A NEW MODEL OF THE e^+e^- BEAM-BEAM INTERACTION *

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Abstract

A new program to simulate the beam-beam interaction between asymmetric e^+ and e^- beams is being developed. Beam bunch distributions are expanded in terms of orthogonal basis functions constructed from solutions to the two-dimensional quantum mechanical harmonic oscillator. Including all solutions corresponding to oscillator energies up to the $N^{\rm th}$ level yields a basis which spans the set of functions composed of a product of a Gaussian times a Hermite polynomial of order N or lower. A consistent and economical description of non-Gaussian beam shapes is thus made possible. In addition, the use of continuous density functions effectively eliminates statistical fluctuations which may arise when beam bunches are modeled by tracking particles. The beam dynamics are encapsulated in matrices which operate on the expansion coefficients of the bunches. These matrices are computed once for each beam with any given set of basis functions and for any particular accelerator. The evolution of a beam distribution is computed by matrix multiplication.

I. INTRODUCTION

Currently used algorithms for the calculation of the effects of the beam-beam interaction in e^+e^- colliders [1], [2], [3] involve tracking representative particles through the machine. An advantage of this approach is that the physics of the bunch dynamics is modeled in a straightforward, clear manner. Also, the freedom of motion of the individual tracked particles permits arbitrary bunch distributions to evolve. However, their use of a finite number of particles allows for the possibility of statistical fluctuations, the magnitudes of which are sensitive to the number of particles tracked. In addition, it is difficult to model with particle tracking the behavior of the bunch core [$\leq \mathcal{O}(\sigma)$] and the behavior of the tails [$> \mathcal{O}(\sigma)$] simultaneously in a self-consistent manner. We have been developing a new description of a bunched beam in which the shapes the bunches can assume are constrained (albeit in an orderly and physically reasonable way) but can be described by a relatively small number of parameters. In addition, model-dependent statistical fluctuations are removed.

The new approach to modeling the time evolution of colliding bunched e^+ and e^- beams is outlined as follows:

- A bunch distribution is expanded in terms of orthogonal basis functions which are constructed from solutions to the two-dimensional quantum harmonic oscillator problem.
- A coordinate transformation is made to a normalized system in which each phase space ellipse of the beam is a circle. This removes the dominant, uncoupled first-order optics.
- The beam-beam interaction is modeled by dividing each beam into transverse slices and then colliding the beams slice by slice. For a given slice, the effect of its interaction with each slice in the other beam is assumed to be small so that a Taylor series expansion of the distribution can be used.
- The extreme relativistic limit is taken to model the electric field of a slice.
- The luminosity is the sum of the luminosities calculated for each slice-slice collision.

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The development of a new code is motivated by the need to reliably evaluate the effects of the beam-beam interaction under highly disruptive conditions such as those in a linac-ring collider. In such a collider, a relatively low energy e^- beam from a linac collides with a relatively high energy e^+ beam in a ring. [4] Because the current in the linac beam is necessarily low compared to the stored beam, very tight focusing is required to achieve a useful luminosity. The configuration is, accordingly, sensitive to beam blowup due to the beam-beam interaction. A reliable calculation of this effect is essential to any assessment of this alternative.

II. THE MODEL

A. Normalized coordinates

We wish to remove the effects of first-order optics on the beam distribution by making a transformation into normalized coordinates. The phase ellipse of a particle, which in one dimension is described by

$$\gamma x^2 + 2\alpha x\theta + \beta\theta^2 = \frac{A_x}{\pi}, \tag{1}$$

is skewed by an angle φ, where A_x is the particle's action in x and θ. The principal axes of the ellipse, $(\bar{x}, \bar{\theta})$, are rotated by this angle and then normalized. The rotation to the principal axes is

$$\bar{\mathbf{x}} = R^T\mathbf{x}, \tag{2}$$

where

$$R = \left(\begin{array}{cc} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{array}\right). \tag{3}$$

The Twiss parameters transform according to

$$\left(\begin{array}{cc} \bar{\gamma} & 0 \\ 0 & \bar{\beta} \end{array}\right) = R^T \left(\begin{array}{cc} \gamma & \alpha \\ \alpha & \beta \end{array}\right) R. \tag{4}$$

