QUANTUM VARIANCES FOR TRANSVERSE SSC INJECTION DYNAMICS

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Abstract

Quantum variances for the transverse dynamics associated with the injection phase of the Susperconducting Super Collider are derived using squeezed state methods.

1 Introduction

During the injection phase into the SSC particles near the edge of the dynamic aperture perform chaotic motion [1]. With an injection amplitude difference as small as 10^{-10} mm chaotic divergences for the x and y transverse betatron motions begin to occur as soon as 15×10^3 turns around the SSC. It is of interest to see what quantum uncertainties are associated with transverse amplitudes. These limits are indeed significant for the SSC, and they are in fact larger than the small differences which lead to chaotic dynamics.

2 Betatron Equations

The equations of motion for transverse betatron oscillations can be found from a Lorentz invariant Hamiltonian [2]. This Hamiltonian which is equal to the rest energy of the proton is invariant for curvilinear coordinate transformations $q^i \to \bar{q}^i$. This allows one to represent equations for betatron motion in the coordinate system related to the ideal orbit of a proton and to see the various approximation made to obtain the usual Courant-Snyder equations. For this invariant Hamiltonian, contravariant coordinates are defined as the four-vector $q^i = (q^o, q^1, q^2, q^3)$ $=(ct, x, y, z) \to (ct, x, y, \bar{s})$ where c is the speed of light and where $\bar{s}$ is the arc length along the ideal orbit. The contravariant components of the four-velocity and four-momentum are

$$u^i = dq^i/ds \quad \text{and} \quad p^i = mcu^i \tag{1}$$

where $ds = \sqrt{dq_i dq^i}$. The invariant Hamiltonian is defined as

$$H(p^i, q^i)/c = p_i u^i - L(u^i, q^i) + mc \tag{2}$$

where the invariant Lagrangian for the motion of a particle of mass m and electric charge e in an electromagnetic field with four-potential $A(q)^i$ is

$$L(u^i, q^i) = mc\sqrt{u_i u^i} + (e/c)A(q)_i u^i. \tag{3}$$

In terms of the conjugate four-momentum, the Lagrangian becomes $L(u^i, q^i) = p_i u^i$, and the invariant action is

$$S = \int p_i u^i ds = \int \bar{p}_i \bar{u}^i ds. \tag{4}$$

Treating $\bar{s}$ as the time coordinate, $\bar{p}_{\bar{s}}$ as the Hamiltonian, and using the method of stationary action, one finds equations which describe transverse betatron motion.

3 Quantum Limits on Injection

As a first approximation for finding the quantum limits, one considers a time-dependent linear oscillator Hamiltonian for an on momentum particle

$$\mathbf{H}(\bar{s}) = \frac{\mathbf{p}^2 + K(\bar{s})\mathbf{q}^2}{2}. \tag{5}$$

With the conditions

$$\begin{aligned} w'' + K(\bar{s})w - \frac{1}{w^3} &= 0 \\ \mathbf{q}'' + K(\bar{s})\mathbf{q} &= 0 \end{aligned} \tag{6}$$

and with $q = we^{\pm i\psi}$ and $\psi' = \frac{1}{w^2}$, one finds the invariant

$$2\mathbf{I} = \gamma(\bar{s})\mathbf{q}^2 + \alpha(\bar{s})(\mathbf{qp} + \mathbf{pq}) + \beta(\bar{s})\mathbf{p}^2 \tag{7}$$

with Courant-Snyder parameters

$$\begin{aligned} \alpha(\bar{s}) &= -ww', \beta(\bar{s}) = w^2, \text{ and} \\ \gamma(\bar{s}) &= \frac{1 + \alpha^2(\bar{s})}{\beta(\bar{s})}. \end{aligned} \tag{8}$$

The quantum states for this system can be constructed with the aid of the squeezing operator defined as

$$\mathbf{S} = exp((\xi^* \mathbf{a}^2 - \xi \mathbf{a}^{\dagger 2})/2). \tag{9}$$

The time-independent rationalized Hamiltonian, $\mathbf{H}_o$, is found from (5) with $K(\bar{s}) = 1$ where the boson operators are found from $\sqrt{2}\mathbf{q} = \mathbf{a} + \mathbf{a}^\dagger$ and $\sqrt{2}i\mathbf{p} = \mathbf{a} - \mathbf{a}^\dagger$. The invariant (7) can be found from $\mathbf{H}_o$ as

$$2I(\bar{s}) = \mathbf{S}\mathbf{H}_o\mathbf{S}^\dagger = \left(\mathbf{b}^\dagger \mathbf{b} + \frac{1}{2}\right) \tag{10}$$

