PRACTICAL CRITERION OF TRANSVERSE COUPLED-BUNCH HEAD-TAIL STABILITY

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Abstract

Analytical outcome of the paper is a few formulae to simplify practical threshold calculations of transverse coupled-bunch head-tail instability caused by narrow-band impedances in a proton synchrotron, which provide a useful quantitative view on how to keep the instability under control with chromaticity and cubic-nonlinearity correctors of the magnetic field. The formulae include: (i) the envelopes of head-tail mode formfactors expressed via a pair of averages over a bunch longitudinal distribution, and (ii) expressions of the effective betatron tune spread in terms of multipole decomposition series. Let the unperturbed bunch be given by its distribution function \( f(J_x, J_z) \) normalized to unit.

II. BASIC SET OF EQUATIONS

Beam dipole moment \( D_y(\vartheta, t) \) and deflecting Lorentz force \( e\mathbf{\nabla}_y(\vartheta, t) \) averaged over beam transverse distribution are decomposed into \( \sum_k D_{yk}\mathbf{e}\mathbf{\nabla}_y(\vartheta, t) = \sum k D_{yk}\mathbf{e}\mathbf{\nabla}_y(\vartheta, t) = \sum k D_{yk}\mathbf{e}\mathbf{\nabla}_y(\vartheta, t) \), \( y = x, z \), with \( \psi_y(J_y) = dy/dt \) being frequency of non-linear oscillations. Let the unperturbed bunch be given by its distribution function \( f(J_x, J_z) \) normalized to unit.

\[ X_{kk}^{(\vartheta)}(\Omega) = \left( \frac{e\mathbf{\nabla}_y(\vartheta, t)}{2\pi R_0} \right) \sum_{y=x,z} Z_{k}^{(\vartheta)}(k\omega_0 + \Omega) D_{yk}(\Omega) \]  

(1)

where \( R_0 \) is the orbit radius, \( \beta \) is beam reduced velocity, \( Z_k^{(\vartheta)}(\omega) \) is the transverse (dipole) coupling impedance. Its \((xy)\) matrix nature accounts for the vacuum chamber cross-section anisotropy, if any. It may result in coupling of coherent motions along \( x, z \). Here we study the standard, axisymmetric case \( Z_k^{(xy)}(\omega) = Z_k(\omega)\delta_{x,y}, y = x, z \).

Consider a beam of average current \( J_0 \) in \( M \leq h \) identical and equiparted bunches, \( h \) is the main RF harmonic number, \( h/M \) is an integer. As it follows from the Vlasov’s linearized Eq., the transverse BTF is

\[ D_{xk}(\Omega) = \left( \frac{i\pi R_0}{2\omega_0} \right) \sum_{k} \delta_{k-k'} \frac{\Delta k M}{\Omega - \omega_{k'}(\Omega)} Y_{kk'}^{(x)}(\Omega) \]

(2)

where \( \beta_{k'} \approx \frac{R(\omega_{k'})}{\omega_{k'}(0)} \) is \( \beta \)-function averaged along the ring, \( E \) is the total energy of the beam, \( \delta_{k-k'} \) is the Kronecker’s delta-symbol. The dispersion integrals \( Y_{kk'}^{(x)}(\Omega) \) are put down in terms of multipole decomposition series

\[ Y_{kk'}^{(x)}(\Omega) = \left( \frac{i\omega_0}{\pi} \right) \sum_{m_x = \pm 1} m_x \sum_{m_y = -\infty}^{\infty} \sum_{\Omega} \int_{0}^{\infty} dJ_x dJ_z \int_{0}^{\infty} dJ_y \frac{\partial^2 f(j_x, j_z, j_y)}{\partial j_y} \]

\[ \int_{0}^{\infty} dJ_\theta dJ_{\psi} \int_{0}^{\infty} dJ_x \int_{0}^{\infty} dJ_z \frac{\partial f(j_x, j_z, j_y)}{\partial j_y} \]

\[ \frac{\partial}{\partial j_y} \theta(j_x, j_z, j_y) \]

Here \( \Delta k = \chi_x/\eta - \omega_{k'}(0)/\omega_0 \); \( \chi_x \equiv (p_x/\omega_0)(\partial \omega_{k'}(0)/\partial p_x) \) is chromaticity of the ring; \( \eta = \alpha - \gamma^{-2} \); \( \alpha \) is orbit compaction factor; \( \gamma \) is relativistic factor; functions \( I_{m_x, m_y, m_z}^{(x)}(j_x, j_z) \) are the coefficients of series which expand a plane wave \( e^{ikx} \delta(\theta, \psi) \) into sum over longitudinal multipoles: \( \sum_{m_x} I_{m_x, m_y, m_z}^{(x)}(j_x, j_z) e^{im_0 \psi} \).

