Analysis of Resonant Longitudinal Instability in a Heavy Ion Induction Linac*

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SUMMARY

A high current beam of subrelativistic ions accelerated in an induction linac is predicted (in some circumstances) to exhibit unstable growth of current fluctuations at high frequencies ($\nu \sim 100$ MHz). The instability is driven by the interaction between the beam and accelerator modules at frequencies close to a cavity resonance. The extent of unstable growth depends on features of the coupling impedance, beam parameters, and total pulse and accelerator lengths. Transient and asymptotic analysis is presented.

Induction Linac Model

We treat a cluster of beams drifting at velocity v, with line charge density λ and current $I = \lambda v$. It is assumed here that all the beamlets ($N \sim 16$) effectively act in concert so that λ and I are the total values and v is the common velocity. The continuity equation, using laboratory frame variables (z,t) is:

$$\frac{\partial \lambda}{\partial t} + \frac{\partial I}{\partial z} = 0 \ . \tag{1}$$

The beam cluster is treated as a cold, 1-d, non-relativistic fluid. An externally imposed field E^{ex} and a smoothed longitudinal field E, induced by interaction of I with the acceleration modules, acts on v:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} = \frac{qe}{m}\left(E + E^{ex}\right) \ . \tag{2}$$

Neglected in this model are velocity spread and a direct space-charge force proportional to $\partial\lambda/\partial z$. These are significant stabilizing features at the high frequencies treated here, so the calculation may be considered pessimistic. The purpose of the present work is to delineate the phenomena of the longitudinal instability occuring in heavy ion induction linacs, and to guide future study. The analysis is similar to that given by V. K. Neil(1) for relativistic electron beams.

The equilibrium beam drifts at constant velocity v_o, so the total equilibrium field $\left(E_o + E_o^{ex}\right)$ vanishes. Equilibrium current I_o and line charge density λ_o are, in general, functions of the retarded time $\tau = t - z/v_o$. However, they are taken to be constant for the duration of the pulse $(0 < \tau < \tau_p)$.

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Let E^{ex} have a small additional component which acts at $z = o$ and therefore perturbs the beam: $E^{ex} = E_o^{ex} + V(t)\delta(z)$. From eqns. (1) and (2) the resulting perturbed beam variables satisfy

$$\frac{\partial \lambda_1}{\partial t} + \frac{\partial I_1}{\partial z} = 0 \quad , \tag{3}$$

$$\frac{\partial v_1}{\partial t} + v_o\frac{\partial v_1}{\partial z} = \frac{qe}{m}\left(E_1 + V\delta(z)\right) \quad , \tag{4}$$

$$I_1 = \lambda_o v_1 + v_o \lambda_1 \ .$$

Module Response

The beam-generated field E_1 is induced by by the passage of the return current $(-I_1)$ through the module impedance Z. If we assume the driven form $I_1 \sim \exp(-i\omega t)$, with induced field $E_1(\omega)$, then impedance is

$$Z(\omega) = -E_1(\omega)/I_1 \ . \tag{5}$$

Specifically, we treat an isolated module resonance characterized by a parallel L-R-C equivalent circuit:

$$C\frac{\partial^2 E_1}{\partial t^2} + \frac{1}{R}\frac{\partial E_1}{\partial t} + \frac{E_1}{L} = -\frac{\partial I_1}{\partial t} \ . \tag{6}$$

The circuit parameters are related to measured resonance features; let R be the real impedance peak (units of Ω/m in the smoothed field model), occuring at angular frequency ω_o. and resonance width is $\Delta\omega = \omega_o/Q$ at 71% of maximum $|Z|$. Then we have

$$C = Q/\omega_o R \ \ (\text{F-m}), \quad LC = \omega_o^{-2} \ . \tag{7}$$

The impedance formula for general (complex) ω is

$$Z = \left(\frac{1}{R} - i\omega C - \frac{1}{i\omega L}\right)^{-1} = \frac{R}{1 - iQ\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)} \ . \tag{8}$$

Typical resonance parameters of interest for the Heavy Ion Fusion application are $\nu_o = \omega_o/2\pi$ = 30–300 MHz, Q = 10-100, and R = 100-1000 Ω/m.

Beam Frame Equations

It convenient to use the retarded time variable $\tau = t - z/v_o$ and z instead of the laboratory frames variables t and z. Then eqns. (3) and (4) become

$$\frac{\partial I_1}{\partial z} = \frac{\partial}{\partial \tau} \frac{\lambda_o v_1}{v_o} , \quad (9)$$

$$\frac{\partial v_1}{\partial z} = \frac{q e}{m v_o} \left[E_1 + \delta(z) V(\tau)\right] . \quad (10)$$

We eliminate v_1 to obtain

$$\frac{\partial^2 I_1}{\partial z^2} = \frac{\partial}{\partial \tau} K^2 C \left[E_1 + \delta(z) V(\tau)\right] , \quad (11)$$

where
$$K^2 = \frac{q e \lambda_o}{m v_o^2 C} . \quad (12)$$

Equation (6) becomes

$$\left(\frac{\partial^2}{\partial \tau^2} + \frac{\omega_o}{Q}\frac{\partial}{\partial \tau} + \omega_o^2\right) E_1 = -\frac{1}{C}\frac{\partial I_1}{\partial \tau} . \quad (13)$$

The natural scale frequencies for z and τ are K and ω_0. Generally the scale magnitude $\omega_0\tau \sim 300$ is much larger than $Kz \sim 10$. Eqns.(11) and (13) describe strongly coupled oscillations in both τ and z, which exhibit growth in both variables. We expect large growth in τ will be produced by an external perturbation $V(\tau)$ which contains any appreciable content near the resonant frequency ω_0. In eqns. (11)-(13), K may be a function of τ, however a numerical treatment would then be required. Henceforth K is taken to be constant.