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where

$$\mathbf{b}(\bar{s}) = \mathbf{S}e^{i\theta}\mathbf{a}\mathbf{S}^{\dagger} = \frac{1}{2}\left(\frac{1}{w} + w - iw'\right)\mathbf{a} + \frac{1}{2}\left(\frac{1}{w} - w - iw'\right)\mathbf{a}^{\dagger} \quad (11)$$

with $\theta(w, w')$. The eigenstates of $I(\bar{s})$ satisfy the eigenvalue equation

$$\mathbf{I}(\bar{s})|n,\bar{s}) = \left(n + \frac{1}{2}\right)|n,\bar{s}) \qquad |n,\bar{s}) = \frac{(\mathbf{b}^{\dagger})^{n}}{\sqrt{n!}}|0), \quad (12)$$

and the units of $I(\bar{s})$ are $(\hbar/|\vec{p}|)rad$. The Schrödinger states [3] are

$$|n,\bar{s})_{s} = e^{i\alpha_{n}(\bar{s})}|n,\bar{s}) \quad (13)$$

where the phase is

$$\alpha_{n}(\bar{s}) = -\left(n + \frac{1}{2}\right)\int^{\bar{s}} \frac{d\bar{s}'}{w^{2}(\bar{s}')}. \quad (14)$$

The coherent state for a time-dependent linear oscillator used to evaluate variances is generated from the squeezed ground state as

$$|\beta,\bar{s})_{s} = e^{\beta\mathbf{b}^{\dagger}(\bar{s}) - \beta^{*}\mathbf{b}(\bar{s})}|0,\bar{s})_{s} \quad (15)$$

where β, related to the classical value of the invariant $I(\bar{s})$, is the eigenvalue of the operator $\mathbf{b}(\bar{s})$.

One can now use the states (15) to obtain results appropriate for the SSC. Using appropriate scaling transformations

$$P \rightarrow \frac{p}{|\vec{p}|},\ H \rightarrow \frac{H}{|\vec{p}|},\ \text{and}\ \hbar \rightarrow \frac{\hbar}{|\vec{p}|} \quad (16)$$

with the momentum $|\vec{p}| \approx \mathcal{E}/c$ for a proton of energy $\mathcal{E}$, one finds the variances and the uncertainty product represented in terms of the Courant-Snyder parameters (8) are

$$\sigma(q) = \sqrt{\frac{\hbar\beta(\bar{s})}{2|\vec{p}|}},\ \sigma\left(\frac{p}{\vec{p}}\right) = \sqrt{\frac{\hbar\gamma(\bar{s})}{2|\vec{p}|}}, \qquad \frac{p}{|\vec{p}|} = \frac{dq}{d\bar{s}},\ \text{and}\ \sigma(q)\sigma\left(\frac{dq}{d\bar{s}}\right) = \frac{\hbar\sqrt{\beta(\bar{s})\gamma(\bar{s})}}{2|\vec{p}|}. \quad (17)$$

Writing the amplitude as $q_{amp} = \sqrt{\epsilon/\pi\beta(\bar{s})}$ with the emittance $\epsilon = 2\pi I(\bar{s})$, one finds the results

$$\frac{\sigma(q_{amp})}{q_{amp}} = \left(\frac{\epsilon_{q}}{\epsilon}\right)^{1/2} \qquad \epsilon_{q}/\pi = \frac{\hbar}{2|\vec{p}|} \approx \frac{\hbar c}{2\mathcal{E}} \approx \frac{\lambda_{proton}}{2} \quad (18)$$

where ϵ_{q}/π, the quantum emittance, represents half the resolution length of a proton in the beam. With the approximations $\hbar c \approx 2 \times 10^{-19}\, TeV\, m$, and $\mathcal{E} \approx 2\, TeV$, one finds

$$\epsilon_{q}/\pi \approx 5 \times 10^{-20}\, m. \quad (19)$$

For a typical SSCTRK tracking result showing chaotic motion with $\beta(\bar{s}) \approx 300\, m$ and with $x_{amp} \approx 3.5\, mm$, one finds

$$\epsilon/\pi \approx 4 \times 10^{-8}\, m$$

and

$$\sigma(x_{amp}) \approx 3.9 \times 10^{-6}\, mm. \quad (20)$$

Similarly, the angular uncertainty is

$$\sigma\left(\frac{dx}{d\bar{s}}\right) \approx 1.3 \times 10^{-11}\, rad. \quad (21)$$

V. Conclusions

The quantized invariant of the linearized betatron equations which are obtained from an invariant action leads to the time-dependent coherent state used to evaluate the uncertainties of the position and momentum operators. The maximum value of $\beta(\bar{s})$ in the SSC arcs during injection is $300\, m$, the value used to obtain the variances (20) and (21). However, larger variances can occur during the injection phase in other regions where $\beta(\bar{s}) > 1800\, m$. It is of interest to note that the emittance defined as $\epsilon = 2\pi I(\bar{s})$ which is the area of the (q, q') phase space ellipse is quantized and that the variance is

$$\sigma(\epsilon) = \pi|\beta|(\hbar/|\vec{p}|)rad.$$

A more complete account may be found in [4].

References

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