ANALYTIC COMPUTATION OF BEAM IMPEDANCES IN COMPLEX HETEROGENOUS ACCELERATOR GEOMETRIES

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Abstract

A general framework has been developed for computing longitudinal and transverse beam impedances in accelerator pipes consisting of several coaxial tubes with non simple transverse geometry, possibly made of composite materials and/or bearing special features like e.g., holes or slots, based on the combined use of Lorentz reciprocity theorem, Debye potentials, extended impedance boundary conditions, and generalized transmission line (waveguide) circuit concepts. The results are applied to the proposed LHC design.

I. INTRODUCTION

Rounded corners, multi-layered or composite walls, pumping holes, etc., make accelerator cross-sectional pipe geometries not simple. Beam coupling impedances must then be computed by numerical methods, analytic solutions being available only for simple (transverse) geometries where, e.g., the Laplacian is separable, and simple (e.g., perfect conductor) boundary conditions. Analytic, even approximate, solutions on the other hand are relatively appealing, as they provide an immediate insight into the role played by the design parameters.

In this paper we briefly summarize a general approach for the analytic computation of beam coupling impedances in complex structures, together with some representative results pertinent to the proposed LHC liner.

II. PERTURBED COUPLING IMPEDANCES

Stationary perturbative formulæ for the beam (complex, frequency dependent) coupling impedances per unit length [1] of pipes with non-simple cross sections and/or boundary conditions can be obtained from the electromagnetic reciprocity (Lorentz) theorem, and relate the beam coupling impedance $Z^0_\|$, $Z^0_\perp$ of a simple, unperturbed pipe assumed known, to that of another pipe differing from the former by some perturbation in the boundary geometry and/or constitutive properties [2], [3]. They read:

$$Z^\perp - Z^\parallel = \frac{\kappa_0}{\beta_0 c} \left\{ Y_0 \int_{\partial S} E_{\|,w}^* E_{\|,w}^{irr} \right\}$$

$$\cdot \left[ \beta_0 E_{\|,w}^{irr} + \beta_0^{-1} E_{\perp,all}^{irr} \right] dt - \int_{\partial S} E_{\|,w}^* E_{\|,w}^{irr} dt \right\}, \quad (1)$$

for the longitudinal impedance, and:

$$\frac{Z^\perp - Z^\parallel}{Z^\perp} = \frac{\kappa_0}{\beta_0 c} \lim_{\rho_0 \to 0} \left\{ Y_0 \int_{\partial S} Z_{w,all} \nabla_{\rho_0} E_{\rho_0,\perp,all}^{irr,*} \left( \rho_0, \rho_0 \right) \nabla_{\rho_0} \left[ \beta_0 E_{\|,w}^{irr} + \beta_0^{-1} E_{\perp,all}^{irr} \right] \left( \rho_0, \rho_0 \right) dt +$$

$$- \int_{\partial S} \left[ \nabla_{\rho_0} E_{\|,w}^* \left( \rho_0, \rho_0 \right) \right] \nabla_{\rho_0} E_{\|,w}^{irr,*} \left( \rho_0, \rho_0 \right) \right\} \quad (2)$$

for the transverse one. In (1) and (2) $\bar{\rho}$ is the transverse position, $\nabla_{\rho_0}$ is the transverse gradient, $\beta_0 = \nu_0 / c$ = beam velocity/light velocity (in vacuum), $\partial S$ is the pipe cross-section boundary, $E_{\|,w}^{irr}$, $E_{\perp,all}^{irr}$ are the (known, $k$-domain) solenoidal and irrotational parts (Helmholtz theorem) of the electric field, in the simple, unperturbed pipe, $Q$ is the beam charge, $Y_0$ is the free-space wave-admittance, $Z_{w,all}$ the (complex, frequency dependent) surface impedance describing the local properties of the pipe wall, and $k = \omega / \beta_0 c$.

