ANALYTIC AND NUMERICAL ANALYSIS OF THE LONGITUDINAL COUPLING IMPEDANCE OF AN ANNULAR CUT IN A COAXIAL LINER

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Abstract

Beam pipes of high-energy superconducting colliders require a shielding tube (liner) with pumping slots to screen cold chamber walls from synchrotron radiation. Pumping slots in the liner walls are required to keep high vacuum inside the beam pipe and provide for a long beam lifetime. As previously discussed [Fedotov and Gluckstern, Phys. Rev. E **54**, 1930 (1996)], for a long narrow slot whose length may be comparable with the wavelength, the usual static approximation for the polarizability and susceptibility which enter into the impedance is a poor one.[1] Our objective is eventually to analyze and obtain numerical values for a rectangular slot of arbitrary dimensions. In this paper we present an analysis, based on a variational formulation, for the impedance of an annular cut in the inner conductor, including both the realistic coaxial structure of the beam-pipe and the effect of finite wavelength. For low frequencies, the numerical results are checked against analytical results, with which they agree.

1 INTRODUCTION

The solution is based on the method of field matching at the liner radius, including the discontinuity. We construct a variational form for the impedance, which is stationary with respect to arbitrary small variations of the field about its true value. The variational approach ensures very good accuracy for the impedance, since the error will be proportional to the square of the error in the chosen trial fields.

We assume the liner thickness to be negligible and call the region inside the inner conductor the "pipe region" and the region outside the inner conductor the "coaxial region". The technique consists of expanding fields in both regions into a complete set of functions. At the common interface the fields have to be matched, yielding equations for the expansion coefficients.

Since the driving current on axis is proportional to $\exp(-jkz)$, the problem is simplified by obtaining $Z_\parallel(k)$ for an even driving current $\cos kz$ and an odd driving current $-j\sin kz$ separately and then taking their sum. We should note that the variational method becomes possible only when the problem is separated into an even and an odd part. In the even problem E_z is even in z, while in the odd problem E_z is odd in z (where $z=0$ is chosen to be the center of the cut). We use the superscript (e) for the even part and the superscript (o) for the odd part.

2 GENERAL ANALYSIS

In the pipe region the fields are given by the source fields plus a general solution of the Maxwell equations for the cylindrical waveguide. In the coaxial region we have the general solution of the Maxwell equations for the coaxial waveguide. Due to the symmetry of the problem we have no θ dependence, and therefore need to consider only the azimuthally symmetric TM modes. For the portion of the problem when $E_z^{(e)}$ is even in z we have

$$E_z^{(e)}(r,z) = \int dq \cos qz A^{(e)}(q)\left[\frac{J_0(\kappa r)}{J_0(\kappa a)}, \frac{F_0(\kappa r)}{F_0(\kappa a)}\right]. \quad (1)$$

Here we use the notation where the first part in square brackets corresponds to the pipe region $r \le a$, and the second part corresponds to the coaxial region $a \le r \le b$, with the function F_0 being the solution of the Maxwell equations for the coaxial region for the TM modes [$F_0(u) = Y_0(u)J_0(\kappa b) - J_0(u)Y_0(\kappa b)$]. Note that we consider the inside and outside surfaces of the liner both to be at $r=a$, since we neglect the thickness of the liner compared to the wavelength. Therefore, the coefficient $A^{(e)}(q)$ is the same for both $r<a$ and $r>a$, since $E_z^{(e)}$ is continuous at $r=a$ within the hole and on both sides of the liner surface, where $E_z^{(e)}=0$. The continuity of $H_\theta^{(e)}$ in the hole gives

$$\int dz' E_{z'}^{(e)}(a,z')K_{11}^{(e)}(z,z') = Z_0 I_0 \cos kz, \quad (2)$$

where

$$\begin{aligned} K_{11}^{(e)}(z,z') &= K_{11}^{(e)}(z',z) \\ &= a\int dq \cos qz \cos qz' k_{11}, \end{aligned} \quad (3)$$

with

$$k_{11} = jkaP(q), \quad (4)$$

and

$$P(q) = \left[\frac{J_0'(\kappa a)}{\kappa a J_0(\kappa a)} - \frac{F_0'(\kappa a)}{\kappa a F_0(\kappa a)}\right]. \quad (5)$$

We now treat $P(q)$ as a function of κa with $\kappa b = (b/a)\kappa a$ and express this function as a sum over the zeros of the respective denominators. The resulting expression for $K_{11}^{(e)}$, in Eq. (3), can then be integrated over q by means of the residue theorem. Using the definition of the impedance for the even part, we can rewrite Eq. (2) as

$$\frac{Z_0}{Z_\parallel^{(e)}} = -\frac{\left(\int\int dz'dz E_{z'}^{(e)}(a,z')E_z^{(e)}(a,z)K_{11}^{(e)}(z,z')\right)}{\left(\int dz E_z^{(e)}(a,z)\cos kz\right)^2}, \quad (6)$$

which is easily seen to be a variational form for the impedance, with trial function $E_z^{(e)}(a,z)$. Expanding $E_z^{(e)}(a,z)$ into a complete set in $|z| < g/2$ and evaluating the integrals in Eq. (6), the solution for the even part of the impedance is obtained by finally truncating and inverting the resulting matrix equations.