The perturbation $V(\tau)$ is assumed to turn on within the beam pulse at some $\tau = \tau_0 > o$. The structure of the coupled equations ensures that perturbed quanatities may be consistently assumed to vanish outside the zones $\tau \geq \tau_0$, $z \geq o$.

General Solution by Laplace Transformation

Taking E_1 and I_1 to vanish for ($\tau < \tau_0$, $z < o$), we find from eqn. (11) the initial conditions at $z = o+$:

$$I_1\,(o+) = o, \quad \frac{\partial I_1}{\partial z}(o+) = \frac{q e \lambda_o}{m v_o^2}\frac{\partial V}{\partial \tau} . \quad (14)$$

Performing the Laplace transformation

$$\left(\widehat{I_1} , \widehat{E_1}\right) = \int_o^{\infty} d\tau\, e^{i\omega\tau} \left(I_1 , E_1\right) , \quad (15)$$

we get

$$\frac{\partial^2 \widehat{I_1}}{\partial z^2} = - K^2 C i\omega \widehat{E_1} = \left(\omega \Gamma K\right)^2 \widehat{I_1} , \quad (16)$$

with
$$\Gamma = \left(\omega_o^2 - \omega^2 - i\omega\omega_o/Q\right)^{1/2} .$$

The solution $\widehat{I_1}$ satisfying the initial conditions (14) is

$$\widehat{I_1} = \left(\frac{q e \lambda_o}{m v_o^2}\right)\left(-i\omega \widehat{V}\right) \frac{\sinh(\omega \Gamma K z)}{(\omega \Gamma K)} . \quad (17)$$

The inverse transformation is $I_1 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\, e^{-i\omega\tau}\, \widehat{I_1}$. (18)

with all singularities below the real ω axis.

Asymptotic Growth Formulas

The general character of resonant growth can be determined by the saddle point method. The form of I_1, given by eqn (18) is

$$I_1 \sim \int_{-\infty}^{+\infty} d\omega\, f(\omega) \exp(g) , \quad (19)$$

with

$$g = -i\omega\tau \pm \omega \Gamma K z . \quad (20)$$

Intrinisic singularities are located at the complex resonance values (poles of Γ^2),

$$\omega = \pm\, \overline{\omega} - i\omega_o/2Q , \quad (21)$$

where

$$\overline{\omega} = \omega_o \sqrt{1-1/(2Q)^2} .$$

Saddle points are located at the six roots of the equation

$$0 = \frac{\partial g}{\partial \omega} = - i\tau \pm K z \Gamma \left[1 + \Gamma^2 \omega(\omega + i\omega_o/2Q)\right] . \quad (22)$$

The integration contour in eqn (19) can be deformed to pass through the saddle points, and the dominant contributions to I_1 are produced at these locatons for sufficiently large z and τ. To simplify the saddle calculaton let

$$u = \omega / \overline{\omega} + i\varepsilon, \quad \varepsilon = \omega_o/2\overline{\omega}_o Q \quad (23)$$

assumed small compared with unity. Then eqns. (20) and (21) become (at the saddles)

$$g = -i\overline{\omega}\tau \frac{u(u-i\varepsilon)^2}{(1-i\varepsilon u)} , \quad (24)$$

$$(u^2-1)^3 = \left(\frac{Kz}{\overline{\omega}\tau}\right)^2 (1-i\varepsilon u)^2 . \quad (25)$$

For mode growth associated with the module resonance we expect $u \approx \pm 1$, which requires $\overline{\omega}\tau >> Kz$, Eqn. (25) gives

$$u^2 = 1 + r \left(\frac{Kz}{\overline{\omega}\tau}\right)^{2/3} (1-i\varepsilon u)^{2/3} , \quad (26)$$

where r is any cube root of unity. The small quantity $\varepsilon \approx (2Q)^{-1}$ is a fixed constant. However we shall find that at peak growth for fixed z, $(Kz/\bar{\omega}\tau)$ is of order $\varepsilon^{3/2}$. To solve eqn (26) by an expansion in the small parameters we formally define

$$\beta \equiv \frac{(Kz / \bar{\omega}\tau)^{2/3}}{\varepsilon}, \tag{27}$$

and regard β as of order unity. The resulting expressions for u and g are, assuming $u \approx + 1$,

$$u = 1 + \frac{\varepsilon\beta r}{2} - \varepsilon^2\left(\frac{\beta^2 r^2}{8} + \frac{i\beta r}{3}\right) + \ldots, \tag{28}$$

$$g = -i\bar{\omega}\tau\left[1 + \varepsilon\left(\frac{3\beta r}{2} - i\right) + \varepsilon^2\left(\frac{3\beta^2 r^2}{8} - i\beta r\right) + \ldots\right]. \tag{29}$$

