The WKB Approximation and the Travelling-Wave Acceleration Cavity

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Abstract

The equations of motion of a charged particle in a travelling-wave accelerating cavity are those of a harmonic oscillator with a varying restoring constant. This restoring constant is negative for transverse motion, resulting in diverging trajectories. The restoring constant is positive for longitudinal motion. For many years the only solution to the equation in the program TRANSPORT [1] was for massless particles. The WKB approximation of quantum mechanics yields a very accurate solution for massive particles. It allows both the amplitude and the wave number of the trajectory to vary as the cavity is traversed. Solutions are approximately 30 times more accurate than previously published. Comparisons are made with numerical integration.

1 Introduction

The original derivation of the transfer matrix for the travelling-wave accelerator cavity was for electrons. It assumed an ultra-relativistic beam in which the mass of any particle was negligible compared to its kinetic energy. For many years this ultra-relativistic approximation was the only option in the computer program TRANSPORT. [1] People who wanted to design proton or ion beams containing accelerating cavities had to work with a very bad approximation.

In the 1987 edition of this conference, Hurd and McGill [2] (henceforth referred to simply as "Hurd") pointed out the need for a representation of an accelerating cavity for massive particles. They also provided a representation which was a substantial improvement over what was previously available.

In the travelling-wave accelerator cavity, both the amplitude and the wavelength of the transverse motion changes as the particle is accelerated. The wavelength of the transverse motion is not the same as the wavelength of the accelerating field. Hurd approximated the solution by holding the wavelength constant and letting the amplitude vary. About the same time the present author produced a solution where the wavelength varied and the amplitude was held constant. The two solutions were of comparable accuracy. An exact solution was also derived by this author using Runge-Kutta integration. The two solutions differed from the exact solution by about one percent.

It is desirable to have a solution which satisfied two criteria: (1) It is accurate to the number of decimal places which are printed out in the transfer matrix by TRANSPORT. (2) It longitudinally segments to the accuracy that the transfer matrix is printed by TRANSPORT. Previously, the accelerating cavity for massive particles was the only element in TRANSPORT for which longitudinal segmentation was not exact.

The WKB approximation of quantum mechanics [3] allows both the amplitude and wavelength of the motion to vary as the particle is accelerated. It is an approximation which may be iterated to any level of accuracy. We found that a single iteration was sufficient to produce the desired accuracy. The discrepancy from the exact solution was reduced from the two previous approximations by a factor which ranged from 20 to 200, depending on the numbers involved and the specific matrix element. Below we give the analytic derivation of the expressions and make comparisons with numerical solutions produced by Runge-Kutta integration.

2 Transverse Motion

The transfer matrix elements are referred to the standard six coordinates as used in TRANSPORT. They are x, x', y, and y' in the transverse plane, and ℓ and δ in the longitudinal direction. The quantity ℓ describes the longitudinal separation between two particles as a function of distance along the reference trajectory. The quantity δ is the fractional momentum deviation from the reference particle.

In the description of the equations of motion we shall follows Hurd's paper. [2] [4] [5] We are unable to improve on his presentation. We shall also employ his notation.

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The electric and magnetic fields are assumed to be constant over the length of the accelerating cavity. They are approximated then by:

$$E_z = \tfrac{1}{q}\Delta E cos\phi, \qquad E_r = \tfrac{\gamma_s}{q}(\tfrac{\pi r}{\gamma_s \beta_s \lambda})\Delta E sin\phi \qquad (1)$$

$$B_\theta = \tfrac{\gamma_s \beta_s}{qc}(\tfrac{\pi r}{\gamma_s \beta_s \lambda})\Delta E sin\phi$$

Here ΔE refers to the maximum possible energy gain of the cavity, and ϕ to the synchronous phase of the particle. The energy gain of the synchronous particle is then $\Delta E cos\phi$. The wavelength of the rf field is λ. The quantities γ and β are relativistic factors and c is the speed of light. The subscript "s" refers to the synchronous particle. The charge of the particle being accelerated is q.

In the derivation we shall also use a quantity η which is the product of γ and β. The additional subscripts "o" and "f" refer respectively to the initial and final values of the quantities to which they are attached. The length of the accelerating cavity will be L. The rest energy of the particle is mc^2.

