Abstract: An accurate knowledge of the beam sizes at collision points is of vital interest for several important aspects of collider's operation. No device is usually available to provide direct information and one relies on the scaling of beam sizes measurements performed elsewhere, using an acquired knowledge of amplitude functions. This paper reviews machine luminosity, for the interpretation of beam-beam effects, for the normalization of cross-sections at collision points is of vital interest for results.

The two Equ. 2 can be solved for $c$ and $B_{m}$, and since $2\beta = B_{m} + c^{2}/B_{m}$ (see Fig. 1) one gets, after some algebraic manipulations:

$$\beta = \frac{1}{2} \left( 1 - (\xi^{2} - 1) \right)$$

with $\xi = \beta(+) + \beta(-)$, $\eta = \beta(+)$ - $\beta(-)$.

Table 1 illustrates the case of LEP where $2 = 3.78$ m. With the assumption that $\beta(\xi)$ can be measured to 1%, the smallest controllable value for the beta difference is $\eta = (\eta)_{c} = 3.8 \beta(\xi)/100$.

The result is very surprising: in the focusing plane the relative error is increased by nearly an order of magnitude, just because the symmetry of the amplitude function around the crossing point cannot be guaranteed!

In practice the numerical application given above is of academic interest because no position monitor exists on the inside the LEP insertion quads. But a similar derivation can be done for BPMs located beyond the quads (see Appendix A) and the resulting accuracy on $\beta_{s}$ is even worse [1].

Table 1 : Numerical application to LEP

<table>
<thead>
<tr>
<th>Plane</th>
<th>Focussing ($\epsilon$)</th>
<th>Defocussing ($\epsilon$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.07 m</td>
<td>1.75 m</td>
</tr>
<tr>
<td>$\beta(\xi)$</td>
<td>195.5 m</td>
<td>9.6 m</td>
</tr>
<tr>
<td>$\eta$</td>
<td>91 m</td>
<td>19.7 m</td>
</tr>
<tr>
<td>$\xi$</td>
<td>2.8 m</td>
<td>0.14 m</td>
</tr>
<tr>
<td>$\delta* / \delta_\xi$</td>
<td>1.86 x 10^{-4}</td>
<td>0.142</td>
</tr>
<tr>
<td>$\delta* / \delta_\eta$</td>
<td>1.79 x 10^{-3}</td>
<td>0.003</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>7.15</td>
<td>1.10</td>
</tr>
</tbody>
</table>
3. Determination of $B^*$ from the Q-shift induced by quadrupole gradient perturbation

When a small change of strength $\Delta k$ is applied to one of the insertion quads the resulting Q-shift is given to first order by [21]

$$\Delta Q = \frac{\Delta k}{4\pi} \int \hat{B}(s) ds$$

or

$$\Delta Q = \frac{\Delta k}{4\pi} \hat{B}(s) ds$$

(5)

with $L$ the magnetic length of the quad. In most cases this measurement is not accurate in the defocusing plane due to the small induced Q-shift and will be disregarded in the following discussion. Let us apply it to the focusing plane to either quad and obtain two independent values called $\Delta B_+$ and $\Delta B_-$. We are now back to the problem of paragraph 2 of computing $E^*$. The exact solution is derived in Appendix B and reads

$$\Delta Q = \frac{\Delta k}{4\pi} \left[ \frac{1}{2} + \frac{n}{P} \right]$$

(6)

where $\xi = \Delta B_+ + \Delta B_-$, $\eta = \Delta B_+ - \Delta B_-$, and $P = 4\xi$ are two constants. Equ. 6 reveals very clearly the respective influence of $\xi$ and $\eta$ on the amplitude function at the collision point.

The accuracy of the result can be analysed as follows. For a value of $\Delta B_+$ one has to measure twice $Q$ and $k$ to get $Q_0$ and $\Delta k$ used in Equ. 5. Since all measurement errors involved are independent, they can be added in quadrature to give

$$\Delta Q = \frac{1}{2} \left( \frac{<k>}{\Delta k} + <Q> \frac{1}{\Delta Q} + \frac{1}{\Delta k} \right)^{1/2}$$

(7)

where $<\Delta k>$, $<Q>$ and $<\Delta Q>$ are the r.m.s. errors. The error propagation is now the same as in paragraph 2 where only $B$ must be replaced by $\hat{B}$ and the partial derivatives of Equ. 6 have to be introduced into Equ. 4.

Numerical application to LEP: The best feasible measuring accuracies are taken as:

- Quadrupole strength: $<\Delta k>/k = 2 \times 10^{-5}$
- Magnetic length: $<\Delta L>/L = 10^{-3}$
- Q Measurement: $<\Delta Q>/Q = 2 \times 10^{-4}$

Simulations with MAD [3] show that for $\Delta k = 10^{-3}$ the Q-shift ($\Delta Q = 0.0413$) is at the limit of linearity for Equ. 5 and should also just be acceptable in the working diamond. $\Delta k = 10^{-3}$ is used for computing the best performances shown in Table 2. Other relevant parameters are [4]: $\beta = 3.7$ m, $L = 2$ m, $k = 0.166$ m$^{-2}$, for the insertion with nominal $B^*_0 = 1.75$ m and $B^*_0 = 7$ cm.