Treated jointly, Eqs.1,2 yield \( M \) eigenvalue problems

\[ \lambda(\omega) D_{xk} = R_{xk}^{-1} \sum_{\Omega} Y_{kk'}^{(x)}(\Omega) Z_{k'}^{'(\vartheta)}(\omega_0 + \Omega) D_{xk'}(\Omega) \]

(3)

\[ (\Omega + 1) e^{ikx} \delta(\theta, \psi) \]

(4)

\[ k^l = n + i l M \]

Each of these stands for one of \( M \) normal coupled-bunch modes labeled by, say, \( n = 0, 1, \ldots \). \( R_{xk} \) has the dimension Ohm/m of a transverse impedance,

\[ R_{xk} = \left( A_{\beta} E / (\epsilon J_0(\beta_{x})) \right) \]

Generally, the characteristic Eq. of coherent oscillations is

\[ 1 = \lambda_\ell(\Omega), \quad \ell = (n, \ldots) \]

\[ \lambda_\ell(\Omega) \]

being an eigenvalue of Eq.4. On solving this Eq. w.r.t. \( \Omega \), one arrives at an eigenfrequency of the \( \ell \)-th coherent mode, the unstable ones having \( \text{Im} \Omega > 0 \).
III. A SINGLE-MODE APPROACH

To simplify the problem, we make specific the within-bunch mode subindices \( m_s, m_r, m_d, r \) that follow the coupled-bunch mode index \( n \) in \( \ell = (n, \ldots) \), and state conditions under which such a mode can exhibit itself solely.

1. Derivation of Eq.3 tacitly implies \( m_s = 0 \) which is due to the ‘smooth’ treatment of the uncoupled betatron head-tail modes nonresonant items, the so called approximation of uncoupled \( m \)-initeness, we take the upper sideband \( m \) would not overlap, and either can be treated separately. For def-

2. Put the working point far from 2-nd order SBRs,

3. Take bunches with a small nonlinearity,

Then, at each sideband \( \omega \approx k \omega_0 + \omega_r(0) + m_d \omega_r(0) \) near instability threshold (Im\( \omega_0 \to +0 \)) a single resonant term whose \( m_d = m_{d'} \) would dominate in the BTF. On dropping the rest, nonresonant items, the so called approximation of uncoupled head-tail modes \( m_d \) is arrived at.

4. Assume \( F_x(\mathcal{J}_r, \mathcal{J}_c) = F_{x}(\mathcal{J}_0) \cdot F_{x2}(\mathcal{J}_r, \mathcal{J}_c) \). On applying to Eq.8, ignore the longitudinal tune spread, \( \omega_0(\mathcal{J}_0) \approx \omega_0(0) \). Then, characteristic Eq.6 factorizes to

5. Effective (instability driving) impedance \( \zeta(\Omega) \) of mode \( \ell = (n, m_s=0, m_r=1, m_d, r) \) is the \( r \)-th eigenvalue of

6. with coupling resistance \( R_c \), resonant frequency \( \omega_c \) and band-
width \( \Delta \omega_c \), the latter two complying the restrictions

