THE DRIVEN PENDULUM, AND E778 TUNE MODULATION

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Introduction - the driven differential pendulum

The angle θ that a rigid gravity pendulum driven by a sinusoidal torque makes with the vertical satisfies

$$\frac{d^2\theta}{dt^2} + (2\pi Q_I)^2 \sin(\theta) = D \cos(2\pi Q_M t) \qquad [1]$$

The undriven average tune Q_{free} is Q_I for small angle oscillations, but decreases to zero as the amplitude is increased to π. In general the driven system behavior depends on D and Q_M, the amplitude and tune of the driving torque. One of four dynamical phases which exist in the (D,Q_M) parameter space exhibits massive chaos. The tune modulation part of the E778 nonlinear dynamics experiment[1] tests the validity of the theoretical models which describe these phases, in an accelerator representation of this extremely simple system. This paper describes the models, and presents some of the experimental results. If the motion is not chaotic, but periodic, the general form of the solution to [1] is a double Fourier series expansion in Q_{free} and Q_M. However, the experimental observable in E778 is the signal observed at a beam position monitor (BPM), which is averaged over the free phases of a distribution of particles. Hence the interesting solutions are expanded in only the driving tune. There is a family of possible solutions, labeled by the integer k,

$$\theta = k\, 2\pi (Q_M t) + \sum_{n=1}^{\infty} c_n \cos(n\, 2\pi\, Q_M t) \qquad [2]$$

where the coefficients c_n are functions of D, Q_M, and Q_I. The pendulum rotates exactly k complete turns in one modulation period. Depending on the value of (D,Q_M), the k-th solution is stable or unstable to small free oscillations.

In the tune modulation system, the solutions correspond to a family of resonance sidebands. Determining the stability of the sidebands is a central problem for the theory. Just as longitudinal motion in RF buckets is an interesting representation of the universally recurring *standard map*, the effect of tune modulation on accelerator resonances is interesting as a representation of the driven differential pendulum. In contrast to the longitudinal problem, tune modulation plays an important role in limiting the performance of proton storages rings like the SSC [2-6].

Five unperturbed islands - the single resonance Hamiltonian, H_5

First, it is necessary to develop a vocabulary for the one dimensional motion near the 2/5 resonance which is observed in E778. The convenient action angle variables, (J,ϕ), are related to the horizontal displacement and angle, (X,X'), at a fixed point, by

$$\begin{pmatrix} (2J)^{1/2}\sin(\phi) \\ (2J)^{1/2}\cos(\phi) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\beta(s)}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} X \\ X' \end{pmatrix} \qquad [3]$$

where α and β are Twiss parameters. That is, J behaves much like the betatron amplitude, while ϕ is the betatron phase of a trajectory under study. If the base tune Q_0 is close to 2/5, motion is well approximated by the the *single resonance Hamiltonian* [7]

$$H_5 = 2\pi (Q_0 - \tfrac{2}{5}) J + V_{40} J^2 - V_{55} J^{5/2} \cos(5\phi + \phi_5) \qquad [4]$$

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This is just shorthand for the five-turn difference equations of motion

$$\begin{pmatrix} J \\ \phi \end{pmatrix}_{t+5} = \begin{pmatrix} J \\ \phi \end{pmatrix}_t + 5 \begin{pmatrix} -\frac{\partial H_5}{\partial \phi} \\ \frac{\partial H_5}{\partial J} \end{pmatrix}_t \qquad [5]$$

demonstrating that t is an integer divisible by five.

The meaning of the three terms in H_5 becomes clear when the partial differentiations in [5] are performed. The first term leads to a small constant net phase advance of $10\pi\,(Q_0 - 2/5)$, but the second term causes an advance of $10\, V_{40} J$, linearly proportional to the action. That is, the amplitude dependent tune is

$$Q(J) = Q_0 + \left(\frac{V_{40}}{\pi}\right) J \qquad [6]$$

consistent with the experimentally observed variation

$$Q = Q_0 - 7 \times 10^{-4}\, a^2 \qquad [7]$$

when $Q_0 \approx 0.42$, and the amplitude a is in millimeters. The resonance action J_I is found by solving [6] with $Q(J_I) = 2/5$. It is now convenient to rewrite H_5 as an expansion around J_I,

