A New Approach to Potential Well Bunch Deformation

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Abstract

A stationary solution of the Vlasov equation fulfills a nonlinear integral equation of Volterra type known as Haissinski's equation. It describes the shape of a bunch of particles as function of the current and the impedance of the surrounding structure. For a linear accelerating voltage, the zero current bunch has a Gaussian shape, while it is deformed for finite currents. This paper describes a new analytic technique for solving this integral equation which is based on transforming the Volterra integral equation into one of Fredholm type with fixed integration limits, using a Fourier expansion for the discontinuous integrand. The results are applied to LEP at injection energy and are found to be in excellent agreement with measurements of the bunch length in the potential well regime.

1 Introduction

The longitudinal phase-space distribution of an electron bunch in a high energy storage ring or accelerator is described by a Fokker-Planck equation. In particular, the energy distribution is determined by the equilibrium between radiation damping and quantum excitation of the synchrotron motion. In the absence of additional perturbing forces, the energy distribution is Gaussian.The stationary bunch shape or line density $\lambda(\tau)$ was originally written as an integral equation containing a double integral [1]. However, by redefining the kernel, it can be reduced to an equation with a single integral[2],[3]

$$\lambda(\tau) = K \exp[-U_0(\tau) - \xi \int_{-\infty}^{\tau} dt\, S(\tau - t)\, \lambda(\tau)] \qquad (1)$$

where $U_0(\tau)$ is the potential of the applied RF voltage. For a (locally) linear RF voltage - usually a very good approximation for electron storage rings with short bunches - the potential is parabolic

$$U_0(\tau) = \frac{\tau^2}{2\sigma_0^2} \qquad (2)$$

where σ_0 is the zero-current RMS bunch length. The parameter ξ is proportional to the bunch current and given by

$$\xi = \frac{2\pi I_b}{h_{RF} V_{RF} cos\phi_s \omega_0^2 \sigma_0^2} \qquad (3)$$

$S(\tau)$ stands for the "step-function response", defined as the integral over the wake function which itself is the Fourier transform of the impedance $Z(\omega)$, i.e..

$$S(\tau) = \frac{1}{2\pi} \int_0^{\tau} dt \int_{-\infty}^{+\infty} d\omega\, Z(\omega) \exp(i\omega t) \qquad (4)$$

Finally, K is a normalization constant defined by

$$\int_{-\infty}^{+\infty} dt\, \lambda(t) = 1 \qquad (5)$$

As can be seen from Eqs.(1) and (2), the line density $\lambda(\tau)$ is Gaussian for vanishing bunch current ($\xi = 0$). However, for $\xi > 0$ the Gaussian is deformed. In the following sections we introduce an analytic procedure converting the nonlinear Volterra equation to an integral equation of Fredholm type which can solved by the method of "degenerate kernels". In Fig. 1 we show the bunch shape in LEP at injection energy for various bunch currents assuming a resonator impedance. The curves represent the analytic results produced by the theory described in this contribution.

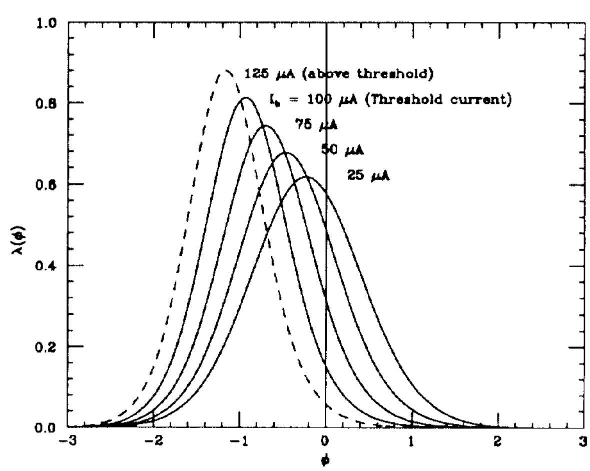

Fig.1: Bunch deformation in LEP as function of current

2 Analytic Solution of the Haissinski Equation

Haissinski's integral equation (1) has been solved in closed form only for two special types of impedances: purely **in-**