For the portion of the problem when $E_z^{(o)}$ is odd in z we perfom similar analysis, and obtain the variational form for the odd part of the impedance

$$\frac{Z_0}{Z_\parallel^{(o)}} = -\frac{(\int\int dz'dz E_{z'}^{(o)}(a,z')E_z^{(o)}(a,z)K_{11}^{(o)}(z,z'))}{(\int dz E_z^{(o)}(a,z)\sin kz)^2}, \tag{7}$$

where

$$\begin{aligned} K_{11}^{(o)}(z,z') &= K_{11}^{(o)}(z',z) \\ &= a\int dq \sin qz \sin qz' k_{11}. \end{aligned} \tag{8}$$

3 ANALYTIC DERIVATION FOR LOW FREQUENCIES

In our particular case where the slot is a narrow annular cut, for low frequencies the leading non-vanishing term in the odd part is a factor of $(g/a)^2$ less than the same term in the even part. To present the analytic result for a narrow annular cut it is therefore sufficient to consider only the even part of the impedance.

For small ka and g/a, using the static approximation for $E_{z'}$, E_z, we evaluate Eq. (6) analytically and obtain

$$\begin{aligned} \frac{Z_\parallel}{Z_0} &= \frac{\ln(b/a)}{\pi}\left[1 + j\frac{4}{\pi^2}kag/a\right. \\ &- \left. 2jka\frac{\ln(b/a)}{\pi}\Big(C_1 + C_2 + 2\ln(4a/g)\Big)\right]. \end{aligned} \tag{9}$$

In this result we supressed the superscript (e), since the odd part of the impedance is negligible. Numerical study shows that $[C_1 + C_2]$ can be replaced by $[\ln(b/a-1) - 1.78]$ for the range of b/a from 1 to 3.

4 NUMERICAL RESULTS AND DISCUSSION

The even part of the impedance is calculated using Eq. (6) and the odd part is calculated using Eq. (7). Finally, one sums together the even and the odd parts to obtain the coupling impedance. The general behavior of the real and imaginary parts with respect to frequency is presented in Figs. 1 and 2.

For low frequencies we obtain the leading terms for the real and imaginary parts analytically, according to Eq. (9). The agreement between the analytic and numerical results is very good. As an example, results for $b/a = 2$ are presented in Table 1. For other values of b/a, the analytic and numerical results are also in good agreement.

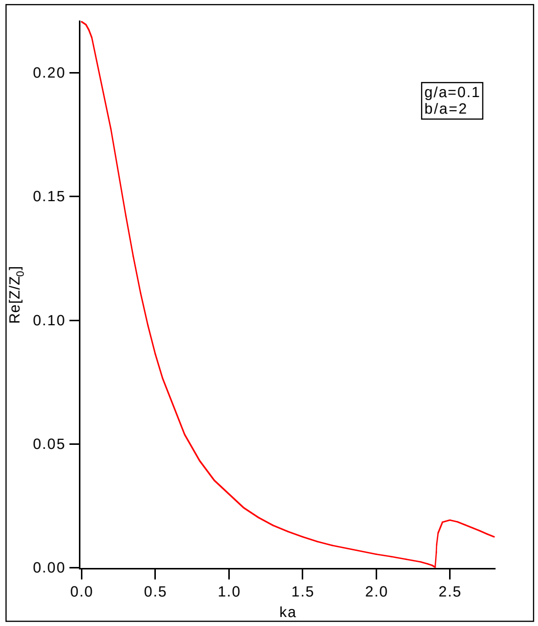

Figure 1: Real part of the coupling impedance of an annular cut in a coaxial liner.

	g/a	Analytic approximation	Numerical result
	0.01	0.02976	0.02907
b	0.03	0.02332	0.02308
	0.05	0.02025	0.02022
	0.07	0.01823	0.01830

Table 1: $Im[Z_\parallel/Z_0]$ for $b/a = 2$ at frequency $ka = 0.03$, $C_1 = -0.667$, $C_2 = -1.117$.

The real part of the impedance in the limit of zero frequency becomes finite, and is equal to $\ln(b/a)/\pi$, which agrees with the result obtained by Palumbo.[2] Its physical origin is the energy radiated in the TEM mode in the coaxial region.

For low frequencies, the coupling impedance of the narrow annular cut can be easily presented in terms of an equivalent circuit. Specifically, for $g \ll a$, we can write for the admittance

$$Y = R^{-1} + j\omega C, \tag{10}$$

where R is given by $Z_0\ln(b/a)/\pi$, and C is given by $2a\epsilon_0(C_1 + C_2 + 2\ln(4a/g))$, corresponding to the parallel combination of the resistance R and the capacitance C. In Figs. 3 and 4 we present the real and imaginary parts of the admittance Y as a function of ka. As one can see, the real part of the admittance is purely $1/R$ until $ka = 2.405$, the cutoff of the TM_{01} mode. At this cutoff the singularity corresponds to the fact that power starts to dissipate not just in the coaxial region, but also in the pipe region.