$$H_5 = \frac{1}{2} U I^2 - V \cos(5\phi) \qquad [8]$$

where the value of ϕ_5 has been arbitrarily set to zero, and

$$I = J - J_I, \quad U = 2V_{40}, \quad V = V_{55} J_I^{5/2} \qquad [9]$$

Substitution of [8] into [5] (with J replaced by I) shows that $(I,\phi) = (0,0)$ is a fixed point – a trajectory launched there is stationary. In a region close enough to $I = 0$, then, H_5 may be considered as representing differential equations of motion, continuous in t, which agree well with the difference motion whenever t divides by five. In this approximation

$$\begin{pmatrix} \frac{dI}{dt} \\ \frac{d\phi}{dt} \end{pmatrix} = \begin{pmatrix} -\frac{\partial H_5}{\partial \phi} \\ \frac{\partial H_5}{\partial I} \end{pmatrix} = \begin{pmatrix} -5V\sin(5\phi) \\ U\ I \end{pmatrix} \qquad [10]$$

or, in terms of a single second order equation of motion,

$$\frac{d^2\phi}{dt^2} + 5\, VU \sin(5\phi) = 0 \qquad [11]$$

which is very reminiscent of [1]. The small amplitude motion is

$$\begin{pmatrix} I \\ \phi \end{pmatrix} = \phi_0 \begin{pmatrix} 5\left(\frac{V}{U}\right)^{1/2} \sin(2\pi Q_I t) \\ \cos(2\pi Q_I t) \end{pmatrix} \qquad [12]$$

where the *island tune* Q_I is given by

$$Q_I = \frac{5}{2\pi} (U\, V)^{1/2} \qquad [13]$$

Figure 1 shows the presence of five resonance islands (in normalized X,X' space) under simulated E778 experimental conditions.

The apparently continuous sequence of dots which follow a single trajectory in Figure 1 are represented in the theory by contours of H_5, which describe a parabolic valley along the I-axis, modulated along the ϕ-axis by the $\cos(5\phi)$ term. This leads to five local minima separated by

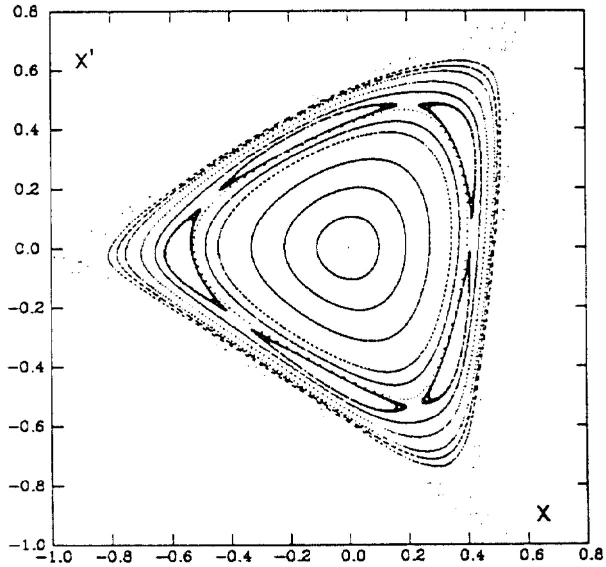

Figure 1. Surface of section plot from a simulation of E778, with islands at an amplitude of about 5 mm ($\beta \approx 100$ meters).

five saddle points, corresponding to five stable and five unstable fixed points. The island half width is found by equating the saddle point elevation with the elevation at $\phi = 0$,

$$I_W = 2\left(\frac{V}{U}\right)^{1/2} \qquad [14]$$

Figure 2 shows the BPM signal observed when a proton beam is kicked into a location which overlaps a resonance island. After an initial gaussian decay, due to the tune spread across the beam, a *persistent signal* remains. The ratio of the persistent signal to the kick amplitude approximately equals the fraction of beam trapped in the island. This offers an experimental means to measure I_W [1].

