PHASE TRAJECTORY ANALISIS AT THE NONLINEAR RESONANCES

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Abstract: The phase trajectories close to the parametric resonance and to the nonlinear resonances of third and fourth order, which play an important role in the cyclic accelerators particle dynamic, are investigated. The consideration is carried out in the canonical variables X,Y related by a simple way to the angle and the displacement of the circulating particle with respect to the reference orbit at a given accelerator azimuth. The problem is reduce to the construction of the phase trajectories H(X,Y)=const, that are third or fourth order curves and determine the mode of motion in the vicinity of the resonance. The phase trajectories build-up is performed in the Klein's perturbation method.

Introduction

The problems of nonlinear dynamics play an important role in different fields of the modern physics such as particle physics, nuclear physics, plasma physics, quantum electronics and, of course, in accelerator physics. An excitation of nonlinear resonance is widely used for particle extraction from cyclic accelerators [1] , action of nonlinear resonances determines the dynamical aperture of large accelerators and storage rings [2,3]. One of the most important problems in study of the nonlinear oscillations is construction of the trajectory of motion of representative points on phase plane and determination of stable motion regions. In present paper a geometric method of investigation of particle motion at nonlinear resonances is developed.

General consideration

The equation of one-dimensional particle motion in an accelerator in the presence of a disturbance is the following

$$\frac{d^2x}{d\varphi^2} + \nu^2 x = \varepsilon\ f(\varphi, x, dx/d\varphi) \qquad , \qquad (1)$$

where x the transverse displacement of the circulating particle with respect to the equilibrium orbit, ν is the betatron oscillation frequency, $\varphi = \int (ds/\nu\beta)$ is the generalized azimuthal angle, $f(\varphi, x, dx/d\varphi)$ is the periodic function in φ with period equal to 2π, β is the betatron function. Taking the smallness of the perturbation into account the solution of the equation (1) can be represented in the form

$$x = a(\varphi)\ \cos[\nu\varphi + \psi(\varphi)] \qquad (2)$$

with the amplitude a and the phase ψ depending on φ. For further analysis of the motion it is convenient to make the transformation to new variables [4,5]

$$X = a\ \cos\psi,\ Y = -a\ \sin\psi \qquad (3)$$

In this variables the equations of particle motion, take the canonical form

$$\frac{dX}{d\varphi} = \frac{\partial H}{\partial Y}\ ,\quad \frac{dY}{d\varphi} = -\frac{\partial H}{\partial X}\ , \qquad (4)$$

where H is the Hamiltonian. The particle motion is mapped on the phase plane by the H(X,Y)=const trajectories.

Third order resonance

The third order resonance $\nu = q/3$ is excited by the suitable q harmonic of the quadratic magnetic field $f(\varphi, x, dx/d\varphi) = -A_2 x^2 \cos q\varphi - Bx^3$, B is the constant component of qubic magnetic field. The Hamiltonian in this case has the following form [5]

$$H = \frac{A_2}{16}(Y^3 - 3X^2Y) + \frac{1}{2}(\nu - \frac{q}{3})(X^2+Y^2) + \frac{B}{2}(X^2+Y^2)^2 \quad . \qquad (5)$$

Let us introduce the new designations

$$s = Y - \sqrt{3}X + \frac{2\sqrt{3}}{3}X_0,$$
$$t = Y + \sqrt{3}X + \frac{2\sqrt{3}}{3}X_0, \qquad (6)$$
$$u = Y - \frac{\sqrt{3}}{3}X_0,$$

where $X_0 = \frac{8\sqrt{3}}{3}(\nu - \frac{q}{3})/A_2$, and in accordance with the sign of the tune shift $\delta = (\nu - \frac{q}{3})$ it takes positive or negative value. Taking (6) into account one can cast (5) in the form

$$stu + \frac{8B}{A_2}\ (X^2+Y^2)^2 = \frac{16}{A_2}H - \frac{4}{9}\ X_0^3 \qquad (7)$$