Equation 1 in these coordinates becomes

$$\bar{\gamma}\bar{x}^2 + \bar{\beta}\bar{\theta}^2 = \frac{A_x}{\pi}. \tag{5}$$

Then the normalized coordinates, (μ, ν), are defined such that

$$\mu^2 + \nu^2 = \frac{A_x}{\pi}, \tag{6}$$

and

$$\begin{pmatrix} \mu \\ \nu \end{pmatrix} = \begin{pmatrix} \sqrt{\gamma}\cos\varphi & \sqrt{\gamma}\sin\varphi \\ -\sqrt{\beta}\sin\varphi & \sqrt{\beta}\cos\varphi \end{pmatrix} \begin{pmatrix} x \\ \theta \end{pmatrix}. \qquad (7)$$

In all six dimensions, the transformation at a location s in the accelerator is given by

$$\begin{pmatrix} \mu \\ \nu \\ \epsilon \\ \omega \\ \zeta \\ \xi \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & & & & \\ a_{21} & a_{22} & \multicolumn{2}{c}{0} & \multicolumn{2}{c}{0} \\ & & b_{11} & b_{12} & & \\ \multicolumn{2}{c}{0} & b_{21} & b_{22} & \multicolumn{2}{c}{0} \\ & & & & c_{11} & c_{12} \\ \multicolumn{2}{c}{0} & \multicolumn{2}{c}{0} & c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} x \\ \theta \\ y \\ \phi \\ l \\ \delta \end{pmatrix}. \qquad (8)$$

B. Beam description

The distribution of a single bunch of an electron or positron beam is represented by

$$\rho = \sum_{nmhkij} C_{nmhkij}\, \rho_{nmhkij}(\mu, \nu, \epsilon, \omega, \zeta, \xi; s), \qquad (9)$$

where

$$\begin{aligned} \rho_{nmhkij}(\mu, \nu, \epsilon, \omega, \zeta, \xi; s) &= N_{nmhkij}\, H_n(\mu/b_x)\, H_m(\nu/b_x) \\ &\times H_h(\epsilon/b_y)\, H_k(\omega/b_y) H_i(\zeta/b_z)\, H_j(\xi/b_z) \\ &\times \exp\left[-\frac{(\mu^2+\nu^2)}{2b_x^2} - \frac{(\epsilon^2+\omega^2)}{2b_y^2} - \frac{(\zeta^2+\xi^2)}{2b_z^2}\right]. \end{aligned} \qquad (10)$$

These are the basis functions of the beam, which are the solutions to the two-dimensional quantum harmonic oscillator problem [5]. The functions $H_p(q)$ are Hermite polynomials in q of order p, and b_x, b_y, and b_z are arbitrary oscillator parameters chosen to yield the most compact description of a bunch. The normalization of the basis functions N_{nmhkij} is defined by

$$\int_0^\infty d^6\mu\, \rho_{nmhkij}(\mu, \nu, \epsilon, \omega, \zeta, \xi; s) = \left(\frac{1}{2}\right)^6, \qquad (11)$$

so that

$$\begin{aligned} &\int_{-\infty}^{\infty} d^6\mu\, \rho(\mu, \nu, \epsilon, \omega, \zeta, \xi; s) \\ &= \sum_{nmhkij} C_{nmhkij} = \begin{cases} 1 & n, m, h, k, i, \text{ and } j \text{ even} \\ 0 & n, m, h, k, i, \text{ or } j \text{ odd} \end{cases} \end{aligned} \qquad (12)$$

Including s as an argument of ρ is a reminder that the transformation from normalized to unnormalized coordinates is a function of the Twiss parameters evaluated at s.

The sum of the indices $\{n, m, h, k, i, j\}$ is limited by the order of the expansion of basis functions,

$$n + m + h + k + i + j \leq N. \qquad (13)$$

Because the solutions are that of the two-dimensional harmonic oscillator, the indices for each two-dimensional pair will share the relation

$$n = N' - m, \qquad (14)$$

where N' is an integer less than or equal to N. The number of C_{nmhkij} coefficients for an order N is given in Table I.

