ROBINSON INSTABILITY AND LONGITUDINAL MODE COUPLING*

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Abstract

Robinson's stability conditions are rederived by using Sacherer's integral equation. The relation between the maximum stable beam current and the cavity detuning is reproduced with a correction to account for the equilibrium phase-space distribution of beam particles. The coupling between the longitudinal dipole and quadrupole modes is also studied. It is found that the coupling modifies Robinson's stability conditions significantly in the small cavity detuning region.

Introduction

More than thirty years ago, K. W. Robinson derived the stability conditions for a beam-cavity interaction system in a synchrotron or a storage ring.[1] For machines operated below transition ($\gamma < \gamma_t$), these stability conditions are

$$0 < \sin(2\phi_y) \quad , \tag{1}$$

and

$$I < \frac{V_m \cos\psi_s}{\mathcal{R}\sin(2\phi_y)} \quad , \tag{2}$$

where $\phi_y = \tan^{-1}[2Q(\omega_R - \omega_{RF})/\omega_R]$ is the radio-frequency (RF) detuning angle, Q is the quality factor of the cavity, ω_R is the resonant frequency of the cavity, ω_{RF} is the frequency of the applied RF power, I is the averaged beam current, V_m is the maximum voltage on the cavity, ψ_s is the synchronous angle between the beam current and the voltage of the cavity, and $\mathcal{R}$ is the shunt resistance of the cavity. For machines operated above transition ($\gamma > \gamma_t$), the relational sign in Inequality (1) needs to be reversed.

After Robinson's work on the stability of a beam-cavity system, the same or similar problems were examined by using different approaches and formalisms other than the equivalent circuit model originally used by Robinson, including the use of the more elaborate formalism of the Vlasov equation.[2,3] One of the advantages of using the Vlasov equation over the equivalent circuit model is that higher synchrotron sidebands (or harmonics) are included in the formalism in a natural way. The mode originally examined by Robinson was identified as the dipole mode in Sacherer's formalism of the bunched-beam longitudinal instability. The next sidebands correspond to the quadrupole mode perturbation in the phase space of the beam particles' distribution. For narrow-band resonators, only the first few synchrotron sidebands are important.

Among the publications using the Vlasov equation, only Inequality (1) has been reproduced, while an explicit rederivation of Inequality (2) by this technique still has not been seen in the literature. Recently, the consistency between the use of the equivalent circuit model and the use of Sacherer's integral equation for examining the stability of the beam-cavity system has been questioned.[4] The purpose of this paper is twofold. First, we will rederive Robinson's stability limits, including Inequality (2), by using Sacherer's integral equation. Second, we will apply the same technique to examine the coupling effect between the dipole and quadrupole modes. For simplicity, only the case of $\gamma < \gamma_t$ will be considered because the $\gamma > \gamma_t$ case can be treated by the same procedure.

Robinson's Stability Criteria

Consider the case of M equally spaced particle bunches circulating with angular revolution frequency Ω_o in a circular accelerator or a storage ring. We choose a coordinate system such that the z-axis is along the direction of particle propagation. For the purpose of our discussion, we can neglect the repulsive Coulomb force between particles in the equilibrium state, and we assume that the RF focusing force can be approximated by a linear function, $-\omega_s^2 z$, where the synchrotron frequency ω_s is defined according to

$$\omega_s^2 = -\frac{q\eta h V_m \cos\psi_s}{2\pi m_o \gamma R^2} \quad , \tag{3}$$

where q and m_o are the charge and the rest mass of a beam particle, respectively; h is the RF harmonic number; R is the effective machine radius; and $\eta = (1/\gamma_t^2) - (1/\gamma^2)$. We also assume that all bunches have the same equilibrium particle distribution described by a distribution function $f_o(z, v_z)$ in the phase space, where z and v_z are the distance and the relative velocity with respect to the reference particle at the bunch center. Neglecting the relative motion among bunches, the following Sacherer's integral equation for the *coherent modes* of longitudinal perturbations can be derived from the linearized Vlasov equation:[3]