In general that is the equation of a fourth order curve. In absence of the constant component of qubic magnetic field (B=0) it defines a third order curve. The phase trajectories are specified by the equation

$$stu = \eta \quad , \qquad (8)$$

where $\eta = 16\ H/A_2 - 4\ X_0^3/9$ is a constant. By analogy with electrostatic one can call the η quantity as charge [6]. Then the complete

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phase plane splits into two domains: one carrying positive charge $\eta>0$, and other carrying negative charge $\eta<0$. The boundary between these domains is the separatrix

$$stu=0 \quad , \tag{9}$$

on separatrix $H=\frac{\sqrt{3}}{36}A_2X_0^3$. This equation is satisfied by the set of three straight lines: s=0, t=o, u=o forming the equilateral triangle at its intersection. The vertexes of this triangle determine the position of three unstable fixed points

$$1)\ X = 0, \quad Y = -\frac{2\sqrt{3}}{3}X_0;$$

$$2)\ X = -X_0, \quad Y = \frac{\sqrt{3}}{3}X_0;$$

$$3)\ X = X_0, \quad Y = \frac{\sqrt{3}}{3}X_0.$$

The separatrix splits the phase plane into regions of stable and unstable motion. Let us investigate the nature of the phase curves. It is previously worth to note that each of the straight lines s=0, t=0, u=0 divides the plane into two half-plane having positive and negative values of corresponding variables s, t, u. In accord with this fact the sign of the stu production is determined. For a small η value such that $|\eta| \ll |X_0^3|$ the corresponding phase trajectories place close to the separatrix in the domain defined by the η sign. It gives the qualitative technique for graphic build-up of the phase trajectories [7] (Fig.1, where $X_0>0$). There is the largest

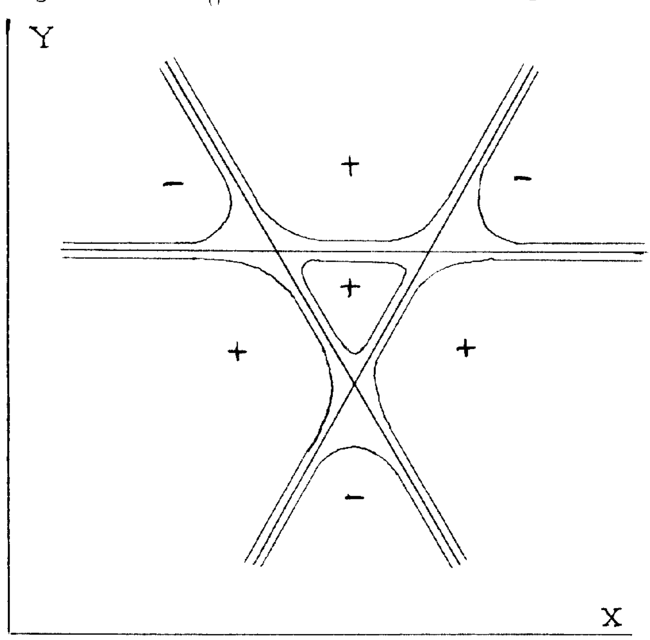

Fig. 1. Phase trajectories at third order resonance, B=0.

detachment of the curves from separatrix near to the fixed points. In the central region there are the close curves contained the stable fixed points. At $X^2+Y^2=R^2 \gg X_0^2$ the phase curve distance from the separatrix decreases as η/R^2. At B not equal to zero at the large distance from the center of the coordinate system $R^2 \gg X_0^2$ term in (7) proportional to $(X^2=Y^2)^2$ dominates and determines the sign of the left-hand side of the equation keeping it constant everywhere outside of some central circumference of a large enough radius. This gives rise to the coalescence of the separatrix branches. Moreover there is combining of the branches bounded the sectors with η opposite in sign to B.