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ductive [1],[4] or purely **resistive** [5]. For purely **capacitive** impedances an approximate solution has been published recently [6].
However, these impedance models are not very realistic for short bunches in a high-energy storage ring. Here we illustrate the method for a **resonator impedance**, although it can be applied to any impedance. The resonator impedance has three parameters: shunt impedance R_s, resonator frequency $\omega_r/2\pi$ and quality factor $Q > 1/2$, and its step-function response $S(\phi)$ can be expressed by the imaginary part of a complex exponential

$$S(\phi) = \frac{2R_s}{W}\,\mathrm{Im}[\mathrm{e}^{\mathrm{i}\nu_r\phi}] \tag{6}$$

where

$$\nu_r = \nu_0(i + W) \tag{7}$$
$$\nu_0 = \omega_r\sigma_0/(Q\sqrt{2}) \tag{8}$$
$$W^2 = 4Q^2 - 1 \tag{9}$$

and where we introduced the normalized independent variable

$$\phi = \tau/(\sigma_0\sqrt{2}) \tag{10}$$

With the parameter

$$\epsilon = \frac{2R_s\xi}{W} = \frac{R_s I_b}{\pi h V_{RF} cos\phi_s f_0^2\sigma_0^2 W} \tag{11}$$

the Haissinski equation for a resonator impedance becomes

$$\lambda(\phi) = K e^{-\phi^2 - \epsilon\,\mathrm{Im}\left[\mathrm{e}^{\mathrm{i}\nu_r\phi}\int_{-\infty}^{\phi}\mathrm{e}^{-\mathrm{i}\nu_r\varphi}\lambda(\varphi)\mathrm{d}\varphi\right]} \tag{12}$$

The basic method to solve Eq. (12) consists in transforming it from a **Volterra** to a **Fredholm** type integral equation with integration limits independent of ϕ. For this purpose we use the identity

$$\int_{-\infty}^{\phi} F(\varphi)d\varphi = \int_{-\infty}^{+\infty} F(\varphi)H(\varphi - \phi)d\varphi \tag{13}$$

where $H(x)$ is the Heavyside step function, i.e. $H(x) = 1$ for $x < 0$ and $H(x) = 0$ when $x > 0$. Then the Haissinski equation in the Fredholm representation reads

$$\lambda(\phi) = K e^{-\phi^2 - \epsilon\,\mathrm{Im}\left[\mathrm{e}^{\mathrm{i}\nu_r\phi}\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{i}\nu_r\varphi}\lambda(\varphi)\mathrm{H}(\varphi-\phi)\mathrm{d}\varphi\right]} \tag{14}$$

We now replace $H(x)$ by a periodic square wave function with a period $T = 2\pi/\nu$ assumed to be large compared to the bunchlength σ_0. In particular, we may take the machine circumference which is the actual period of the line density. We then use the Fourier expansion of $H(x)$ (or of the product of H with the step-function response for a general impedance)

$$H(x) = \frac{1}{2} + \frac{i}{\pi}\sum_{k=-\infty}^{+\infty}\frac{e^{(2k-1)i\nu x}}{2k-1} \tag{15}$$

Finally inserting (15) into (14) and rearranging the terms leads to the following system of equations

$$\lambda(\phi) = K\exp\{-\phi^2 - \epsilon\,\mathrm{Im}\,[\exp(\mathrm{i}\nu_r\phi)\mathrm{G}(\phi)]\} \tag{16}$$

$$G(\phi) = \frac{C_0}{2} + \frac{i}{\pi}\sum_{k=-\infty}^{+\infty} C_{2k-1}\frac{e^{-(2k-1)i\nu_r\phi}}{2k-1} \tag{17}$$

$$C_k = \int_{-\infty}^{+\infty}\lambda(\varphi)\exp\{-i(\nu_r - k\nu)\varphi\}d\varphi \tag{18}$$

which form an infinite set of algebraic equations for the coefficients C_k.