Table 2 : Performance for $B^*_0$ measurement

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>$\eta$</th>
<th>$\epsilon$</th>
<th>$B^*_0$</th>
<th>$\Delta B^<em>_0$/$B^</em>_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>m</td>
<td>m</td>
<td>cm</td>
<td>cm</td>
<td>%</td>
</tr>
<tr>
<td>518</td>
<td>3</td>
<td>1.4</td>
<td>7.3</td>
<td>7.7</td>
</tr>
<tr>
<td>518</td>
<td>5</td>
<td>2.1</td>
<td>7.8</td>
<td>12.3</td>
</tr>
<tr>
<td>518</td>
<td>10</td>
<td>4.6</td>
<td>10.1</td>
<td>18.6</td>
</tr>
<tr>
<td>518</td>
<td>20</td>
<td>9.3</td>
<td>19.2</td>
<td>19.5</td>
</tr>
<tr>
<td>518</td>
<td>30</td>
<td>13.9</td>
<td>27.4</td>
<td>27.5</td>
</tr>
<tr>
<td>518</td>
<td>40</td>
<td>18.6</td>
<td>35.2</td>
<td>35.5</td>
</tr>
</tbody>
</table>

4. The antisymmetric gradient perturbation method

In order to apply higher gradient perturbations without boosting $\Delta Q_2$ beyond 0.04, one can apply simultaneously $+\Delta k$ on one quad and $-\Delta k$ on the other quad of the insertion. The observed tune will then be:

$$Q' = Q_0 + \frac{L}{4\pi} \left[ \Delta k + \Delta k^2 + O(\Delta k^3) \right]$$

(8)

where $Q_0$ corresponds to the unperturbed case, $n = \Delta B_+ - \Delta B_-$, $\Delta k^2$ and $O(\Delta k^3)$ are higher order terms in $\Delta k$. If the signs of the perturbations are reversed, the measured tune becomes:

$$Q' = Q_0 - \frac{L}{4\pi} \left[ \Delta k + \Delta k^2 + O(\Delta k^3) \right]$$

(9)

and the difference $Q = Q' - Q_0$ reads:

$$Q = \frac{L}{2\pi} \left[ \Delta k + O(\Delta k^3) \right]$$

(10)

In Equ. 10, $Q_0$ has disappeared and the second order terms vanish so that the linear expression holds up to third order in $\Delta k$ and provides for an accurate determination of $n$. Simulations made with MAD [3] have shown that the linearity is preserved up to $\Delta k \leq 5 \times 10^{-3}$. Therefore using antisymmetric perturbations leads through Equ. 10 to values of $n$ 20 times more precise than achieved in paragraph 3 since $<\Delta n>$ now reads:

$$<\Delta n> = \frac{2}{\sqrt{\pi}} <\Delta Q>/ L \Delta k$$

This result can be used to get a much higher accuracy in the determination of $B^*_0$ given by Equ. 9, when the value of $\xi$ is obtained by the traditional method of paragraph 3 and the value of $\eta$ is given by Equ. 10.

The accuracies on $B^*_0$ get by both methods are shown in Fig. 2 versus $\eta$, for three values of the Q-measuring resolution $<\Delta Q>$.

5. Conclusions

The three procedures discussed above are fully complementary to provide $B^*$ with the best accuracy. They all rely, in the analytic derivation, on the assumption that there is no beam beam effect and should thus preferably be performed with a single beam.
References


APPENDIX A: Propagation of the beta functions from the waist to the nearest BPM's

Let the waist near a collision point be characterised by \( \beta_m \) and \( c \) (see Fig. 1). The transfer matrix from the waist to the BPM reads:

\[
\text{M} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, \quad \text{with } d = \frac{A1}{A2},
\]

where \( A1, A2 \) are defined in Fig. 1 and \( aij \) the quad matrix elements.

Twiss parameters transform through \( M \) like \[5\]:

\[
\xi = \frac{A2 + A1}{Am}, \quad \eta = \frac{A1 - A2}{Am},
\]

and we can write the betas at the two BPM's as:

\[
\beta = \frac{A2}{A1} \beta + \left[ A (1 + c) + B \right] / A1 - \beta
\]

(33)

with \( A = a11 + a21 \) and \( B = a12 + a22 \). Using again \( \xi = \beta_+ + \beta_- \) and \( \eta = \beta_+ - \beta_- \) the two Equs. A3 become

\[
\eta = -4Ac (\xi + B)/A1
\]

(34)

which reduces to:

\[
\frac{\eta \beta}{\xi} = \frac{-4Ac (\xi + B)/A1}{\beta}
\]