<u>Five islands with tune modulation</u>

If a set of quadrupoles is perturbed by a small sinusoidal current, the base tune is modulated according to

$$Q_0 = Q_{00} + q \sin(2\pi Q_M t) \qquad [15]$$

where q and Q_M are the tune modulation amplitude and tune. Tune modulation is included in the resonance Hamiltonian by adding a single term to equation [8], giving

$$H_5 = 2\pi q \sin(2\pi Q_M t)\, I + \frac{1}{2} U I^2 - V \cos(5\phi) \qquad [16]$$

Now H_5 is time dependent, so it is no longer conserved, and it is not possible to picture the motion by plotting its contours. The two first order equations of motion are

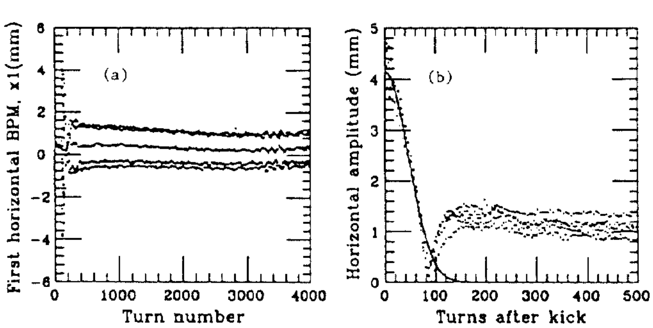

Figure 2. Typical data from the E778 experiment, showing both Gaussian decoherence and a persistent signal.

$$\begin{pmatrix} \frac{dI}{dt} \\ \frac{d\phi}{dt} \end{pmatrix} = \begin{pmatrix} -5V \sin(5\phi) \\ 2\pi q \sin(2\pi Q_M t) + UI \end{pmatrix} \qquad [17]$$

while the single second order differential equation in ϕ,

$$\frac{d^2\phi}{dt^2} + (2\pi Q_I)^2 \frac{\sin(5\phi)}{5} = (2\pi)^2 q Q_M \cos(2\pi Q_M t) \qquad [18]$$

is explicitly analogous to the driven pendulum equation, [1]. The family of solutions periodic in Q_M is now

$$5\phi = k\, 2\pi (Q_M t) + \sum_{n=1}^{\infty} c_n \cos(n\, 2\pi Q_M t) \qquad [19]$$

identical to [2], with 5ϕ replacing θ. The tune of the k-th solution

$$Q_k = \frac{2}{5} + \frac{1}{2\pi} < \frac{d\phi}{dt} > = \frac{2}{5} + k\frac{Q_M}{5} \qquad [20]$$

demonstrates the possibility of a family of stable sidebands. Each sideband has five islands, at an action I_k given by $Q(I_k) = Q_k$, so

$$I_k = k\frac{2\pi Q_M}{5U} \qquad [21]$$

If solution k is stable, then persistent signals should be observed at Q_k when a beam is kicked on top of one of the sideband islands.

The small angle $k=0$ solution is illuminating. It is given, for all Q_M, by

$$I = -\frac{Q_I^2}{Q_I^2 - Q_M^2} \frac{2\pi q}{U} \sin(2\pi Q_M t) \qquad [22]$$

$$\phi = \frac{Q_M^2}{Q_I^2 - Q_M^2} \frac{q}{Q_M} \cos(2\pi Q_M t) \qquad [23]$$

At constant q, the action amplitude goes to $(2\pi q)/U$ for small Q_M and to zero for large Q_M, while the phase amplitude goes to zero and q/Q_M for slow and fast modulation. This explains the "amplitude modulation" and "phase modulation" labels in Figure 3, which shows the four dynamical phases in the (q, Q_M) parameter space. The solid line with the pole in the figure,

$$\left|\frac{q Q_M}{Q_I^2 - Q_M^2}\right| = \frac{1}{5} \qquad [24]$$

is the small angle boundary below which [23] applies. Rigorous analysis (below) shows that this is the stability boundary for the $k=0$ solution in the slow modulation limit. Rigorous analysis in the large Q_M limit shows that, although the $k \neq 0$ sideband islands are stable, their size is insignificant below the small angle boundary. It is necessary to rely on numerical iterative solutions and simulations when $Q_M \approx Q_I$. Simulations and iterative solutions of [19] in [18] appear to agree that below Q_I [24] marks the limit of stability of the $k=0$ fundamental. Just above Q_I the $k=0$ solution is stable for all values of q. Thus, the small angle boundary has different physical implications above and below the pole. The iteration scheme also indicates that none of the $k \neq 0$ sideband solutions are stable below Q_I, but that all of the solutions are stable above it, with the possible exception of a small region near the pole.