Table I
Number of basis function coefficients for order N

N		N		N	
0	1	3	84	6	924
1	7	4	210	7	1716
2	28	5	462	8	3003

C. Slicing the beam

To calculate the effects of the beam-beam interaction between the colliding bunches, it is necessary to divide each beam into slices and to calculate the incremental changes as the beams pass through each other. The transverse beam distribution for each slice is given by

$$\begin{aligned} \rho_\lambda(\mu, \nu, \epsilon, \omega; s) &= \sum_{nmhkij} C^\lambda_{nmhkij}\, \rho_{nmhk}(\mu, \nu, \epsilon, \omega; s) \\ &\times \sum_{ij} \int_{s_\lambda - t/2}^{s_\lambda + t/2} dl \int_{-\infty}^{\infty} d\delta\, \rho_{ij}(\zeta, \xi; s) \end{aligned} \qquad (15)$$

where t is the thickness of the slice. The value of the integral for each i and j is tabulated and the constant C_{nmhkij} becomes C^λ_{nmhkij} to distinguish the slices from one another. After the two beams have been stepped through one another, the sliced distribution is reassembled into a single bunch with a functional form describing the dependence on l and δ. The new coefficients of the reassembled bunch are determined by a χ^2 fit, which gives the expression

$$C_{nmhkij} = \frac{\sum_\lambda C^\lambda_{nmhkij} \int_{s_\lambda - t/2}^{s_\lambda + t/2} dl \int_{-\infty}^{\infty} d\delta\, \rho_{ij}(\zeta, \xi; s)}{\sum_\lambda \int_{s_\lambda - t/2}^{s_\lambda + t/2} dl \int_{-\infty}^{\infty} d\delta\, \rho_{ij}(\zeta, \xi; s)}. \qquad (16)$$

D. The beam-beam interaction

The effects of the beam-beam interaction are determined by making a Taylor expansion of each slice, keeping only the linear terms in the expansion. This is justified if the disruption of a single slice is small for each slice-slice interaction. Each time a slice collides with another slice, the resulting particle distribution, ρ', is

$$\rho' = \rho \;+\; \frac{\partial \rho}{\partial \theta}\,\delta\theta \;+\; \frac{\partial \rho}{\partial \phi}\,\delta\phi. \qquad (17)$$

The extreme relativistic limit is taken such that the electric field lines are entirely transverse. Each slice in the oncoming beam $\tilde{\rho}_\lambda(\mathbf{x}; s)$ is treated as part of an infinite line charge to calculate the transverse impulse on a beam [6],

$$(\delta\theta, \delta\phi) = -\frac{e^2}{2\pi\epsilon_0 E_0} \int \tilde{\rho}_\lambda(\mathbf{x}'; s) \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^2}\, d^2x', \qquad (18)$$

where all but horizontal and vertical coordinates have been integrated over,

$$\tilde{\rho}_\lambda(\mathbf{x}; s) = \int d\theta\, d\phi\, \rho_\lambda(\mu, \nu, \epsilon, \omega; s). \qquad (19)$$

A new set of coefficients describes the changed slice. Substituting Equation 15 into Equation 17 gives

$$\begin{aligned}\sum_{nmhkij} C'^{\lambda\,(1)}_{nmhkij}\,\rho_{nmhk} = \sum_{nmhkij} C^{\lambda\,(1)}_{nmhkij} \Bigg\{ \rho_{nmhk} \\ -\frac{e^2}{2\pi\epsilon_0 E_0}\left[\frac{\partial\mu}{\partial\theta}\frac{\partial}{\partial\mu}\rho_{nmhk} + \frac{\partial\nu}{\partial\theta}\frac{\partial}{\partial\nu}\rho_{nmhk}\right] \\ \times \int \tilde{\rho}^{(2)}_{\lambda'}(\mathbf{x}';s)\,\frac{x-x'}{|\mathbf{x}-\mathbf{x}'|^2}\,d^2x' \\ -\frac{e^2}{2\pi\epsilon_0 E_0}\left[\frac{\partial\epsilon}{\partial\phi}\frac{\partial}{\partial\epsilon}\rho_{nmhk} + \frac{\partial\omega}{\partial\phi}\frac{\partial}{\partial\omega}\rho_{nmhk}\right] \\ \times \int \tilde{\rho}^{(2)}_{\lambda'}(\mathbf{x}';s)\,\frac{y-y'}{|\mathbf{x}-\mathbf{x}'|^2}\,d^2x' \Bigg\}, \qquad (20)\end{aligned}$$

