Using the Panofsky-Wenzel Theorem in the Analysis of Radio-Frequency Deflectors*

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Abstract

In a 1956 paper Panofsky and Wenzel considered the transverse momentum imparted to a fast particle moving parallel to the axis of a cavity excited in either a TE (no component of the electric field parallel to the axis) or TM mode (no component of the magnetic field parallel to the axis). One conclusion of this paper was that in a TE mode the deflecting impulse of the electric field exactly cancels the impulse of the magnetic field. This result is sometimes misinterpreted as concluding that if the electric field acting on a particle is purely transverse, the deflection impulses from the electric and magnetic fields must cancel one another. This conclusion is false.

Instead

$$\mathbf{p}_\perp = \left(\frac{e}{\omega_0}\right) \int_0^d (-i) \nabla_\perp E_z \, dz, \tag{1}$$

implicitly derived in the 1956 paper, is a more useful form of the theorem for deflecting cavities. In this equation, $\mathbf{p}_\perp$ is the transverse momentum imparted to the particle, d is the length of the cavity, e is the charge of the particle, ω_0 is the angular frequency of the cavity, and $\nabla_\perp E_z$ is the transverse gradient of the z component of the electric field along the path of the particle. The $-i$ represents a 90° phase advance of the integrand with respect to the electric field. In other words, the integrand has the same phase as the magnetic field.

Eq. (1) is not restricted to TE or TM modes. In particular, it applies to two similar-looking modes in a high-energy deflector that was studied for the Accelerator Transmutation of Waste (ATW) project at Los Alamos National Laboratory.

I. INTRODUCTION

In a 1956 paper, Panofsky and Wenzel showed that no transverse momentum is imparted to particles traveling parallel to the axis of a cavity excited in a TE mode [1]. This result for TE modes is sometimes misinterpreted as concluding that if the electric field acting on a particle is purely transverse, then the deflection impulses from the electric and magnetic fields must cancel one another. In other words, no transverse momentum will be imparted to the particle. We will show why this conclusion is false by rederiving Panofsky and Wenzel's result and showing that their restricting their work to cavities with pure TE or TM modes is unnecessary.

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II. DERIVING THE THEOREM

The transverse momentum $\mathbf{p}_\perp$, imparted to a particle with velocity v and charge e and traveling in the z direction through a radio-frequency cavity of length d, is given by

$$\mathbf{p}_\perp = \int_{t(z=0)}^{t(z=d)} \mathbf{F}_\perp dt = (e/v) \int_0^d \left[\mathbf{E}_\perp + (\mathbf{v} \times \mathbf{B})_\perp\right] dz, \tag{2}$$

if $\mathbf{v}$ is large enough to allow the particle direction to remain essentially unchanged by the transverse force. Panofsky and Wenzel have shown that we can simplify the above equation by expanding the right hand side of it in terms of the vector potential. To see this recall that in general

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \mathbf{V}, \tag{3}$$

where $\mathbf{A}$ is the magnetic vector potential, i.e.

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{4}$$

and where $\mathbf{V}$ is the scalar potential. Since $\mathbf{V}$ is constant inside a cavity,

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \text{ and} \tag{5}$$

$$\mathbf{E}_\perp = -\frac{\partial \mathbf{A}_\perp}{\partial t}. \tag{6}$$

Expressing $(\mathbf{v} \times \mathbf{B})_\perp$ in terms of $\mathbf{A}$, we get

$$\begin{aligned}(\mathbf{v} \times \mathbf{B})_\perp &= \left[\mathbf{v} \times (\nabla \times \mathbf{A})\right]_\perp = \left[\nabla (\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \nabla) \mathbf{A}\right]_\perp \\ &= \nabla_\perp (\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \nabla) \mathbf{A}_\perp. \end{aligned} \tag{7}$$

Thus we know that

$$\mathbf{p}_\perp = (e/v) \int_0^d \left[\left(-\frac{\partial \mathbf{A}_\perp}{\partial t} - (\mathbf{v} \cdot \nabla) \mathbf{A}_\perp\right) + \nabla_\perp (\mathbf{v} \cdot \mathbf{A})\right] dz. \tag{8}$$

Because $\mathbf{v}$ is essentially constant and is in the z direction,

$$(\mathbf{v} \cdot \nabla) \mathbf{A}_\perp = v \frac{\partial \mathbf{A}_\perp}{\partial z} \text{ and} \tag{9}$$

$$\nabla_\perp (\mathbf{v} \cdot \mathbf{A}) = v \nabla_\perp A_z, \text{ so} \tag{10}$$

$$\mathbf{p}_\perp = (e/v) \int_0^d \left[-\left(\frac{\partial \mathbf{A}_\perp}{\partial t} + v \frac{\partial \mathbf{A}_\perp}{\partial z}\right) + v \nabla_\perp A_z\right] dz \tag{11}$$

$$= e \int_0^d \left[-\left(\frac{1}{v}\frac{\partial \mathbf{A}_\perp}{\partial t} + \frac{\partial \mathbf{A}_\perp}{\partial z} \right) + \nabla_\perp A_z \right] dz. \quad (12)$$