$$(\omega - l\omega_s)R_l(r) = \frac{q^2 M\eta\Omega_o l}{2\pi m_o \gamma r}\left(\frac{df_o}{dr}\right)\sum_{n,m}\frac{Z_n(\omega + n\Omega_o)}{n} i^{m-l-1} \times J_l\left(\frac{nr}{R}\right)\int_0^\infty R_m(r')J_m\left(\frac{nr'}{R}\right)r'dr' \quad , \tag{4}$$

where $r = \sqrt{z^2 + (v_z/\omega_s)^2}$ is the amplitude of the synchrotron oscillation of a beam particle, ω is the frequency to be solved, l and m are integers designating the azimuthal harmonics of the perturbation in the phase space of an individual bunch, $R_l(r)$ is the Fourier content of the lth harmonic of the perturbation in the phase space, n is the azimuthal harmonic number around the ring, $Z_n(\omega + n\Omega_o)$ is the longitudinal impedance at the frequency $\omega + n\Omega_o$, and $J_k(x)$ is the kth order Bessel function of the first kind with argument x. In arriving at Eq. (4) we have assumed that the equilibrium distribution function depends on r only, and we have neglected the *time-of-flight* effect.[5]

For a narrow-band resonator impedance, we need only to consider those modes with frequencies very close to the resonant frequency of the cavity. Thus, we need only to consider the cases of $n = \pm h$, $l = \pm 1$, and $m = \pm 1$ in Eq. (4). One therefore derives the following two equations:

$$(\omega - \omega_s)R_1(r) = -\frac{iA}{r}\left(\frac{df_o}{dr}\right)J_1\left(\frac{hr}{R}\right)\mathcal{Z}(\omega)(\Gamma_1 - \Gamma_{-1}) \quad , \tag{5}$$

and

$$(\omega + \omega_s)R_{-1}(r) = -\frac{iA}{r}\left(\frac{df_o}{dr}\right)J_{-1}\left(\frac{hr}{R}\right)\mathcal{Z}(\omega)(\Gamma_1 - \Gamma_{-1}) \quad , \tag{6}$$

where $A = q^2 M\eta\Omega_o/(2\pi h m_o \gamma)$,

$$\mathcal{Z}(\omega) = Z_{+h}(\omega + h\Omega_o) - Z_{-h}(\omega - h\Omega_o) \quad , \tag{7}$$

and

$$\Gamma_m = \int_0^\infty R_m(r)J_m\left(\frac{hr}{R}\right)r\,dr \quad . \tag{8}$$

Multiplying both sides of Eqs. (5) and (6) by $rJ_1(hr/R)$ and integrating over r, we obtain two linear equations of Γ_1 and Γ_{-1}. For nontrivial solutions of Γ_1 and Γ_{-1}, the determinant of the coefficients of these quantities must be equal to zero. We therefore have following dispersion relation:

$$s^2 + \omega_s^2 - 2i\vartheta_1\omega_s\mathcal{Z}(s) = 0 \quad , \tag{9}$$

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where

$$\vartheta_m = A\int_0^\infty \left[J_m\left(\frac{hr}{R}\right)\right]^2 \frac{df_o}{dr}dr \quad . \tag{10}$$

In obtaining Eq. (9), we have made a change of variable of $s = -i\omega$ in order to make it easy to apply Routh's criterion[6] for investigating the roots of Eq. (9). If the real parts of all the roots are equal to or less than zero, then the system is stable; otherwise there is an instability.

For a narrow-band resonator impedance, we can use the following approximations:

$$Z_{\pm h}(\omega \pm h\Omega_o) \approx \mathcal{R}\Big/\left(1 + \frac{s}{\alpha} \pm i\tan\phi_y\right) \quad , \tag{11}$$

where $\alpha = \omega_R/(2Q)$. Substituting the above impedances into Eq. (9), yields

$$s^4 + 2\alpha s^3 + (\omega_s^2 + \alpha^2\sec^2\phi_y)s^2 + 2\alpha\omega_s^2 s + \alpha^2\omega_s^2\left(\sec^2\phi_y - \frac{F_1 I\mathcal{R}\tan\phi_y}{V_m\cos\psi_s}\right) = 0 \quad , \tag{12}$$

where the *reduced form factor* F_m is defined as

$$F_m = -\frac{4R^2}{h^2}\left\{\int_0^\infty \frac{df_o}{dr}\left[J_m\left(\frac{hr}{R}\right)\right]^2 dr\right\}\Big/\left[\int_0^\infty f_o(r)rdr\right] \quad . \tag{13}$$