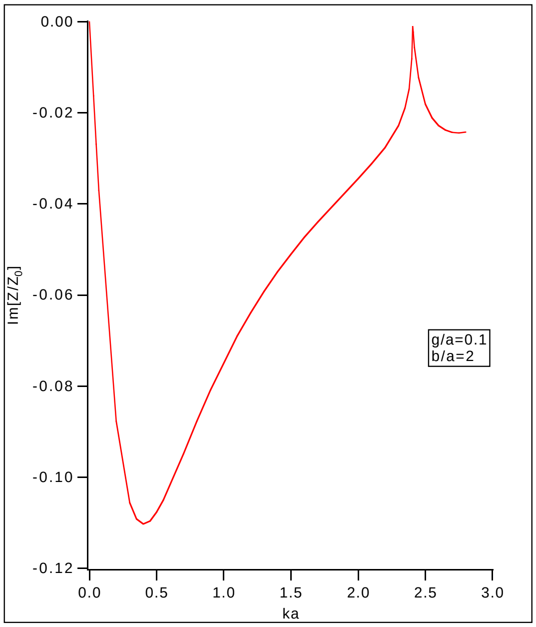

Figure 2: Imaginary part of the coupling impedance of an annular cut in a coaxial liner.

5 SUMMARY

The purpose of this paper is to present the analysis of the calculation of the coupling impedance of an annular cut in a coaxial liner of negligible wall thickness. We obtain equations for calculating the even and odd parts of the impedance, expressed in variational form. The use of the variational method makes numerical study fast and accurate. In order to check the developed technique, an analytic calculation is performed for low frequencies and compared with the numerical results. The agreement between the analytic and numerical results is very good. As previously mentioned, the present calculation is a first step in the development of an analysis for the coupling impedance of a rectangular slot with finite azimuthal length.[3]

6 ACKNOWLEDGMENTS

The authors wish to thank Dr. S. Kurennoy for helpful comments. This work was supported by the U.S. Department of Energy.

7 REFERENCES

[1] A.V. Fedotov and R.L. Gluckstern, Phys. Rev. E, **54**, 1930 (1996).

[2] L. Palumbo, Analytic Calculation of the Impedance of a Discontinuity, Particle Accelerators, 25, 1990.

[3] A.V. Fedotov and R.L. Gluckstern, General Analysis of the Longitudinal Coupling Impedance of a Rectangular Slot in a Thin Coaxial Liner, following paper.

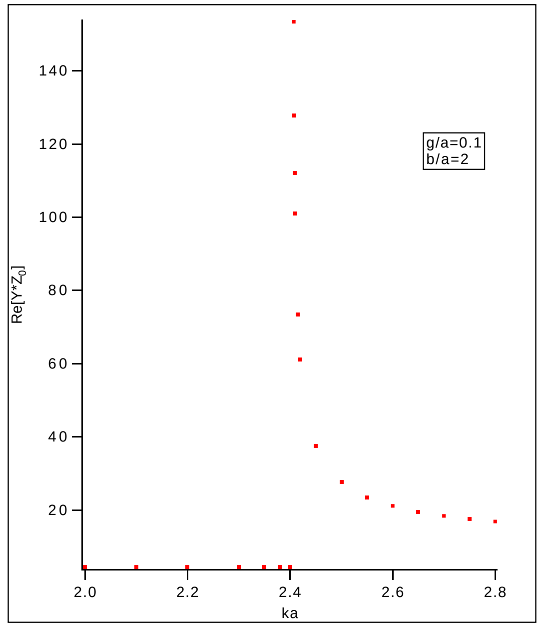

Figure 3: Real part of the admittance of an annular cut in a coaxial liner.

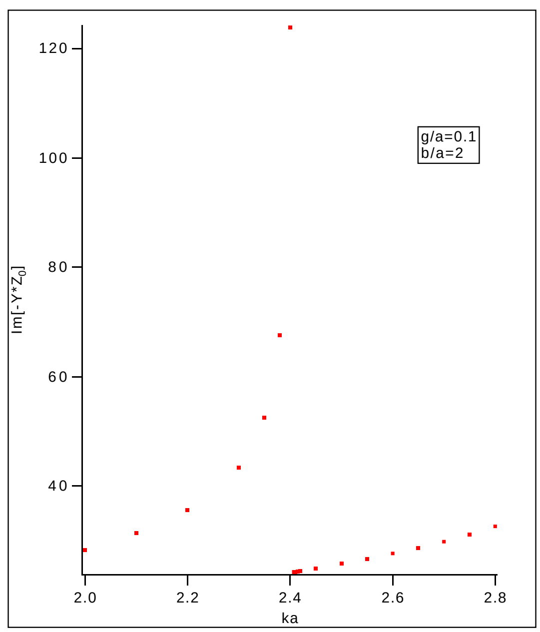

Figure 4: Imaginary part of the admittance of an annular cut in a coaxial liner.