Fourth order resonance

The fourth order resonance $\nu=q/4$ is excited by the suitable q harmonic of the qubic magnetic field $f(\varphi, x, dx/d\varphi)=-A_3x^3\cos q\varphi - Bx^3$. The Hamiltonian in this case has the following form [4,5]

$$H = \frac{A_3}{48}(X^4-6X^2Y^2+Y^4)+ \frac{1}{2}(\nu-\frac{q}{4})(X^2+Y^2)+ \frac{B}{2}(X^2+Y^2)^2 \quad . \tag{10}$$

It defines the phase trajectoiries to be the fourth order curves. These curves, as follows from (10), are symmetric about both axes. H(X,Y) in (10) may be represented in the following form

$$f_1(X,Y) \quad f_2(X,Y)=\eta, \tag{11}$$

$$f_1 = g(X^2 + \frac{1}{k_i} Y^2 + \frac{\operatorname{sgn} \delta}{(1+k_i)(1+D)} X_o^2) \ ,$$

$$f_2 = g^{-1}((1+D)X^2+(1+D)k_iY^2+\frac{\operatorname{sgn} \delta\ k_i}{(1+k_i)} X_o^2) \ ,$$

$$\eta = \frac{k_i}{(1+k_i)(1+D)} X_o^4 + \frac{48}{A_3} H$$

where g is scale factor, $\delta=\nu-q/4$, $D= 24B/A_3$, $X_0=2(6|\delta|/A_3)^{1/2}$, $k_{1,2}=(-3+D\pm\sqrt{8(1-D)})/(1+D)$, $k_1k_2=1$. The equations

$$f_1(X,Y)=0, \ f_2(X,Y)=0 \tag{12}$$

are the second order curve equations. At $D>1$ there are the unintersecting closed phase trajectories. The intersecting points of the curves are real for $D \leqslant 1$. Hence for such D one has a couple intersecting phase trajectories. The equation

$$f_1(X,Y) \quad f_2(X,Y)=0 \tag{13}$$

determines the separatrix. And mentioned points define position of the fixed points. At $-1 \leqslant D \leqslant 1$ there are two hyperbolae [4,5,8]. Let us consider in more detail the case $D<-1$, when our curves are two ellipses (Fig.2).

The phase trajectory bild-up may be performed in "the small variation method" explained by F.Klein []. The product $f_1 f_2$ is positive in the piece of the plane lying within or outside both ellipses; in four crescent pieces belonging only to one ellips it is negative. For small η the phase trajectories are arranged close to the separatrix in domains defined by the η sign. This treatment, as one can see, offers to seek the fixed points positions and to analyze its nature.

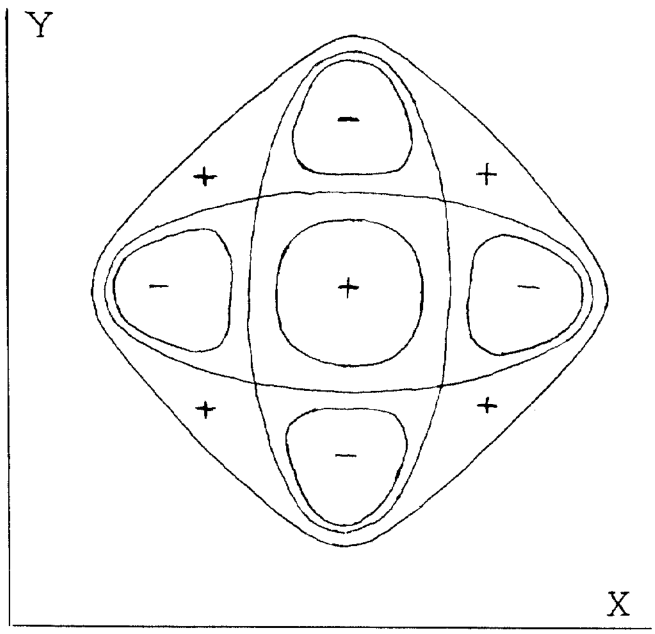

Fig. 2. Phase trajectories at fourth order resonance, D < -1.

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