The conditions for stability, by Routh's criterion[6], are

$$\sin(2\phi_y) < \frac{2V_m\cos\psi_s}{F_1 I\mathcal{R}} \quad . \tag{14}$$

and

$$\tan\phi_y > 0 \quad . \tag{15}$$

We note that, except for the factors $2/F_1$, Eq. (12) is the same as the dispersion relation obtained from the equivalent circuit formalism, and the stability conditions derived here are the same as the stability conditions in Inequalities (1) and (2).

Dipole-Quadrupole Mode Coupling

If the quadrupole modes are included, we need to consider the cases of $n = \pm h$, $l = \pm 1$, $l = \pm 2, m = \pm 1$, and $m = \pm 2$. Using the similar procedure in deriving Eq. (12), we obtain the dispersion relation

$$s^6 + 2\alpha\, s^5 + (5\omega_s^2 + \alpha^2\sec^2\phi_y)\, s^4 + 10\alpha\omega_s^2\, s^3 + [4\omega_s^4 + 5\omega_s^2\alpha^2\sec^2\phi_y - 4\mathcal{R}\omega_s\alpha^2(\vartheta_1 + 4\vartheta_2)\tan\phi_y]s^2 + 8\alpha\omega_s^4 s + 4\alpha^2\omega_s^2[\omega_s^2\sec^2\phi_y + 16\mathcal{R}^2\vartheta_1\vartheta_2 - 4\mathcal{R}\omega_s(\vartheta_1 + \vartheta_2)\tan\phi_y] = 0 \quad . \tag{16}$$

The conditions for stability, by Routh's criterion, are

$$\tan\phi_y > \frac{16\mathcal{R}\vartheta_2}{3\omega_s} \quad , \tag{17}$$

and

$$\left(\tan\phi_y - \frac{4\mathcal{R}\vartheta_1}{\omega_s}\right)\left(\tan\phi_y - \frac{4\mathcal{R}\vartheta_2}{\omega_s}\right) + 1 > 0 \quad . \tag{18}$$

Combining Inequalities (17) and (18), we can derive the following stability conditions:

[1] If $\sin\phi_y \le [2\sqrt{\xi}/(1+\xi)]$, where ξ is defined in Eq. (22) below, then

$$I < \frac{3\tan\phi_y V_m\cos\psi_s}{4F_2\mathcal{R}} = I_d T \quad , \tag{19}$$

where

$$I_d = \frac{2V_m\cos\psi_s}{F_1\mathcal{R}\sin(2\phi_y)} \tag{20}$$

is the maximum stable current for the dipole mode when there is no coupling with the quadrupole mode,

$$T = \frac{3\sin^2\phi_y}{4\xi} \quad , \tag{21}$$

and

$$\xi = \frac{F_2}{F_1} \quad . \tag{22}$$

[2] For $[2\sqrt{\xi}/(1+\xi)] \le \sin\phi_y \le [4\sqrt{\xi/(3+12\xi)}]$, there are two subcases:

(a) for $1/2 \le \xi$:

$$I < I_d T \quad , \tag{23}$$

(b) for $\xi \le 1/2$, there are two stable regions:

$$I < I_d T_- \quad , \tag{24}$$

and

$$I_d T_+ < I < I_d T \quad , \tag{25}$$

where

$$T_\pm = \sin^2\phi_y\left[1 + \xi \pm \sqrt{(1-\xi)^2 - 4\xi\cot^2\phi_y}\right]\Big/(2\xi) \quad . \tag{26}$$

[3] If $[4\sqrt{\xi/(3+12\xi)}] \le \sin\phi_y$, then

$$I < I_d T_- \quad . \tag{27}$$

The condition described in Inequality (27) corresponds to the limit of the beam current in the majority of practical cases. It can be shown easily that when $\xi\cot^2\phi_y$ is very small, the conditions in Inequality (27) reduce to Robinson's stability conditions.

