Cavity RF Mode Analysis Using a Boundary-Integral Method

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ABSTRACT

A 3-dimensional boundary-integral method has been developed for rf cavity mode analysis. A frequency-dependent, homogeneous linear matrix equation is generated from a variant of the magnetic field integral equation (MFIE) where the domain of integration is a closed surface specifying the rf envelope of the cavity. Frequencies at which the MFIE has non-zero solutions are mode frequencies of the cavity, and the solutions are the corresponding surface magnetic field distributions. The MFIE can then be used to calculate the electric and magnetic field at any other point inside the cavity. Forward iteration is used to find the largest complex eigenvalue of the matrix at a specific frequency. This eigenvalue is 1 when the frequency corresponds to a cavity rf resonance. The matrix equivalent of the MFIE is produced by approximating the cavity surface by a set of perfectly conducting surface elements, and assuming that the surface magnetic field has constant amplitude on each element. The method can handle cavities with complex symmetries, and be easily integrated with finite-element heat-transfer and stress analysis codes.

INTRODUCTION

Computer codes capable of calculating the full 3-dimensional electromagnetic field distributions are now commonly used for designing rf cavities for accelerators. However, most of these codes need large computational resources to compute sufficiently detailed distributions for further mechanical engineering analysis of cavities. Furthermore, the mesh often used by these codes is not compatible with finite-element meshes commonly used in commercial thermal-stress analysis codes used in the mechanical design. Past work at AECL in the mechanical design of high-power rf cavities for a variety of applications showed the desirability of a technique to compute the rf field distribution over the cavity surface on the same surface mesh used in mechanical thermal-stress analysis. This paper presents such a technique that uses a boundary-integral method to compute the surface rf magnetic field distribution in cavities filled with a homogeneous medium. A similar approach using a different boundary-integral relation has been developed at the University of Pavia.[1] In addition to being particularly convenient for coupled rf-mechanical design problems, these techniques usually require considerably less computational effort.

BOUNDARY-INTEGRAL EQUATIONS

Although Maxwell's equations are usually posed as a set of coupled differential equations, many alternative, equivalent sets of integral equations can be derived. One useful integral equation[2] for harmonically varying fields with angular frequency ω is:

$$H(r') = \frac{1}{4\pi}\int_V (J \times \nabla\phi)\,dv + \frac{1}{4\pi}\int_S [i\omega\epsilon(n\times E)\phi - (n\times H)\times\nabla\phi - (n\cdot H)\nabla\phi]\,da \tag{1}$$

where

$$\phi = \frac{e^{ik|r'-r|}}{|r'-r|}, \tag{2}$$

E and H are the electric and magnetic fields, S is any surface enclosing a medium with volume V, dielectric constant ϵ, internal current distribution J and propagation constant k, and n denotes an inward-facing unit vector normal to the surface. If S specifies a cavity with a perfectly conducting surface enclosing a volume with no internal current distribution (i.e., $J = 0$), then (1) simplifies to:

$$H(r') = -\frac{1}{2\pi}\int_S n \times H(r) \times \nabla\phi\, da \tag{3}$$

for all r', r on the surface. The simplicity of (3) arises since only $n \times H$, the component of H transverse to the surface, is non-zero on a perfect conductor.

This equation is a variation of the magnetic field integral equation[3] (MFIE) commonly used for the analysis of antennas of arbitrary shape. Equation (3) is a homogeneous Fredholm integral equation,[4] which has non-zero solutions for H only at discrete values of ω where the equation is said to be singular. The frequencies where (3) becomes singular are the cavity resonance frequencies, and the corresponding solutions, H, are the resonance field distributions.

NUMERICAL SOLUTION OF THE MFIE

Equation (3) is transformed into a matrix equation using standard techniques in finite-element analysis,[5] in this case using the collocation technique with pulse basis functions. The surface is approximated by a set of 4-node quadrilateral elements with the tangential rf magnetic field assumed to be constant within each element. The collocation points are the geometric centres of each element. This results in two complex degrees of freedom for each element.