3 Iterative solutions for $\lambda(\phi)$

Since this system of equations is strongly nonlinear, it cannot be solved in closed form. We therefore use an iterative approach to obtain successive approximations of the solution. For $\epsilon = 0$ the solution is known and given by

$$\lambda^{(0)}(\phi) = \frac{1}{\sqrt{\pi}}\exp[-\phi^2\,] \tag{19}$$

We now use this "unperturbed" solution to construct approximate coefficients $C_k^{(0)}$ from Eq. (18). Then a first correction for λ can be found from (17) and (16). Inserting (19) into (18) gives

$$C_k^{(0)} = \exp\{-\frac{(\nu_r - k\nu)^2}{4}\} \tag{20}$$

All infinite sums occurring after inserting Eqs.(18) and (19) into (17) are of the type

$$\sum_k \frac{e^{-(2ak+b)^2}}{2k-1} = \frac{i\pi}{2}\left[\mathrm{w(a+b)} - \mathrm{e}^{-(\mathrm{a+b})^2}\right] \tag{21}$$

which have been evaluated as described in [7]. For the corrected line density we then obtain

$$\lambda^{(1)}(\phi) = K\exp\{-\phi^2 - \epsilon\exp\{-\phi^2\}\,\mathrm{Im}[\mathrm{w}(\frac{\nu_\mathrm{r}}{2} - \mathrm{i}\phi]\} \tag{22}$$

where w(z) stands for the complex error function. Note that the period of the square-wave function does not occur in the result for $\lambda^{(1)}$. As we shall see this remains also true for the higher iterates $\lambda^{(2)}$ and $\lambda^{(3)}$. In order to get more physical insight we expand the perturbing part of the exponent of (22) into a Taylor series with respect to ϕ

$$\lambda^{(1)}(\phi) = K\exp\{-\phi^2 - a_0 - a_1\phi - a_2\phi^2 ...\} \tag{23}$$

The a_k are given by

$$a_0 = \frac{1}{2}\epsilon\,\mathrm{Im}[\mathrm{w}(\nu_\mathrm{r}/2)] \tag{24}$$
$$a_1 = \frac{1}{2}\epsilon\,\mathrm{Im}[\mathrm{i}\nu_\mathrm{r}\mathrm{w}(\nu_\mathrm{r}/2)] \tag{25}$$
$$a_2 = \frac{1}{4}\epsilon\,\mathrm{Im}[2\mathrm{i}\nu_\mathrm{r}/\sqrt{\pi} - \nu_\mathrm{r}^2\mathrm{w}(\nu_\mathrm{r}/2)] \tag{26}$$

For $\lambda^{(1)}$ we need only the term linear in ϕ of the perturbing part. After normalization it becomes

$$\lambda^{(1)}(\phi) = \frac{1}{\sqrt{\pi}} \exp[-(\phi - \Delta\phi)^2] \tag{27}$$

with

$$\Delta\phi = -\frac{1}{2}a_1 = -\frac{1}{4}\epsilon \mathrm{Re}[\nu_r \mathrm{w}(\nu_r/2)] \tag{28}$$

To first order only a shift of the unperturbed Gaussian is observed when the bunch current differs from zero ($\epsilon > 0$). The next step is to insert $\lambda^{(1)}$ into Eqs.(18) and (17) to obtain the next order correction $\lambda^{(2)}$.

$$\lambda^{(2)}(\phi) = Ke^{-\phi^2 - \epsilon \exp\{(-\phi-\Delta\phi)^2\} \, \mathrm{Im}[\mathrm{w}(\frac{\nu_r}{2} - \mathrm{i}(\phi-\Delta\phi)]} \tag{29}$$