where we have labeled the beams 1 and 2. A similar equation will exist for beam 2. The closure relationship between Hermite polynomials allows the solution of $C'^{\lambda\,(1)}_{nmhkij}$ to take the form

$$\begin{aligned}C'^{\lambda\,(1)}_{nmhkij} &= C^{\lambda\,(1)}_{nmhkij} \\ &+ \sum_{\tilde{n}\tilde{m}\tilde{h}\tilde{k}} \sum_{n'\cdots j'} C^{\lambda\,(1)}_{\tilde{n}\cdots\tilde{j}} C^{\lambda'\,(2)}_{n'\cdots j'} F_{\tilde{n}\cdots\tilde{k},n\cdots j,n'\cdots j'}(s). \qquad (21)\end{aligned}$$

$F_{\tilde{n}\cdots\tilde{k},n\cdots j,n'\cdots j'}(s)$ is the tabulated result of the integral. The integral of Equation 20 can be broken down into several integrals of exponentials times polynomials, except for a separable one-dimensional integral which is integrated numerically and tabulated. The evolution of the beam distribution due to the beam-beam interaction is reduced to a sum over the coefficients of the basis functions.

The luminosity, $\mathcal{L}$, can now be calculated as the sum of the luminosities for each interaction between all pairs of slices. We also utilize the closure of the Hermite polynomials to reduce the luminosity to a form containing tabulated integrals, $G_{nmhk}(s_{\lambda\lambda'})$, and expansion coefficients,

$$\begin{aligned}\mathcal{L} &= f\sum_{\lambda\lambda'}\int d^4\mu\;\rho^{(1)}_{\lambda}(\mu,\nu,\epsilon,\omega;s_{\lambda\lambda'})\;\rho^{(2)}_{\lambda'}(\mu,\nu,\epsilon,\omega;s_{\lambda\lambda'}) \\ &= f\sum_{\lambda\lambda'}\sum_{nmhkij}\sum_{i'j'} C^{\lambda\,(1)}_{nmhkij} C^{\lambda'\,(2)}_{nmhki'j'} G_{nmhk}(s_{\lambda\lambda'}). \qquad (22)\end{aligned}$$

Equation 22 is a sum over all slices interacting at the point $s_{\lambda\lambda'}$ where the $\lambda^{\rm th}$ slice of beam 1 collides with the $\lambda'^{\rm\ th}$ slice of beam 2.

E. Longitudinal phase space

Four effects which directly influence the longitudinal bunch distribution are being included in the model: synchrotron radiation damping (in both transverse and longitudinal phase spaces), quantum excitation, RF acceleration, and beam energy changes during beam-beam collisions. Longitudinal damping is being modeled by a convolution of the beam distribution and the radiation distribution functions. [7] Only terms contained in a description of the beam to a fixed order N are retained. Quantum excitation is being modeled in a parallel fashion. Transverse damping is modeled using a longitudinal momentum impulse at the RF cavity. The magnitude of the kick is a function of l only. Finally, we have begun to investigate the inclusion of energy changes during the beam-beam collisions following the approach of Hirata $et\ al.$ [6] as far as applicable.

III. SUMMARY

By expressing the distribution of a single beam bunch as an expansion of the two-dimensional quantum harmonic oscillator basis functions in normalized coordinates, the beam dynamics for Gaussian and non-Gaussian beams can be calculated. The beam blowup due to the beam-beam interaction and the luminosity can be computed directly as a sum over the coefficients in the expansion and tabulated integrals. The characteristics of this approach complement those of currently used algorithms.

References

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