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A constant, purely tangential, rf magnetic field is assumed over each surface element. An element basis is defined for each element i, where $\boldsymbol{n}_i^1$ and $\boldsymbol{n}_i^2$ are orthogonal unit vectors tangential to the element and $\boldsymbol{n}_i$ is a unit vector normal to the element. Since the four nodes defining a surface element are not constrained to lie in a plane, the unit vector $\boldsymbol{n}_i$ is defined to be parallel to the cross product of vectors joining opposite corners of the element. The other two unit vectors, $\boldsymbol{n}_i^1$ and $\boldsymbol{n}_i^2$, are then defined to be orthogonal to each other and to $\boldsymbol{n}_i$. Equation (3) then becomes

$$\begin{aligned} H_j^1 &= \sum_{i=1}^{N} \left(A_{i,j}^{11} H_i^1 + A_{i,j}^{21} H_i^2 \right) \\ H_j^2 &= \sum_{i=1}^{N} \left(A_{i,j}^{12} H_i^1 + A_{i,j}^{22} H_i^2 \right) \end{aligned} \tag{4}$$

where

$$\boldsymbol{H} = H^1 \boldsymbol{n}^1 + H^2 \boldsymbol{n}^2 . \tag{5}$$

The boundary-integral method is used to calculate the terms in the four matrices

$$\begin{aligned} A_{i,j}^{11} &= \boldsymbol{n}_j^1 \cdot \int_{S_i} \boldsymbol{n}_i \times \boldsymbol{n}_i^1 \times \nabla\phi(\boldsymbol{r}-\boldsymbol{r}_j)\, da , \\ A_{i,j}^{12} &= \boldsymbol{n}_j^1 \cdot \int_{S_i} \boldsymbol{n}_i \times \boldsymbol{n}_i^2 \times \nabla\phi(\boldsymbol{r}-\boldsymbol{r}_j)\, da , \\ A_{i,j}^{21} &= \boldsymbol{n}_j^2 \cdot \int_{S_i} \boldsymbol{n}_i \times \boldsymbol{n}_i^1 \times \nabla\phi(\boldsymbol{r}-\boldsymbol{r}_j)\, da , \\ A_{i,j}^{22} &= \boldsymbol{n}_j^2 \cdot \int_{S_i} \boldsymbol{n}_i \times \boldsymbol{n}_i^2 \times \nabla\phi(\boldsymbol{r}-\boldsymbol{r}_j)\, da . \end{aligned} \tag{6}$$

Each surface integration uses 1-point integration for distant elements, and 4-point Gaussian integration for nearby elements. The transition between 1-point and 4-point integration occurs when the distance between collocation points exceeds four times the average dimension of the element. The results are very insensitive to the exact threshold.

Any symmetry that exists in the problem may be incorporated in the surface integration by extending the integral in (6) over all elements that map into element i by symmetry while performing appropriate transformations of the rf magnetic field, $\boldsymbol{H}_i$. Possible transformations include rotation, and symmetry or anti-symmetry under reflection.

Matrices A^{11}, A^{21}, A^{12} and A^{22} may be combined into a single $2N$-by-$2N$ matrix:

$$K = \begin{bmatrix} A^{11} & A^{21} \\ A^{12} & A^{22} \end{bmatrix} , \tag{7}$$

and the two N-vectors H^1 and H^2 may be combined into a single vector of length $2N$:

$$H = \begin{bmatrix} H^1 \\ H^2 \end{bmatrix} . \tag{8}$$

The result is an approximation of (3) in the form:

$$H = K(k) H , \tag{9}$$

where the equality only holds, for non-zero H, at values of k corresponding to cavity resonant modes.

These mode frequencies are found by the following method. Equation (9) is generalized to be a standard matrix eigenvalue problem with k-dependent eigenvalues:

$$\lambda(k) H = K(k) H . \tag{10}$$

Since K is a non-symmetric complex matrix, all λ are generally complex as well. The eigenvalues of (10) are searched while varying k for values meeting the conditions

$$\Re(\lambda(k_m)) \approx 1, \qquad \Im(\lambda(k_m)) = 0 \tag{11}$$

when there is a resonant mode with $k = k_m$. The corresponding eigenvector is the rf magnetic field distribution for the mode.

In practice, when searching for the lowest frequency mode, the eigenvalue of (10) that will satisfy (11) is usually the eigenvalue with largest magnitude. In this case, direct iteration[6] is a convenient technique to find $\lambda(k)$. Frequently, variants of forward iteration can be used to find a significant number of low-frequency modes.

RIGHT-CIRCULAR CYLINDER EXAMPLE

The lowest frequency rf modes of a right-circular cylinder with 100 mm radius and 200 mm height have been found using the boundary-integral method. The symmetry in the problem permits modelling of one octant of the cavity. Only 103 surface quadrilateral elements are used. Unlike most finite-element meshes, a connected mesh is not required for this simple implementation of the boundary-integral method, which permits more flexibility in mesh generation.

Figure 1 shows the distribution of eigenvalues of (10) for a frequency of 1150 MHz. A comparison of several low-order mode frequencies for this cavity computed using this boundary-integral-method mesh, to frequencies calculated from exact analytic expressions,[7] is shown in Table I. Figures 2 and 3 show the rf surface magnetic field distribution for the two lowest frequency modes.