For unstable growth the relevant cube root of unity is $r = (\sqrt{3}\, i-1)/2$. We get, keeping only terms through order ε:

$$g_r = \frac{3\sqrt{3}}{4}(\bar{\omega}\tau)^{1/3}(Kz)^{2/3} - \frac{\omega_o\tau}{2Q}, \tag{30}$$

$$g_i = -\bar{\omega}\tau + \frac{3}{4}(\bar{\omega}\tau)^{1/3}(Kz)^{2/3}. \tag{31}$$

At specified z the maximum value of g_r is readily found from eqn (30):

$$0 = \frac{\partial g_r}{\partial\tau} = \bar{\omega}\left[\frac{\sqrt{3}}{4}\left(\frac{Kz}{\bar{\omega}\tau}\right)^{2/3} - \varepsilon\right]. \tag{32}$$

At this point $\beta = 4/\sqrt{3}$, which is of order unity, as assumed. The maximum growth factor is

$$(g_r)_{max} = \left(\frac{\sqrt{3}}{2}\right)^{3/2}\left(\frac{\bar{\omega}Q}{\omega_o}\right)^{1/2} Kz. \tag{33}$$

Application to Heavy Ion Fusion

The maximum growth is calculated here at a medium energy position in a fusion driver, with ion parameters (T = 1000 MeV, m = 200 amu, q = 1), and the typical pulse parameters ($I_o = 10^3$ A, τ_p = 500 ns.) For the module response we take $\bar{\omega} \equiv \omega_o = 2\pi\times10^8\ \mathrm{s}^{-1}$, Q = 30, R = 300 Ω/m. Then we have

$$v_o = .104c,\ \lambda_o = 32.2\ \mu\mathrm{C/m},$$

$$C = 10^{-9}/2\pi\ \mathrm{F\text{-}m},\ K = .0100\ \mathrm{m}^{-1}.$$

At the pulse end $\bar{\omega}\tau_p = 314$ and the maximum growth point is

$$z = \frac{\bar{\omega}\tau_p}{K}\left(\frac{2}{\sqrt{3}\,Q}\right)^{3/2} = \frac{2.36}{K} = 236\ \mathrm{m},$$

$$(g_r)_{max} = \left(\sqrt{3}/2\right)^{3/2} Q^{1/2} Kz = \omega_o\tau_p/Q = 10.5.$$

This calculated total growth [exp(10.5) = 36300] is large enough to be of concern, even though the initial disturbance $V(\tau)$ may be very small in the unstable band. A small rms velocity spread $\left(\Delta v/v_o \approx v_o(g_r)_{max}/\omega_o z_{max} \approx .0022\right)$ would be sufficient to eliminate growth, but would constrain the focal spot radius achievable in a fusion reactor.

Dispersion Relation

A Laplace transformation in both τ and z on eqns (11)-(13) yield

$$I_1 \sim \int d\omega \int d\Omega \frac{F(\omega)}{D} e^{-i(\omega\tau+\Omega z)}, \tag{34}$$

where the dispersion relation is

$$D(\omega,\Omega) = -K^2\omega^2 + i\omega\Omega^2/CZ(\omega) \tag{35}$$

$$= \left(\omega^2 - \omega_o^2 + i\omega\omega_o/Q\right)\left(\Omega^2 - K^2\right) - K^2\left(\omega_o^2 - i\omega\omega_o/Q\right). \tag{36}$$

The latter form of D clearly indicates that a pair of strongly coupled resonances are present, and appear symmetrically when $Q = \infty$. A growth formula, valid for near resonance $\Omega^2 \approx K^2$, may therefore be obtained from eqn. (30) by interchanging $\omega_o\tau$ with Kz. For $Kz >> \omega_o\tau$ we have

$$g_r = \left(3\sqrt{3}/4\right)(\omega_o\tau)^{2/3}(Kz)^{1/3}, \tag{37}$$

$$g_i = -Kz + (3/4)(\omega_o\tau)^{2/3}(Kz)^{1/3}. \tag{38}$$

The roots of the dispersion equation $D(\omega,\Omega) = 0$ can be used to find the growth in z for given real ω; we find for the imaginary part of Ω

$$\Omega_i = \left[\frac{K^2C\omega}{2}\left(|Z| - Z_i\right)\right]^{1/2}, \tag{39}$$

with

$$Z = Z_r + iZ_i = \frac{R\left[1 + iQ\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)\right]}{1 + Q^2\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)^2}.$$

The maximum growth formula eqn. (33) may be recovered from eqn. (39) for large Q by maximizing Ω_i with respect to driving frequency ω.

Reference

[1] V. K. Neil, Interaction of the ATA Beam with the TM_{030} Mode of the Accelerating Cells, LLNL report UCID-20456, 1985.