(35)

with \( \xi = \frac{\beta_m + \xi}{\beta_m} \) and:

\[
\xi = \frac{\beta_+ + \beta_-}{\beta_m} + \frac{\xi}{\beta_+ + \beta_-} + \frac{\xi}{\beta_+ - \beta_-}
\]

(36)

Equ. A6 can be solved for \( \beta_m \) after substitution of \( \xi \) by use of Equ. A4 and reads:

\[
\beta_m = \frac{\xi - \left( \xi^2 - \eta^2 - \xi \right)^{1/2}}{4 A2}\]

(37)

The final result for \( \beta_+ \) is:

\[
\beta_+ = \frac{\xi - \left( \xi^2 - \eta^2 - \xi \right)^{1/2}}{4 A2}
\]

(38)

APPENDIX B: Expressions for \( \beta_m, \xi \) and \( \beta_+ \) in terms of \( \beta_m, \xi \) and \( \beta_+ \)

The variation of \( \beta(s) \) inside a focusing quadrupole can be expressed as [6]:

\[
\beta(s) = \frac{\beta_+ + \beta_-}{2} \left( 1 + \cos \omega s \right) + \frac{\beta_+ - \beta_-}{2 \omega^2} \left( 1 - \cos \omega s \right) + \frac{\beta_+}{\omega} \sin \omega s
\]

(39)

where the origin of \( s \) is taken at the quadrupole entrance, \( \beta_0 \) and \( \beta_0' \) are initial conditions, and \( \omega^2 = 4k \).

The average value \( \bar{\beta} \) obtained by integration of Equ. 31 over the quadrupole length, \( L \), is:

\[
\bar{\beta} = \frac{\beta_0 + \beta_0'}{2} \left( 1 + \frac{\sin u}{u} \right) + \frac{\beta_+ + \beta_-}{2 \omega^2} \left( 1 - \cos \omega s - \frac{\beta_+}{u} \right)
\]

where \( u = \omega L \).

Let us now express the initial conditions in terms of the beta function existing around the collision point and given by Equ. 2:

\[
\beta_\xi = \beta_\xi + \frac{\left( A \xi + B \right)}{A1} \left( 1 - \cos \omega s - \frac{\beta_+}{u} \right)
\]

(32)

Note that on either side one gets

\[
\beta_0 = \beta_0 - \frac{\beta_+}{\omega^2} \left( 1 - \cos \omega s - \omega \frac{\beta_+}{u} \right)
\]

The difference and the sum of the average betas measured on both quadrupoles can be expressed, using Equ. 22 and Equ. 23:

\[
\bar{\beta} = \frac{\beta_0 + \beta_0'}{2} \left( 1 + \frac{\sin u}{u} \right) + \frac{\beta_+ + \beta_-}{2 \omega^2} \left( 1 - \cos \omega s - \frac{\beta_+}{u} \right)
\]

which reduces to:

\[
\frac{\eta \beta}{\xi} = \frac{-4Ac (\xi + B)/A1}{\beta}
\]

(34)

with \( P = -2k(1 + \frac{\sin u}{u}) + \frac{4}{\omega^2} \left( \frac{\cos \omega s - 1}{u} \right) \); and also:

\[
\xi = \frac{\beta_+ + \beta_-}{\beta_m} + \frac{\xi}{\beta_+ + \beta_-} + \frac{\xi}{\beta_+ - \beta_-}
\]

(35)

Equ. 36 can be expressed, after substitution of Equ. 34 for \( \xi \), as:

\[
M(1 + \frac{n^{2}/P^2}) \beta_m - \xi \beta_m + N = 0
\]

(37)

with \( M - 1 + \frac{\sin u}{u} \) and:

\[
N = \left( \frac{\sin u}{u} \right) + \frac{4}{\omega^2} \left( \frac{\cos \omega s - 1}{u} \right)
\]

(38)

and the solution for \( \beta_m \) is:

\[
\beta_m = \frac{\xi - \left( \xi^2 - 4 M \left( 1 + \frac{n^2}{P^2} \right)^{1/2} \right)^{1/2}}{2 M}
\]

(39)

Since \( \beta_m = \beta_+ + \frac{\xi}{\beta_m} \), using Equ. 23 one has:

\[
\beta_+ = \beta_m \left( 1 + \frac{n^2}{P^2} \right)
\]

(40)

so that finally:

\[
\beta_+ = \frac{\xi - \left( \xi^2 - 4 M \left( 1 + \frac{n^2}{P^2} \right)^{1/2} \right)^{1/2}}{2 M}
\]

(41)

Since \( \xi^2 >> 4 M \), Equ. 35 and 36 can be simplified by expanding the square root to first order and one gets:

\[
\beta_+ = \frac{N}{\xi} \left( 1 + \frac{n^2}{P^2} \right)
\]

(42)

\[
\beta_m = \frac{N}{\xi}
\]

(43)

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