^\circ \) in between the IPs. Integrability of the result-
ing dynamics will show in regularity of motion which will 
be bounded by closed invariant curves \( I_M (r, r') = \text{const.} \) 
and free from resonance islands throughout the linear sta-
bility range, \textit{i.e.} for the beam-beam parameter \( |\xi| < 1/2\pi \).

3 EXAMPLE 2: A SPECIAL LONGITUDINAL 
DISTRIBUTION

In the previous example we dealt with a short nonlinear 
kick, the time dependence was represented by the delta-
function. Now we turn to a continuous-time dependence of 
the nonlinear force, and present a dynamical system with 
two invariants, which can be derived by means of usual 
accelerator theory tools. Let us take the 1D equation of 
particle’s motion in an accelerator:

\[ x'' + g(s)x = F(x, s), \] (10)

where \( g(s) \) is the focusing function and \( F(x, s) \) is an arbi-
trary force. This equation can be simplified by using the 
betatron phase \( \psi = \int ds/\beta(s) \) instead of \( s \) and chang-
ing the physical variable \( x \) to the normalized variable \( \tilde{x} = x/\sqrt{\beta(s)} \):

\[ \tilde{x}'' + X = \beta^{3/2} F(X \sqrt{\beta}, s(\psi)). \] (11)

The force due to round counter beam with the transverse 
Gaussian distribution can be presented now in the factor-
zized form, thus separating its dependence on the transverse 
and longitudinal coordinates:

\[ F_{rb} = -\frac{2Ne^2}{\gamma mc^2} \frac{1 - \exp(-r^2/2\varepsilon)}{r/\sqrt{\beta}} \frac{f(\delta - 2s)}{\sqrt{\beta}}, \] (12)

Here \( \varepsilon \) is the emittance of the opposite beam, \( f \) is the 
longitudinal distribution of counter beam \( \int f d\delta = 1 \), \( \delta \) 
is the longitudinal position of the test particle in the weak 
bunch with respect to the bunch center. The “time” \( s = 0 \) 
corresponds to the moment when the central test particle 
(\( \delta = 0 \)) meets the center of the strong bunch.

The equation of particle motion in the interaction region 
in terms of \( r = \sqrt{x^2 + y^2} \) is:

\[ r'' + g(s)r = F_{rb} + M^2/r^3; \] (13)

where the last term means the “centrifugal” force.

Now let us consider a case when the weak bunch of the 
test particles is short with respect to the beta function at 
the IP and we can put \( \delta = 0 \), and at the same moment, 
the longitudinal charge distribution of the strong bunch is 
proportional to the inverse \( \beta \)-function: \( f(2s) = C/\beta(s) \).

For an interaction region which is free of focusing we have:
\( \beta(s) = \beta^* + s^2/\beta^* \), and the perfect distribution is:

\[ f(s) = \frac{C}{1 + (s/2\beta^*)^2}, \] (14)

where \( C \) is a constant, \( \beta^* \) is the \( \beta \)-function value at the IP.

After substitution of the normalized variable \( R = r/\sqrt{\beta(s)} \) and replacement of \( s \) by the phase \( \psi \), one gets:

\[ R'' + R = \frac{M^2}{R^3} - \frac{2Ne^2}{\gamma mc^2} \frac{1 - \exp(-R^2/2\varepsilon)}{R}. \] (15)

One can see, that the force in this equation does not de-
pend on time, and therefore, this 1D equation is integrable.
The coordinate dependence on time can be found using 
conventional 1D formulas.

The trick with obtaining the time-independent force is 
related with the fact, that the ‘centrifugal’ force is invariant 
under substitution of new variables and changing ’time’ to 
the betatron phase. It is easy to see, that the force in (15)
We obtain the same motion for $r$; $r$ integrability of the particles’ dynamics with proving additivity in colliders. The essence of these ways is obtaining The paper presents new ways to improve single particle sta-

deviations time-dependent for particles with large energy off-sets and

motion. In this case we always stay near the integer resonance.

One can see, that the Gaussian shape of the betatron distri-
bution is not important here; the distribution may be any

longitudinally-Gaussian strong bunch: $\sigma = \sqrt{2} \beta^*$. 

Another essential feature of this solution is that the work-
ing point of this system is near the half-integer resonance, when the number of interaction points is odd, and near the integ-
er resonance, when this number is even. So, pertur-
bations of the arcs of collider determine the permissible
distance of the working point from the resonance, and conse-
sequently, determine the accuracy of the conservation of
integrals (this situation is common for integrable systems:

perturbations almost always lead to small stochasticity in

nearly integrable systems, the point is in allowable values

of perturbations).

One more remark is needed. The weak bunch has a small
longitudinal size in our dynamical system (while the lon-
gitudinal distribution of the strong bunch is taken propor-
tional to the inverse $s$-function). If its length is not small
in comparison with the $\beta$-function, then the force becomes
time-dependent for particles with large energy off-sets and
deviations $\delta$ even in the normalized variables. Importance
of this modulation was checked by simulation [5]. 

A formal construction of the integrable distribution for
the above example can be found in [3].

4 CONCLUSION

The paper presents new ways to improve single particle sta-
bility in colliders. The essence of these ways is obtaining integrability of the particles’ dynamics with proving addi-
tional integrals of the particle motion. For example, if the
“round colliding beams” conditions are fulfilled then the

longitudinal component of the angular momentum is the
invariant for colliding beams. In some particular cases of
the RCBs, additionally to the angular momentum $M$, we
can find one more invariant which is quadratic in momen-
tum, and holds for any value of $M$. These two integrals of
motion suffice for a non-autonomous 2D dynamical system
to be integrable.

The both RCB examples above deal with what is called a
“weak-strong” beam-beam model: study of motion of a
test particle affected by a strong nonlinear fields of a strong
counter beam. In the both recipes the arc lattice in between
the IPs must be perfectly linear. The 1st example with a
short counter bunch requires a special transverse distribu-
tion in the strong bunch for integrability. In the 2nd exam-
ple, the strong bunch length is of the order of $\beta^*$. Here, we
can choose an “inverse $\beta$-function” longitudinal bunch pro-
file for reduction of motion to a 1D autonomous dynamics.

The proposed integrable systems with globally regu-
lar motion and without any beam-beam blow-up threshold
have strengthened the concept of round colliding beams.
They were tested in simulations [1, 5] against perturbations
inevitably present in a real machine. The beam emittance
growth becomes mostly determined by the arc lattice non-
linearities and imperfections, so we believe that it will be
possible to achieve a higher luminosity by reducing the im-

5 REFERENCES


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1 Actually, the both $x$ and $y$ phase advances of $\pi$ are also acceptable.
We obtain the same motion for $r$, $r'$ due to symmetry of potential of this motion. In this case we always stay near the integer resonance.