NONLINEAR ACCELERATOR PROBLEMS VIA WAVELETS:
4. SPIN-ORBITAL MOTION

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Abstract
In this series of eight papers we present the applications of methods from wavelet analysis to polynomial approximations for a number of accelerator physics problems. In this part we consider a model for spin-orbital motion: orbital dynamics and Thomas-BMT equations for classical spin vector [9]:

\[ H = H_{orb}(q, p, t) + w(q, p, t) \cdot s. \] (2)

More explicitly we have

\[ \frac{dq}{dt} = \frac{\partial H_{orb}}{\partial p} + \frac{\partial (w \cdot s)}{\partial p}, \]
\[ \frac{dp}{dt} = -\frac{\partial H_{orb}}{\partial q} - \frac{\partial (w \cdot s)}{\partial q}, \]
\[ \frac{ds}{dt} = [w \times s] \] (3)

We will consider this dynamical system also in another paper via invariant approach, based on consideration of Lie-Poisson structures on semidirect products. But from the point of view which we used in this paper we may consider the similar approximations as in the preceding parts and then we also arrive to some type of polynomial dynamics.

3 VARIATIONAL APPROACH IN BIORTHOGONAL WAVELET BASES

Because integrand of variational functionals is represented by bilinear form (scalar product) it seems more reasonable to consider wavelet constructions [10] which take into account all advantages of this structure. The action functional for loops in the phase space is [11]

\[ F(\gamma) = \int_\gamma pdq - \int_0^1 H(t, \gamma(t))dt \] (4)

The critical points of \( F \) are those loops \( \gamma \), which solve the Hamiltonian equations associated with the Hamiltonian...
$H$ and hence are periodic orbits. By the way, all critical points of $F$ are the saddle points of infinite Morse index, but surprisingly this approach is very effective. This will be demonstrated using several variational techniques starting from minimax due to Rabinowitz and ending with Floer homology. So, $(M,\omega)$ is symplectic manifolds, $H : M \to R$, $H$ is Hamiltonian, $X_H$ is unique Hamiltonian vector field defined by $\omega(X_H(x),v) = -dH(x)(v)$, $v \in T_xM$, $x \in M$, where $\omega$ is the symplectic structure. A T-periodic solution $x(t)$ of the Hamiltonian equations $\dot{x} = X_H(x)$ on $M$ is a solution, satisfying the boundary conditions $x(T) = x(0), T > 0$. Let us consider the loop space $\Omega = C^\infty(S^1,R^{2n})$, where $S^1 = R/\mathbb{Z}$, of smooth loops in $R^{2n}$. Let us define a function $\Phi : \Omega \to R$ by setting

$$\Phi(x) = \int_0^1 \frac{1}{2} < -J\dot{x}, x > dt - \int_0^1 H(x(t))dt, \ x \in \Omega$$

The critical points of $\Phi$ are the periodic solutions of $\dot{x} = X_H(x)$. Computing the derivative at $x \in \Omega$ in the direction of $y \in \Omega$, we find

$$\Phi'(x)(y) = \frac{d}{dc}\Phi(x + cy)|_{c=0} = \int_0^1 < -J\dot{x} - \nabla H(x), y > dt$$

Consequently, $\Phi'(x)(y) = 0$ for all $y \in \Omega$ iff the loop $x$ satisfies the equation

$$-J\dot{x}(t) - \nabla H(x(t)) = 0,$$

i.e. $x(t)$ is a solution of the Hamiltonian equations, which also satisfies $x(0) = x(1)$, i.e. periodic of period 1. Periodic loops may be represented by their Fourier series: $x(t) = \sum e^{2\pi i Jk} x_k, x_k \in R^{2k}$, where $J$ is quasivector structure. We give relations between quasivector structure and wavelets in our other paper. But now we need to take into account underlying bilinear structure via wavelets. We started with two hierarchical sequences of approximations spaces \cite{10}:

$$\ldots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \ldots, \quad (8)$$

$$\ldots \tilde{V}_{-2} \subset \tilde{V}_{-1} \subset \tilde{V}_0 \subset \tilde{V}_1 \subset \tilde{V}_2 \ldots, \quad (9)$$

and as usually, $W_0$ is complement to $V_0$ in $V_1$, but now not necessarily orthogonal complement. New orthogonality conditions have now the following form:

$$\tilde{W}_0 \perp V_0, \ W_0 \perp \tilde{V}_0, \ V_j \perp \tilde{W}_j, \ \tilde{V}_j \perp W_j \quad (9)$$

translates of $\psi$ span $W_0$, translates of $\tilde{\psi}$ span $\tilde{W}_0$. Biorthogonality conditions are

$$< \psi_{jk}, \tilde{\psi}_{j'k'} > = \int_{-\infty}^{\infty} \psi_{jk}(x) \tilde{\psi}_{j'k'}(x)dx = \delta_{kk'}\delta_{jj'}, \quad (10)$$

where $\psi_{jk}(x) = 2^{j/2}\psi(2^jx-k)$. Functions $\varphi(x), \tilde{\varphi}(x-k)$ form dual pair: $< \varphi(x-k), \tilde{\varphi}(x-\ell) > = \delta_{kl}, < \varphi(x-k), \tilde{\varphi}(x-\ell) >= 0$. Functions $\varphi, \tilde{\varphi}$ generate a multiresolution analysis. $\varphi(x-k), \psi(x-k)$ are synthesis functions, $\tilde{\varphi}(x-\ell), \tilde{\psi}(x-\ell)$ are analysis functions. Synthesis functions are biorthogonal to analysis functions. Scaling spaces are orthogonal to dual wavelet spaces. Two multiresolutions are intertwining $V_j + W_j = V_{j+1}, \quad V_j + \tilde{W}_j = \tilde{V}_{j+1}$.

These are direct sums but not orthogonal sums.

So, our representation for solution has now the form

$$f(t) = \sum_{j,k} \tilde{b}_{jk}\psi_{jk}(t), \quad (11)$$

where synthesis wavelets are used to synthesize the function. But $b_{jk}$ come from inner products with analysis wavelets. Biorthogonality yields

$$\tilde{b}_{km} = \int f(t)\tilde{\psi}_{km}(t)dt. \quad (12)$$

So, now we can introduce this more complicated construction into our variational approach. We have modification only on the level of computing coefficients of reduced non-linear algebraical system. This new construction is more flexible. Biorthogonal point of view is more stable under the action of large class of operators while orthogonal (one scale for multiresolution) is fragile, all computations are much more simpler and we accelerate the rate of convergence. In all types of Hamiltonian calculation, which are based on some bilinear structures (symplectic or Poissonian structures, bilinear form of integrand in variational integral) this framework leads to greater success. In particular cases we may use very useful wavelet packets from Fig. 1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{wavelet_packets.png}
\caption{Wavelet packets.}
\end{figure}

\section{Evaluation of Nonlinearities Scale by Scale. Non-regular Approximation.}

We use wavelet function $\psi(x)$, which has $k$ vanishing moments $\int x^k\psi(x)dx = 0$, or equivalently $x^k = \sum c_{k}\varphi(x)$.
for each $k$, $0 \leq k \leq K$. Let $P_j$ be orthogonal projector on space $V_j$. By tree algorithm we have for any $u \in L^2(\mathbb{R})$ and $t \in \mathbb{Z}$, that the wavelet coefficients of $P_t(u)$, i.e. the set $\{ u, \psi_{j,k} \}, j \leq \ell - 1, k \in \mathbb{Z}$ can be compute using hierarchical algorithms from the set of scaling coefficients in $V_{t}$, i.e. the set $\{ u, \varphi_{t,k} \}, k \in \mathbb{Z}$ [12]. Because for scaling function $\varphi$ we have in general only $\int \varphi(x)dx = 1$, therefore we have for any function $u \in L^2(\mathbb{R})$:

$$\lim_{j \to \infty, k \to j^{-1}} \int \frac{u}{2^{j/2}} < u, \varphi_{j,k} > = 0$$