The first integral on the r.h.s. of (1) and (2) accounts for the effect of constitutive perturbations of the boundary, and thus is nonzero if and only if $Z_{w,all}$ is not identically zero on $\partial S$. The second integral on the r.h.s. of eqs (1) and (2), on the other hand, accounts for the effect of geometrical perturbations of the boundary, and is non-zero if and only if the unperturbed axial field component $E_{\|,w}$ is not identically zero on $\partial S$. Accordingly the second integral in (1) and in (2) effectively spans only the geometrically perturbed boundary subset $\partial S - \partial S_n$.

III. IMPEDANCE BOUNDARY CONDITIONS

Equations (1) and (2) are based on a simple Leontovich (impedance) boundary condition (BC), at the pipe wall [4]:

$$\bar{n} \times (\bar{n} \times \bar{E} - Z_{w,all} \bar{H}) |_{w,all} = 0 \quad (3)$$

$\bar{n}$ being the local normal unit vector. In the spirit of Leontovich BC, the penetration of EM fields from vacuum into multilayered lossy media can be viewed as lossy transverse electromagnetic (TEM) wave propagation in the direction (locally) normal to the interfaces.

The equivalence between (TEM) waves in stratified media and voltage waves through cascaded transmission lines (TL) can thus be used to compute the wall impedance $Z_{w,all}$ at the inner surface of the beam screen, by repeated application of the impedance transport formula across a homogeneous TL section with length $l$ characteristic impedance $Z_c$ and propagation constant $k$:

$$Z_{in} = Z_c Z_{\ell} + j Z_c \tanh(jk\ell) \quad (4)$$

where $Z_{\ell}$ is the impedance connected to the output port, and $Z_{in}$ is the impedance seen at the input port.

2Note that $\bar{Z}$ is a tensor, in general. See [2], and references quoted therein.
3The impedances are obviously independent of $Q$, since the fields in (1) and (2) are proportional to $Q$.
4Leontovich BC can be applied provided: i) the magnitude of the relative index of refraction of the (first) medium where the field penetrates is large, and ii) the penetration depth is small compared to the (minimum) thickness of the medium and the curvature radius of its boundary [5].

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**Notes:**

1. Equations (1) and (2) are accurate for suitably small perturbations; they are exact whenever the coupling impedances depend linearly on $Z_{w,all}$.

2. Note that $\bar{Z}$ is a tensor, in general. See [2], and references quoted therein.

3. The impedances are obviously independent of $Q$, since the fields in (1) and (2) are proportional to $Q$.

4. Leontovich BC can be applied provided: i) the magnitude of the relative index of refraction of the (first) medium where the field penetrates is large, and ii) the penetration depth is small compared to the (minimum) thickness of the medium and the curvature radius of its boundary [5].
The longitudinal impedance can be used to compute the energy lost by the beam per unit pipe length (parasitic loss, \( \Delta E/L \)) [1]. For a Gaussian bunch with r.m.s. length \( \sigma_z \), using eq. (1), one has [6]:

\[
\frac{\Delta E}{L} = \frac{a^2}{Q^2 \epsilon Z_0} W \left( \frac{\sigma_z}{a} \right) G^{(1)} \left( \frac{d}{a} \right)
\]

where for LHC \( a^2 Q^{-2} \epsilon^{-1} Z_0^{-1} = 41.91 \text{ J} \cdot \text{m}^{-1} \), the function \( G^{(1)} \) has been already defined, and:

\[
W \left( \frac{\sigma_z}{a} \right) = Y_0 \int _{-\infty}^{+\infty} e^{-\frac{\sigma_z^2 y^2}{a^2 \beta_0}} Re \left[ Z_{\text{wall}} \left( \frac{\theta y}{a} \right) \right] \, dy
\]

is displayed in Fig. 5.

V. CONCLUSIONS

We introduced a general and systematic framework for computing beam coupling impedances and related quantities in possibly composite, multilayered, complex-shaped accelerator pipes, yielding accurate results in analytic form. We believe that the above could be a valuable tool for predicting the performance and optimizing the design of planned and/or existing accelerators.

References

Figure 2. The functions $G_{||}^{(1)}$ and $G_{||}^{(2)}$.

Figure 3. Multilayered wall and TL equivalent circuit.

Figure 4. LHC wall impedance (rounded corners), real part.

Figure 5. The function $W(\sigma_z/a)$.