The expressions for the fields lead to the first-order equation for the transverse motion of the particle:

$$\tfrac{d}{dz}\gamma_s \beta_s \tfrac{d}{dz} x = (\tfrac{\pi \Delta E sin\phi_s}{\lambda L mc^2})\tfrac{1}{\gamma_s^2 \beta_s^2} x \qquad (2)$$

The WKB approximation yields solutions for the transfer matrix elements as follows:

$$R_{11} = R_{33} = (\tfrac{\eta_{sf}}{\eta_{so}})^{\frac{1}{4}}[cosh(I_t) \qquad (3)$$

$$-\tfrac{1}{4}\tfrac{\eta'_{so}\eta_{so}^{\frac{1}{2}}}{\sqrt{Q}} sinh(I_t)]$$

$$R_{12} = R_{34} = \tfrac{\eta_{so}^{\frac{3}{4}}}{\sqrt{Q}}(\tfrac{\eta_{sf}}{\eta_{so}})^{\frac{1}{4}} sinh(I_t)$$

$$R_{21} = R_{43} = \tfrac{1}{4}[\tfrac{\eta'_{sf}}{\eta_{sf}^{\frac{3}{4}}\eta_{so}^{\frac{1}{4}}} - \tfrac{\eta'_{so}\eta_{so}^{\frac{1}{4}}}{\eta_{sf}^{\frac{1}{4}}}]cosh(I_t)$$

$$+[(\tfrac{\eta_{sf}}{\eta_{so}})^{\frac{1}{4}}\tfrac{\sqrt{Q}}{\eta_{sf}^{\frac{3}{4}}} - \tfrac{1}{16}\tfrac{\eta'_{sf}\eta'_{so}\eta_{so}^{\frac{1}{4}}}{\eta_{sf}^{\frac{1}{4}}\sqrt{Q}}]sinh(I_t)$$

$$R_{22} = R_{44} = \tfrac{1}{4}\tfrac{\eta_{so}^{\frac{3}{4}}}{\sqrt{Q}}\tfrac{\eta'_{sf}}{\eta_{sf}^{\frac{3}{4}}\eta_{so}^{\frac{1}{4}}} sinh(I_t) + (\tfrac{\eta_{so}}{\eta_{sf}})^{\frac{3}{4}} cosh(I_t)$$

$$\text{where} \qquad I_t = \int_0^z \tfrac{\sqrt{Q}}{\eta^{\frac{3}{2}}} d\zeta \qquad (4)$$

and

$$Q = \tfrac{\pi \Delta E sin\phi_s}{\lambda L mc^2} \qquad (5)$$

We are assuming that the rate of energy gain with distance is uniform over the accelerating cavity. This means that γ changes uniformly with distance and its derivative for the synchronous particle is given by:

$$\gamma' = \tfrac{\Delta E sin\phi_s}{L mc^2} \qquad (6)$$

The integral I_t may then be evaluated by a binomial expansion. Taking out the constant factor $\sqrt{Q}$, we have:

$$\int_o^z \tfrac{d\zeta}{\eta^{\frac{3}{2}}} = \tfrac{1}{\eta_{so}^{\frac{3}{2}}\gamma'}[\Delta\gamma - \tfrac{3}{4}\tfrac{\gamma_{so}}{\eta_{so}^2}(\Delta\gamma)^2 \qquad (7)$$

$$-\tfrac{1}{4\eta_s^2}(\Delta\gamma)^3 + \tfrac{7}{8}\tfrac{\gamma_{so}^2}{\eta_{so}^4}(\Delta\gamma)^3]$$

$$\text{where} \qquad \Delta\gamma = \gamma_{sf} - \gamma_{so} \qquad (8)$$

3 Longitudinal Motion

We first formulate the equations of motion in the longitudinal direction in terms of the particle energy W, and its phase ϕ. The synchronous energy is W_s and the synchronous phase is ϕ_s. The deviations of a given particle from these quantities are given by ΔW and $\Delta\phi$. The first-order equations then become:

$$\tfrac{d\Delta W}{dz} = -\tfrac{\Delta E sin\phi_s}{L}\Delta\phi \qquad (9)$$

$$\tfrac{d\Delta\phi}{dz} = \tfrac{2\pi}{\lambda}\tfrac{\Delta W}{\gamma_s^3 \beta_s^3 mc^2}$$