As before, expanding the perturbing part of the exponential into a Taylor series with respect to $\phi - \Delta\phi$ yields

$$\lambda^{(2)}(\phi) = Ke^{-\phi^2 - a_0 - a_1(\phi-\Delta\phi) - a_2(\phi-\Delta\phi)^2 - \ldots} \tag{30}$$

where a_0, a_1 and a_2 are given by Eqs.(24) - (26). In this order of the approximation we keep terms up to $(\phi - \Delta\phi)^2$. Rearranging the terms and using $a_1 = -2\Delta\phi$ we arrive at the result for $\lambda^{(2)}$

$$\lambda^{(2)}(\phi) = \sqrt{\frac{1+a_2}{\pi}} e^{-(1+a_2)(\phi-\Delta\phi)^2} \tag{31}$$

which is a Gaussian with shift $\Delta\phi$ and a bunchlength depending on ϵ and thus on the bunch current:

$$\sigma(\epsilon) = \frac{\sigma_0}{\sqrt{1 + \frac{\epsilon}{4} \mathrm{Im}\left[\frac{2\mathrm{i}\nu_r}{\sqrt{\pi}} - \nu_r^2 \mathrm{w}(\nu_r/2)\right]}} \tag{32}$$

In Fig.2 we show σ/σ_0 as function of the bunch current for LEP at injection energy[8]. For this case, the coefficient ϵ is related to the bunch current by

$$I_b[\mu A] = 25\epsilon/4 \tag{33}$$

The full line corresponds to the analytic result (32) while the dashed line represents the result obtained from a direct numeric integration of the Haissinski equation. The crosses indicate the results of a series of bunch length measurements taken in the LEP control room[9]. A further improvement can be obtained by performing a third iteration step by inserting $\lambda^{(2)}$ into Eqs.(18) and (17) to obtain from Eq.(16).

$$\lambda^{(3)}(\phi) = Ke^{-\phi^2 - \frac{\epsilon}{2} \exp[-\rho(\phi-\Delta\phi)^2] \, \mathrm{Im}[\mathrm{w}(\nu_r - 2\mathrm{i}\rho(\phi-\Delta\phi)/(2\sqrt{\rho})]} \tag{34}$$

where $\rho = 1 - a_2$. For increasing currents the bunch length decreases, due to the capacitive impedance seen by a bunch originally shorter than the wavelength of the resonator impedance. For larger currents also a slight asymmetry of the line density becomes visible.

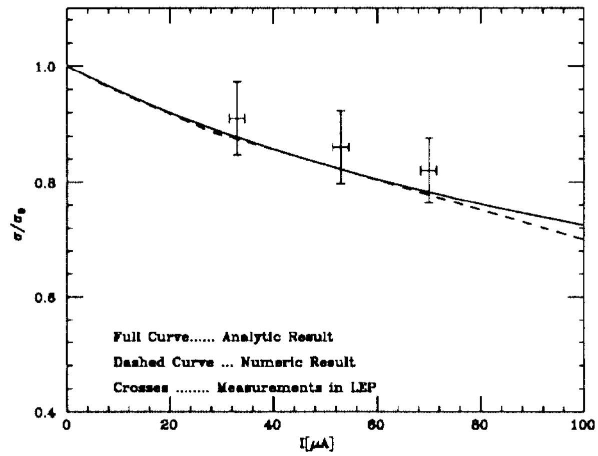

Fig.2 Comparison of the analytic expression for the bunchlength as function of current with numeric integration and measurements

4 Conclusions

Analytic expressions for the deformation of an electron bunch as function of current have been derived by expanding the solution of the Haissinski equation valid in the potential well region. The strongest effect is a shift of the stable phase angle, which increases the potential energy of the bunch until it exceeds the total energy of the time-dependent solution of the Fokker Planck equation. The threshold for "turbulent bunch lengthening" occurs approximately when the shift equals one sigma[10], which rule-of- thumb agrees quite well with observations in LEP.

5 References

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