However, it is also true that

$$v = dz/dt, \text{ so} \quad (13)$$

$$dz/v = dt, \text{ and} \quad (14)$$

$$\left(\frac{1}{v}\frac{\partial \mathbf{A}_\perp}{\partial t} + \frac{\partial \mathbf{A}_\perp}{\partial z} \right) dz = \frac{\partial \mathbf{A}_\perp}{\partial t} dt + \frac{\partial \mathbf{A}_\perp}{\partial z} dz = d\mathbf{A}_\perp, \quad (15)$$

because the path of the particle is fixed in the transverse direction. Thus we know that

$$\mathbf{p}_\perp = e \int_{A_\perp(z=0)}^{A_\perp(z=d)} -(d\mathbf{A}_\perp) + e \int_0^d \nabla_\perp A_z dz. \quad (16)$$

In order for this relationship to be useful, we need to express **A** in terms of **E**. If we assume an $e^{-i\omega_0 t}$ time dependence for **E**, then

$$\mathbf{A} = -\frac{i}{\omega_0}\mathbf{E} \quad (17)$$

is a valid choice for **A**. Notice that $-i = e^{-i\pi/2}$, so **A** has a time dependence of $e^{-i(\omega_0 t + \pi/2)}$. Thus **A** is shifted 90° in time from **E** and has the same phase as the magnetic field, as we would expect.

As Panofsky and Wenzel point out, the first term of Eq. (16) vanishes for any cavity having ends perpendicular to its axis, because $\mathbf{A}_\perp = \mathbf{E}_\perp = 0$ in metal. It can also vanish for a cavity having a beam tube, as long as $\mathbf{E}_\perp = 0$ at $z = 0$ and $z = d$. Thus for these cases we have

$$\mathbf{p}_\perp = e \int_0^d \nabla_\perp A_z dz. \quad (18)$$

Substituting Eq. (17) into Eq. (18), we obtain

$$\mathbf{p}_\perp = \left(\frac{e}{\omega_0} \right) \int_0^d (-i) \nabla_\perp E_z dz. \quad (19)$$

Notice that Eq. (19) is not restricted to TE or TM modes. The change in transverse momentum is indeed zero for a TE mode, because for that mode E_z, and hence $\nabla_\perp E_z$, is identically zero everywhere in the cavity. Notice, however, that for a non-TE mode E_z can be zero everywhere along the path of the particle and still not be identically zero everywhere in the cavity. In such a non-TE mode $\nabla_\perp E_z$ does not have to be zero along the path of the particle, so the particle can be deflected.

III. EXAMPLES

The Panofsky-Wenzel theorem can be useful in gaining insight as to whether a mode in a radio-frequency cavity will deflect the beam. For instance, consider the cavity shown in Fig. 1. (The material around the cavity has been left transparent for clarity.) This 350-MHz, $\beta\lambda/2$ structure is of the type proposed by Leeman and Yao [2]; it has been modified for a 1-GeV, high-current proton beam and modeled using the three-dimensional MAFIA codes [3]. The cavity is 33.8 cm long and 26 cm in diameter. The rods are 15.57 cm long and 5 cm in diameter; the vertical distance between the rods is 5 cm, and the gap between the ends of the rods is 2.67 cm. The beam pipe is 5 cm in diameter. The beam is moving from left to right (in the $+z$ direction) through the center of the cavity.

The deflecting mode is a TEM-like mode, with the magnetic fields curving around the rods, adding in the beam region. Figure 2 shows plots, taken in a transverse cross section, of the MAFIA-generated electric and magnetic fields for this deflecting mode.

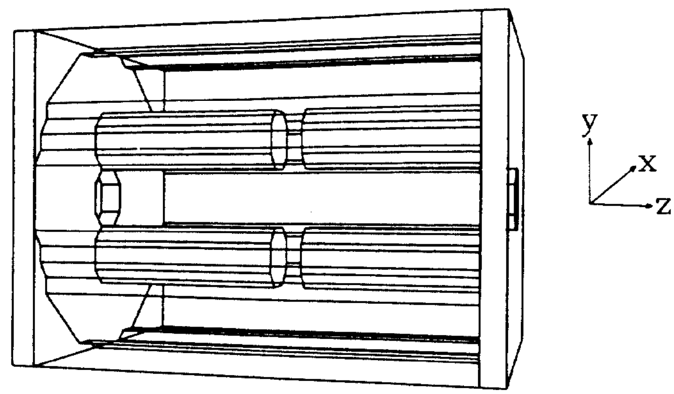

Fig. 1. MAFIA model of deflecting cavity.

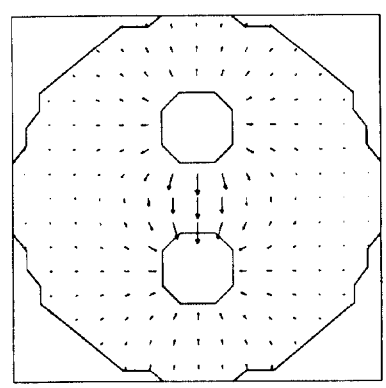

Fig. 2. Electric field of the deflecting mode, taken in an x-y plane.