The quantities used are related, in first order, to the standard TRANSPORT variables by:

$$\ell = \tfrac{\beta_s \lambda}{2\pi}\Delta\phi \qquad \text{and} \qquad \tfrac{\Delta p}{p_s} = \tfrac{1}{\beta_s^2}\tfrac{\Delta W}{W_s} \qquad (10)$$

The WKB approximation yields solutions for the longitudinal transfer matrix elements as follows:

$$R_{55} = \frac{\beta_{sf}}{\beta_{so}}[(\frac{\eta_{so}}{\eta_{sf}})^{\frac{3}{4}} cos(I_\ell)$$

$$+\frac{3}{4}\eta'_{sf}\sqrt{\eta_{so}}(\frac{\eta_{so}}{\eta_{sf}})^{\frac{1}{4}}\frac{1}{\sqrt{Q_\ell}} sin(I_\ell)] \tag{11}$$

$$R_{56} = -\frac{\beta_{sf}\beta_{so}\lambda}{2\pi}\frac{Lmc^2}{\Delta E sin\phi_s}\{\frac{3}{4}[(\frac{\eta_{so}}{\eta_{sf}})^{\frac{1}{4}}\eta'_{sf}$$

$$-(\frac{\eta_{so}}{\eta_{sf}})^{\frac{3}{4}}\eta'_{so}]cos(I_\ell)$$

$$-[\frac{9}{16}\frac{\eta'_{so}\eta'_{sf}}{\sqrt{Q_\ell}}\frac{\eta_{so}^{\frac{3}{4}}}{\eta_{sf}^{\frac{1}{4}}} + \sqrt{Q_\ell}\frac{\eta_{so}^{\frac{1}{4}}}{\eta_{sf}^{\frac{3}{4}}}]sin(I_\ell)\}$$

$$R_{65} = -\frac{\eta_{so}^{\frac{3}{4}}}{\eta_{sf}^{\frac{1}{4}}}\frac{1}{\beta_{so}\beta_{sf}}\sqrt{Q_\ell}\, sin(I_\ell)$$

$$R_{66} = (\frac{\eta_{so}}{\eta_{sf}})^{\frac{1}{4}}\frac{\beta_{so}}{\beta_{sf}}[cos(I_\ell) - \frac{3}{4}\frac{\eta_{so}^{\frac{1}{2}}\eta'_{so}}{\sqrt{Q_\ell}} sin(I_\ell)]$$

Here $$I_\ell = \int_o^z \frac{\sqrt{Q_\ell}}{\eta^{\frac{3}{2}}} d\zeta \tag{12}$$

and $$Q_L = 2Q \tag{13}$$

4 Comparison of Results

Numerical comparison of the values of the transfer matrix elements derived from Runge-Kutta integration and from the old and new versions of TRANSPORT. The example used is from Hurd's article. The initial energy is 100 MeV, the energy gain is 3.19 MeV over a length of 290 cm, and the synchronous phase is 30 degrees. There is some slight discrepancy with the exact values as printed in Hurd's article, which is probably attributable to rounding error in taking the parameters of the accelerating cavity directly from his article.

Element	Numerical Integration	Transport (Old)	Transport (New)
$R_{11} = R_{33}$	1.197	1.0	1.197
$R_{12} = R_{34}$	0.307	0.286	0.307
$R_{21} = R_{43}$	1.386	0.0	1.386
$R_{22} = R_{44}$	1.179	0.973	1.179
R_{55}	0.653	1.000	0.653
R_{56}	2.059	0.0	2.060
R_{65}	-0.281	-0.026	-0.281
R_{66}	0.624	0.997	0.624

References

[1] K.L. Brown, F. Rothacker, D.C. Carey, and Ch. Iselin, "TRANSPORT, A Computer Program for Designing Charged Particle Beam Transport Systems", SLAC Report SLAC91.

[2] J.W. Hurd and J. McGill, "Modification of Acceleration Element in TRANSPORT, IEEE Particle Accelerator Conference, p. 1198, 1987.

[3] J. Mathews and R.L. Walker, <u>Mathematical Methods of Physics</u>, Benjamin/Cummins Publishing Company, 1970.

[4] K. Mittag, "On Parameter Optimization for a Linear Accelerator," Kernforschungszentrum Karlsruhe GmbH, Report KfK 2555, Jan. 1978.

[5] M. Stanley Livingston and John P. Blewett, <u>Particle Accelerators</u>, McGraw-Hill, New York, 1962.