The boundary-integral calculations were performed on a microcomputer using a 25 MHz Intel 386SL microprocessor with a 387 numeric co-processor. The time to calculate the matrix K for a specified frequency was 45 seconds; 50 direct iterations, taking 35 seconds, were sufficient to converge on the maximum eigenvalue. Typically, three to four evaluations of K at different frequencies are required to find each resonant mode. It is not necessary, although often convenient, to find the mode frequencies in increasing order.

SUMMARY

A new approach for analyzing 3-dimensional rf resonant modes using a boundary-integral technique has been presented. The technique was used to compute the several rf mode frequencies and field distributions for a right-circular cylinder. The technique is being applied to several complex rf cavities for particle accelerators.

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Table I RF mode frequencies of a cylindrical cavity 100 mm high and 200 mm in diameter

Resonant Mode	Frequency	
	Analytic	Boundary-Integral Method
TM010	1147.43	1152.23
TE111	1737.42	1735.30
TM110	1828.24	1833.90
TM011	1887.72	1889.36
TE211	2090.59	2089.00
TE011	2364.18	2363.10
TM111	2364.18	2368.52
TM210	2450.38	2457.10
TE311	2503.01	2503.46

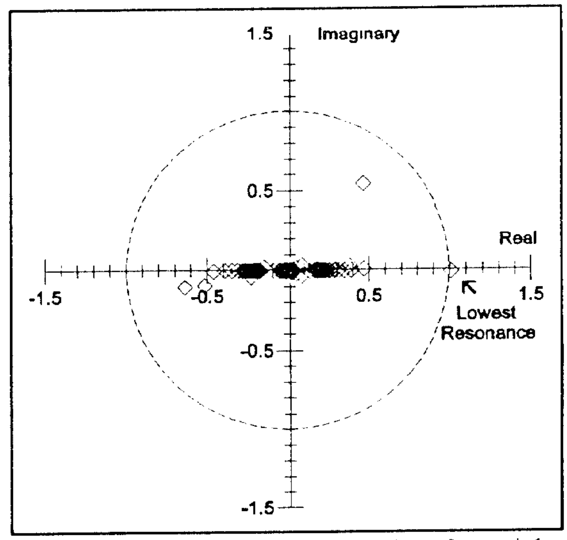

Figure 1. Distribution of eigenvalues for a right-circular cylindrical cavity at 1150 MHz.

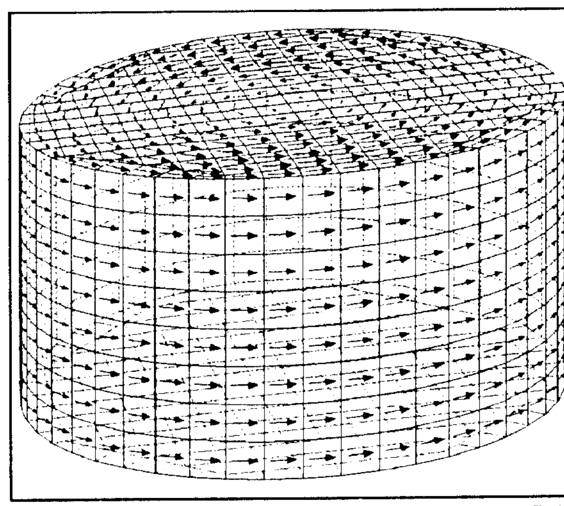

Figure 2. TM_{010} mode surface rf magnetic field distribution calculated using the boundary-integral method.

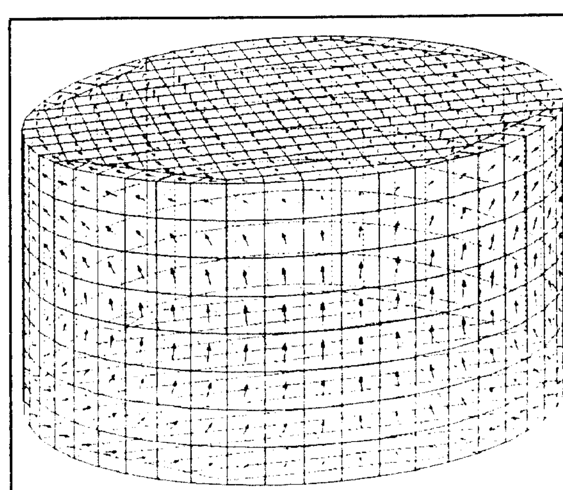

Figure 3. TE_{111} mode surface rf magnetic field distribution calculated using the boundary-integral method.