<u>Slow modulation,</u> $Q_M << Q_I$

If the tune changes adiabatically slowly, it is reasonable to approximate the rate of change as constant, at its most stringent maximum. The Hamiltonian in equation [16] then becomes

$$H_5 = (2\pi)^2 q Q_M t\, I + \frac{1}{2} U I^2 - V \cos(5\phi) \qquad [25]$$

This is still time dependent, but now a canonical coordinate transformation is possible, from (I, ϕ, H_5) to $(\bar{I}, \bar{\phi}, \bar{H}_5)$, making $\bar{H}_5$ time independent. Specifically, the generating function

$$F_3(I, \bar{\phi}, t) = -I\bar{\phi} - \varepsilon t \bar{\phi} - \frac{1}{6}\varepsilon^2 t^3 \qquad [26]$$

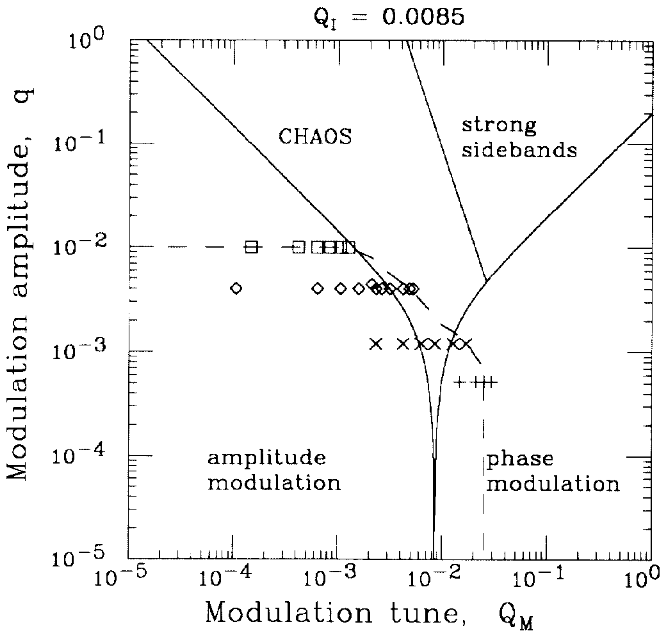

Figure 3. The four dynamical phases in the tune modulation parameter space (Q_M, q), for a typical $Q_I = 0.0085$. The region accessible to the E778 experiment is shown by the dashed line.

$$\varepsilon = \frac{(2\pi)^2 qQ_M}{U} = 25\,V\,\frac{q\,Q_M}{Q_I^2}$$

gives, by its definition[8],

$$\bar{I} \equiv -\frac{\partial F_3}{\partial \bar{\phi}} = I + \varepsilon t, \qquad \phi \equiv -\frac{\partial F_3}{\partial \bar{I}} = \bar{\phi} \qquad [27]$$

$$\bar{H}_5 \equiv H_5 + \frac{\partial F_3}{\partial t} = \frac{1}{2}U\bar{I}^2 - V\cos(5\bar{\phi}) - \varepsilon\,\bar{\phi} \qquad [28]$$

While the old and new phases are identical, the new action drifts relative to the old action at a constant speed. The $\varepsilon\bar{\phi}$ term in the new Hamiltonian has serious consequences for the stability of the k=0 fundamental island chain. Pictorially, this non-periodic term corresponds to a constant slope along the quadratic Hamiltonian valley. If this slope is steep enough, there are no longer any local minima. There are no stable islands at all if $|\varepsilon| > 5V$, or

$$\frac{qQ_M}{Q_I^2} > \frac{1}{5} \qquad [29]$$

In the slow limit, this corresponds to the small angle boundary [24].