(13)

If the integer $n(\varphi)$ is the largest one such that

$$\int x^{\alpha} \varphi(x)dx = 0 \quad \text{for} \quad 1 \leq \alpha \leq n$$

(14)

then if $u \in C^{n+1}$ with $u^{(n+1)}$ bounded we have for $j \to \infty$ uniformly in $k$:

$$\lim_{j \to \infty, k \to j^{-1}} \int \frac{2^{j/2}}{u} < u, \varphi_{j,k} > = O(2^{-j(n+1)})$$

(15)

Such scaling functions with zero moments are very useful for us from the point of view of time-frequency localization, because we have for Fourier component $\tilde{\Phi}(\omega)$ of them, that exists some $C(\varphi) \in \mathbb{R}$, such that for $\omega \to 0$ $\tilde{\Phi}(\omega) = 1 + C(\varphi) \mid \omega \mid^{2r+2}$ (remember, that we consider $r$-regular mutiresolution analysis). Using such type of scaling functions lead to superconvergence properties for general Galerkin approximation [12]. Now we need some estimates in each scale for non-linear terms of type $u \to f(u) = f \circ u$, where $f$ is $C^\infty$ (in previous and future parts we consider only truncated Taylor series action). Let us consider non regular space of approximation $\mathbb{V}$ of the form

$$\mathbb{V} = V_q \oplus \sum_{q \leq j \leq p-1} \mathbb{W}_j$$

(16)

with $\mathbb{W}_j \subset W_j$. We need efficient and precise estimate of $f \circ u$ on $\mathbb{V}$. Let us set for $q \in \mathbb{Z}$ and $u \in L^2(\mathbb{R})$

$$\prod_{k \in \mathbb{Z}} f_q(u) = 2^{-q/2} \sum_{k \in \mathbb{Z}} f(2^{q/2} < u, \varphi_{q,k} > \cdot \varphi_{q,k}$$

(17)

We have the following important for us estimation (uniformly in $q$) for $u, f(u) \in H^{n+1}$ [12]:

$$\| P_q f(u) \|_{L^2} - \prod f_q(u) \|_{L^2} \equiv O(2^{-(n+1)q})$$

(18)

For non regular spaces (16) we set

$$\prod_{j=0}^{\infty} f_{qj}(u) = \prod_{j=0}^{\infty} \prod_{j=qj \to \infty} f_{qj}$$

(19)

Then we have the following estimate:

$$\| P_{\mathbb{V}} f(u) \|_{L^2} - \prod f_{\mathbb{V}}(u) \|_{L^2} \equiv O(2^{-(n+1)q})$$

(20)

uniformly in $q$ and $\mathbb{V}$ (16). This estimate depends on $q$, not $p$, i.e. on the scale of the coarse grid, not on the finest grid used in definition of $\mathbb{V}$. We have for total error

$$\| f(u) - \prod f_{\mathbb{V}}(u) \|_{L^2} = \| f(u) - P_{\mathbb{V}} f(u) \|_{L^2}$$

$$+ \| P_{\mathbb{V}} f(u) - \prod f_{\mathbb{V}}(u) \|_{L^2}$$

(21)

and since the projection error in $\mathbb{V}$: $\| f(u) - P_{\mathbb{V}} (f(u)) \|_{L^2}$ is much smaller than the projection error in $V_q$ we have the improvement (20) of (18). In concrete calculations and estimates it is very useful to consider approximations in the particular case of c-structured space:

$$\mathbb{V} = V_q + \sum_{j=q}^{p-1} \text{span}\{\psi_{j,k}, k \in [2^{(j-1)} - c, 2^{(j-1)} + c] \mod 2^j\}$$

(22)

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5 REFERENCES