In this case only one \( k_1 \gg -\omega_0/\omega_0 \) or \( k_2 \ll -\omega_0/\omega_0 \) azimuthal harmonic of coupled-bunch mode \( n \) would fall inside the HOM bandwidth. Thus, Eq.11 reduces to

the unstable harmonic being \( k_2 \) (the slow betatron wave). As Re\( Z^{-1}_c(\omega) \approx \text{const} \) at \( \omega \approx \pm \omega_c \), the point \( R_x/\zeta(\Omega) \) which represents HOM’s effect at \( k = n + M \) in the threshold map moves almost parallel to imaginary axis of the complex plane \( Y \), the distance from the axis being \( |R_x|/(A_{k_2}^0 R_c) \) (it does vary insignificantly due to \( A_{k_2}^0 \)). Thus, the beam stability is surely guaranteed given

where \( A_x \) is a maximal \( \text{Re} \) -extension of threshold map,

Being a sufficient stability criterion, inequality Eq.16 becomes a necessary one in large rings with \( \Delta \omega_c \approx \zeta \omega_0 \). Up to HOM bandwidth \( \Delta \omega_c \) and \( \omega_0 \approx \omega_c \), one can insert \( k_2 \approx -(\omega_c + \omega_r)/\omega_0 \) into \( A_{k_2}^0 \) to transform it into the longitudinal formfactor,

which is a function of the external parameters only: \( F_0(\mathcal{J}_0), \omega_c/\omega_0, \chi/\eta \). On adopting the above assumptions, one finally arrives at the stability criterion

with two bunch formfactors \( \Lambda_0, \Lambda_x \) left to be estimated.

IV. FORMFACTORS

A. Longitudinal Formfactor

According to Eq.8, \( |\delta \omega_0| \ll \omega_0(0) \) and the law of motion along \( \vartheta \) is just \( \vartheta(\mathcal{J}_0, \vartheta_0) \approx \sqrt{\mathcal{J}_0} \cos(\vartheta_0 + \varphi_0) \). Hence,

with \( J_m(y) \) denoting Bessel functions of the \( m \)-th order, \( \Delta \vartheta_0 = \Delta \vartheta(\mathcal{J}_0) \) being longitudinal half-width of the bunch (in
other words, oscillation amplitude along \( \theta \) at a phase-plane trajectory \( J_\theta = J_{\theta_0} \). It implies the following reflection properties

\[
\Lambda_\theta^{(-m_\theta)} = \Lambda_\theta^{(m_\theta)}; \quad \Lambda_\theta^{(m_\theta)}(k, \Delta \theta) = \Lambda_\theta^{(m_\theta)}(k, \Delta \theta). \tag{22}
\]

Globally, formfactor \( \Lambda_\theta^{(0)} \) of the rigid-bunch head-tail mode \( m_\theta = 0 \) dominates, envelope Eq.19 thus coinciding with \( \Lambda_\theta^{(0)} \) (except for a small region near \(|k, \Delta \theta| \approx 3–5\) where mode \( |m_\theta| = 1 \) may exhibit itself).

Replace \( J_{\theta}^{(n)}(y) \) in Eq.18 by their quadratic small-argument and trigonometric large-argument (with \( 1/2 \) substituted for \( \cos^2(\ldots) \)) asymptotes. On integrating, one obtains with accuracy sufficient for practical purposes,

\[
\Lambda_\theta \approx \Lambda_\theta^{(0)} \approx \left\{ \begin{array}{ll}
1 - \frac{1}{2} \langle \theta^2 \rangle |k, \Delta \theta|^2, & |k, \Delta \theta| \lesssim 2; \\
\frac{1}{2} \langle \theta^{-1} \rangle |k, \Delta \theta|^{-1}, & |k, \Delta \theta| \gtrsim 3.
\end{array} \right. \tag{23}
\]

Here, numerical factors \( \langle \theta^2 \rangle \lesssim 1 \) and \( \langle \theta^{-1} \rangle \gtrsim 1 \) with \( \theta = \theta / |\Delta \theta| \) are, respectively, mean-square and mean-reciprocal reduced half-widths of a bunch,

\[
\frac{\langle \theta^2 \rangle}{\langle \theta^{-1} \rangle} = \int_0^\infty \frac{J_\theta(\theta)}{J_\theta(\theta)} \, dJ_\theta \tag{24}.
\]

B. Transverse Formfactor

Let us introduce normalized to unit 1-D transverse distributions \( F_x(J_x) \) and \( F_z(J_z) \) where, \( F_x(J_x) \) is

\[
F_x(J_x) = \int_0^\infty F_x(J_x, J_z) \, dJ_z. \tag{25}
\]

Take into account the cubic nonlinearity of the magnetic field which results in betatron tune spread

\[
\omega_x(J_x, J_z) \approx \omega_x(0) + \frac{\delta \omega_x}{\partial J_x}(0) J_x + \frac{\delta \omega_x}{\partial J_z}(0) J_z, \tag{26}
\]

coefficients at \( J_x \) and \( J_z \) being controlled with the octupole correctors.