<u>Fast modulation.</u> $Q_M >> Q_I$

In this region, a time independent Hamiltonian is found by first applying the generating function

$$F_3(I,\bar{\phi},t) = -I\bar{\phi} - \frac{q}{Q_M}\cos(2\pi Q_M t)\,I \qquad [30]$$

which gives

$$\bar{H}_5 = \frac{1}{2}U\bar{I}^2 - V\cos\left(5\bar{\phi} + \frac{5q}{Q_M}\cos(2\pi Q_M t)\right) \qquad [31]$$

$$= \frac{1}{2}U\bar{I}^2 - V\sum_i J_i\left(\frac{5q}{Q_M}\right)\cos(5\bar{\phi} + i\,2\pi Q_M t)$$

where the J_i are integer order Bessel functions. The Hamiltonian is made time independent by concentrating on the vicinity of the k-th sideband, near the action I_k, and averaging the sum in [31] over one modulation period, to give

$$H_{5k} = \frac{1}{2}U(I - I_k)^2 - V J_k\left(\frac{5q}{Q_M}\right)\cos(5\phi) \qquad [32]$$

(without overbars, and with a shift of origin). This differs from the simple form [8] by the J_k factor, which determines whether or not the k-th sideband is significant. As a rule of thumb, $J_k(A) \approx 0$ if $|A| < |k|$, so sideband k is significant if

$$q > |k|\frac{Q_M}{5} \qquad [33]$$

The right hand side of [33] is the separation of the sideband tune from the fundamental resonance tune, corresponding to the sensible physical condition that, for the resonance to be felt near an action I_k, the tune must be modulated far enough to cross the fundamental.

The sidebands are isolated from each other if their separation in action, given by [21], is larger than the sideband width, given by [14] with $J_k V$ replacing V. Chaos appears if the sidebands overlap, spanning the action range of sidebands of significant size. It is easily shown[3-6] by further approximating J_k, that sideband overlap is expected if [33] is true, and if

$$Q_M^{3/4}(5\,q)^{1/4} < \frac{4}{\pi^{1/4}}Q_I \qquad [34]$$

This boundary is the nearly vertical solid line in Figure 3. Because of the casual Bessel function approximation, sidebands overlap a little earlier or later than this semi-quantitative condition suggests.

Figure 4 shows the effect that entering the chaotic region has on the measured lifetime of E778 persistent signals. A decay time of 47,000 turns is approximately one second in the Tevatron. The dramatic increase in the decay rate when the boundary is crossed is consistent with a fit to the data of $Q_I = 0.0085$. This method of measuring Q_I is time intensive, since each data point corresponds to a two minute injection cycle of the Tevatron. Other measurements of Q_I have been made, by Fourier analysis of the phase, and of BPM signal sidebands, and will be reported elsewhere. Measurement of Q_I in a single machine cycle is hopefully expected in the near future, opening up the possibility of a rapid comprehensive scan of resonances across a relatively wide range of tunes.

<u>Acknowledgements</u>

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<u>References</u>

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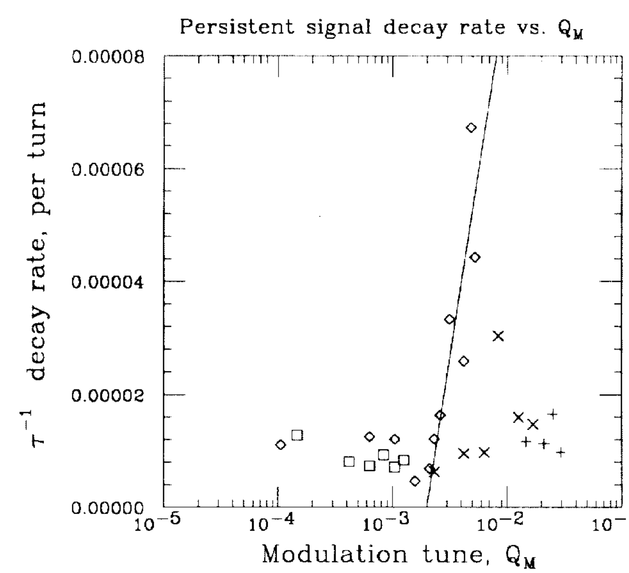

Figure 4. The effect of tune modulation on the persistent signal, for the data plotted in Figure 3. The decay rate is significantly larger in the chaotic region.