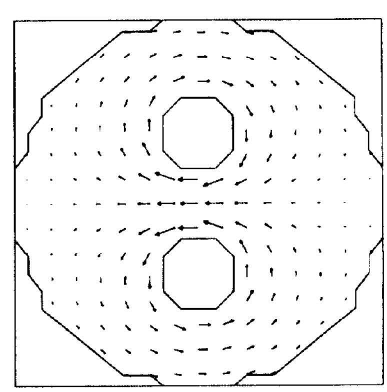

Fig. 3. Magnetic field of the deflecting mode, taken in an x-y plane.

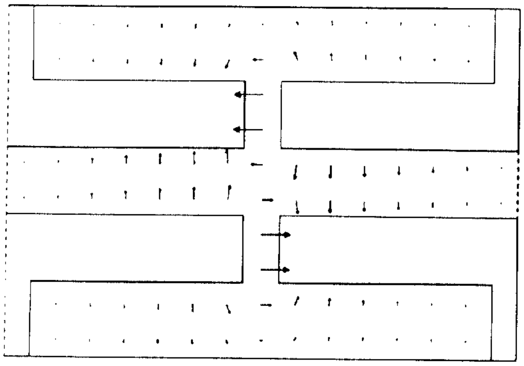

Fig. 4. Electric field of the deflecting mode, taken in a z-y plane.

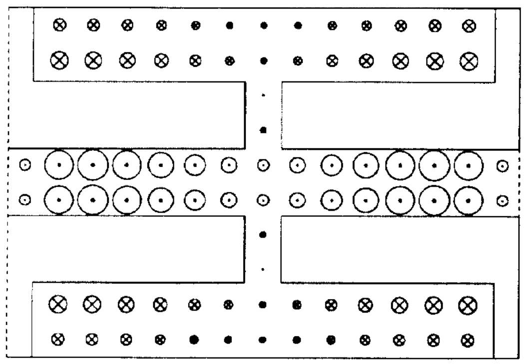

Fig. 5. Magnetic field of the deflecting mode, taken in a z-y plane.

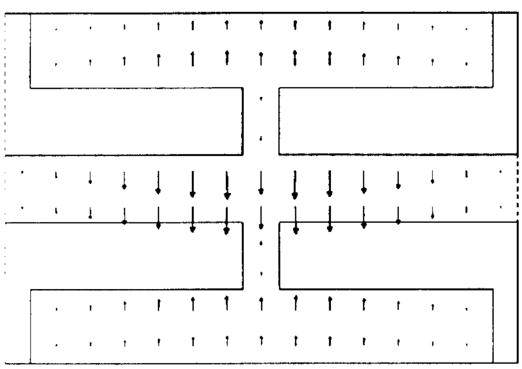

Fig. 6. Electric field of the next-highest mode, taken in a z-y plane.

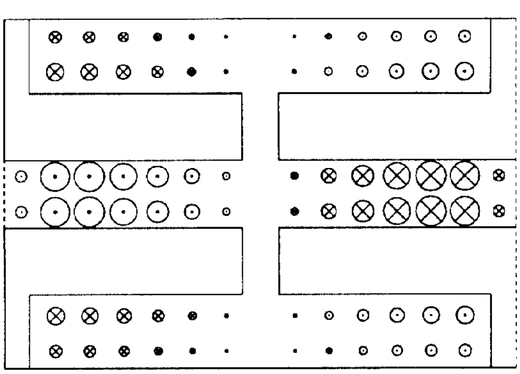

Fig. 7. Magnetic field of the next-highest mode, taken in a z-y plane.

The transverse fields of the next-highest mode are similar to those of the deflecting mode in the region away from the z midplane of the cavity. But as we can see in Figs. 4-7, the z variation is quite different for the two modes. The magnetic field is symmetric about the z midplane for the deflecting mode, but it is antisymmetric for the next-highest mode. The electric fields also have different patterns for the two modes. In particular, the deflecting mode has a large gradient of E_z in the middle of the cavity; the next-highest mode has a much smaller gradient. We can use Eq. (19) to inspect which of the two modes will be the better deflector. In fact, if we calculate the relative deflections using MAFIA, we find that the deflection caused by the deflecting mode is more than twenty times greater than that of the next-highest mode.

IV. CONCLUSIONS

The Panofsky-Wenzel theorem has more general applications than its authors stated in their original paper. The form

$$\mathbf{p}_\perp = \left(\frac{e}{\omega_0}\right) \int_0^d (-i) \nabla_\perp E_z \, dz$$

does not depend upon the cavity being excited in a TE or a TM mode. It depends only upon the assumptions that (1) the particles are rigid enough that the particle orbit is not substantially affected in its passage through the cavity and (2) the transverse electric field vanishes at each end of the cavity. This form of the theorem is useful in distinguishing between deflecting and nondeflecting modes in a resonant cavity.

V. ACKNOWLEDGMENTS

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References

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