Formfactor \( \Lambda_x \) is amenable to straightforward calculations in two particular cases. Indeed, for \( \delta \omega_x / \partial J_z = 0 \)

\[
\Lambda_x = \frac{b_{x,x}}{\delta \omega_x / \omega | \right) \frac{\partial \omega_x}{\partial J_x}(0) J_x, \tag{27}
\]

\[
b_{x,x} = \max_{J_z \geq 0} \left( J_x \right) \right) \right). \tag{28}
\]

On the other hand, for \( \delta \omega_x / \partial J_x = 0 \) it follows that

\[
\Lambda_x = \frac{b_{x,x}}{\delta \omega_x / \omega | \right) \frac{\partial \omega_x}{\partial J_z}(0) J_z, \tag{29}
\]

\[
b_{x,x} = \max_{J_z \geq 0} \left( F_z \right) \right) \right) = J_z \right) F_z(0). \tag{30}
\]

Here \( J_{x}, J_{z} \) are the action variables at the (effective) edge of the bunch; \( \delta \omega_x / \omega \) are the partial betatron tune spreads, both having an arbitrary sign.

On inserting Eq.26 into Eq.17, one sees that \( \Lambda_x \) is kept intact by a simultaneous reversal of signs in \( \delta \omega_x / \omega \) and \( \delta \omega_z / \omega \). Therefore, taking into account the exact Eqs.27–30 and inflicting no loss to generality, rewrite \( \Lambda_x \) as

\[
\Lambda_x = f_x \left( \frac{\delta \omega_x / \omega}{ \omega_{,0} b_{x,x}} \right) \left( \frac{\delta \omega_x / \omega}{ \omega_{,0} b_{x,x}} \right) -1/2 \tag{31}
\]

Dots in \( f_x \) show its dependence on details of joint distribution \( F_x(J_x, J_z) \). Fortunately, the calculations show that \( f_x \) is rather insensitive to \( \omega_x / \omega \) for realistic distributions. With a good accuracy Eq.31 can be used with \( f_x \approx 1 \), which plainly puts down transverse formfactor as a reciprocal of an effective betatron tune spread,

\[
\Lambda_x \approx \left( \frac{\delta \omega_x / \omega}{ \omega_{,0} b_{x,x}} \right) -1/2 \tag{32}
\]

Eqs.20, 23, 32 are the sought-for tool for practical estimates of head-tail instability thresholds.

V. EXAMPLE OF APPLICATION

Consider the UNK 1-st Stage which is to be equipped with \( N = 8 \times 2 = 16 \) conventional copper cavities, their length being \( L = 0.5 \) m; radius \( r_0 \approx 0.577 \) m; surface resistance \( R_\text{c} = 1.7 \times 10^{-8} \) Ohm-m. The figure shows coupling impedances per one cavity for dipole HOMs \( E_{1p} \). Tolerable values of \( R_\text{c} \) are found with Eqs.20,23,32; \( j_0 = 1.4 \) A; \( \alpha = 4.95 \times 10^{-4}; \) \( J_x / \omega_0 = 55.7; \delta \omega_x / \omega_0 = \delta \omega_x / \omega_0 = 0.5 \times 10^{-2}; \) \( \beta_x = 30 \) m. Curve A: injection at \( E = 65 \) GeV with \( h \Delta \theta / \pi = 0.54 \) and standard \( \chi_x \approx +3 \). Curve B: the same for \( \chi_x \approx +3 \) at \( E = 600 \) GeV, \( h \Delta \theta / \pi = 0.38 \). Curve C: large negative \( \chi_x \approx -30 \) as required by a slow extraction scheme.
Evidently, at least nine of the UNK cavity transverse HOMs are to be damped with a dedicated system. More details on the topic can be found in Ref.[1].

References