Numerical Results and Discussions

The reduced form factor F_1 and the parameter ξ are plotted in Fig. 1 and Fig. 2, respectively, as functions of $g = (hL/2R)$ for some equilibrium distribution functions. As we can see in Fig. 1, the values of the reduced form factor F_1 for different distribution functions all converge to the value of two when the value of g or the bunch length approaches zero. As the bunch length increases, the value of F_1 goes down for all distribution functions, and the separations between curves tend to increase. Therefore, for long bunches, the maximum stable current is actually higher than that given in Inequality (2). The plot in Fig. 2 indicates that except for long bunches, the value of ξ is usually much smaller than one, and the values of the ξ for different equilibrium functions all converge to zero when g goes to zero.

The quantities T_-, T_+, and T represent the ratios between the maximum stable beam current with quadrupole–dipole mode coupling and the maximum stable beam current

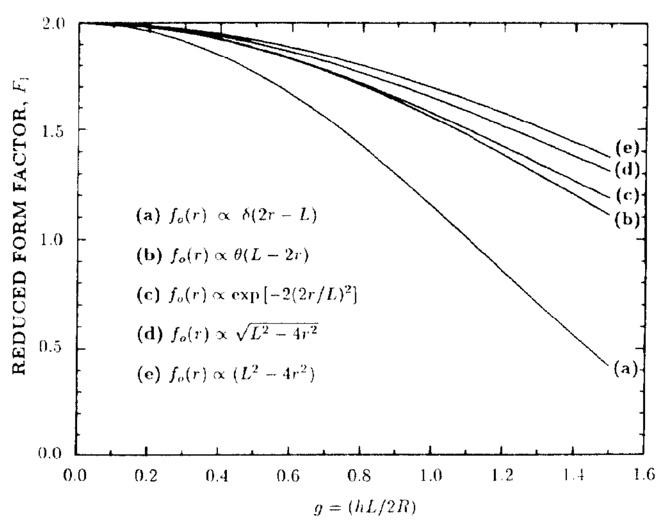

Fig. 1. The reduced form factor F_1 as a function of the parameter g for the distribution functions.

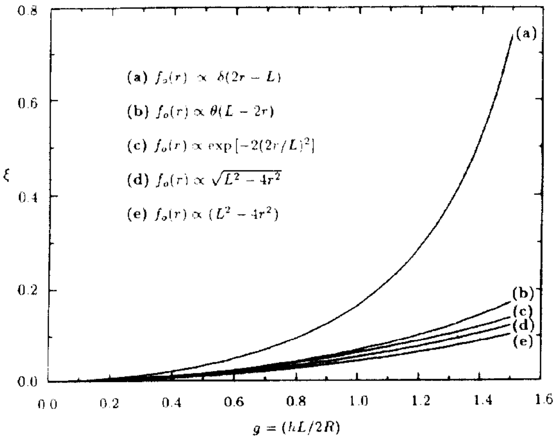

Fig. 2. The quantity ξ as a function of g for different distribution functions.

dipole mode coupling and the maximum stable beam current for dipole mode perturbation only. We note that only T can have values less than one, and this happens only when $3\sin^2\phi_y < 4\xi$. We also note that the maximum values of T_+ and T can be as large as four. Thus, if the region of $3\sin^2\phi_y < 4\xi$ can be avoided, then the maximum stable current actually can be higher than that predicted by Robinson's criteria.

The numerical values of T_-, T_+, and T are shown in Figs. 3 to 5 as functions of ξ for various values of the detuning angle

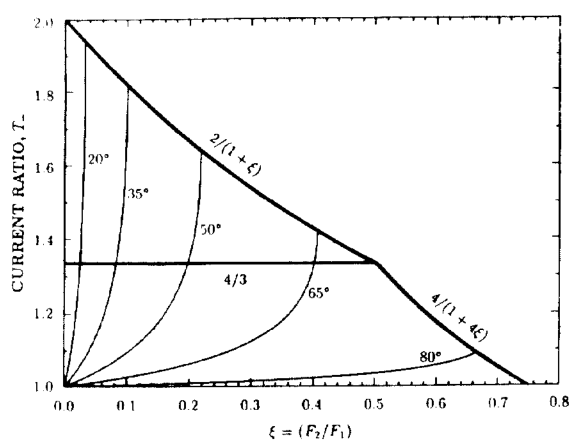

Fig. 3. The current ratio T_- as a function of ξ for various values of the detuning angle ϕ_y. Thick curves correspond to the extreme values of T_-.

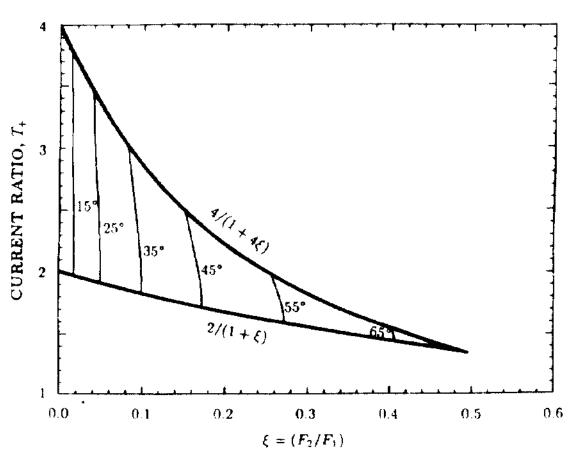

Fig. 4. The current ratio T_+ as a function of ξ for various values of the detuning angle ϕ_y. Thick curves correspond to the extreme values of T_+.

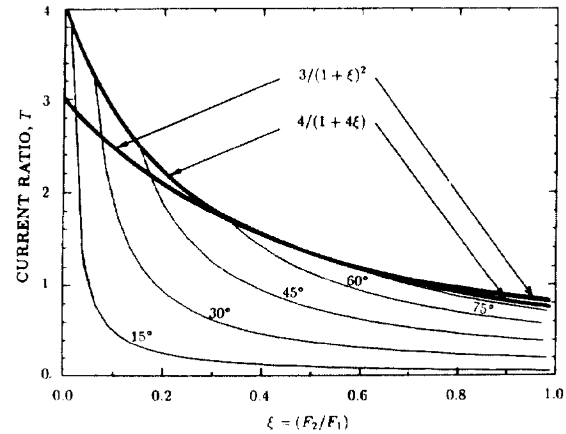

Fig. 5. The current ratio T as a function of ξ for various values of the detuning angle ϕ_y. Thick curves correspond to the extreme values of T.

ϕ_y. As can be seen in the figures, the highest stable currents are always in the regions of small ξ and small cavity detuning. We remind the readers that cavity detuning is necessary because finite RF power is used for acceleration or bunching, and because the required phase between the total voltage of the cavity and the beam current is maintained by compensating the beam loading with the RF power. Thus, in most situations, small cavity detuning implies higher cavity voltage, and the power consumption is higher than minimally required.

Conclusion

We have rederived Robinson's stability criteria by using the Vlasov equation. The results derived from the Vlasov equation also include the form factor correction for different equilibrium distributions in the phase space. We have also studied the coupling between the dipole and quadrupole modes with a resonator impedance. It is shown that for small cavity detuning, the maximal stable beam current can be higher than Robinson's limit without coupling. When $3\sin^2\phi_y < 4\xi$, the threshold current is lower than Robinson's limit and the threshold current drops to zero like $\tan\phi_y/\xi$.

References

1. K. W. Robinson, "Radio Frequency Acceleration II," Report No. CEA-11, Cambridge Electron Accelerator, Cambridge, Massachusetts (September 1956).

2. T. F. Wang, "Robinson Instability Revisited," Accelerator Theory Note, AT-6:ATN-88-21, Accelerator Theory and Simulation Group, Accelerator Technology Division, Los Alamos National Laboratory, November 1988. (Additional references are listed in this note.)

3. F. J. Sacherer, "Bunch Lengthening and Microwave Instability," CERN Internal Report CERN/PS/BR 77-6 (1977).

4. G. H. Rees, "Another Look at Coherent Longitudinal Instability of Bunched Beams," Proc. First European Particle Accelerator Conference, Rome, Italy, June 7-11, 1988, to be published.

5. T. Suzuki, "Hamiltonian Formulation for Synchrotron Oscillations and Sacherer's Integral Equation," Particle Accelerators, Vol. 12, p. 237 (1982).

6. E. J. Routh, *A Treatise on the Dynamics of a System of Rigid Bodies* (Macmillan, London, 1877).