RELATION BETWEEN FIELD ENERGY AND RMS EMITTANCE IN INTENSE PARTICLE BEAMS

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Summary

An equation is presented for continuous beams with azimuthal symmetry and continuous linear focusing, which expresses a relationship between the rate of change for squared rms emittance and the rate of change for a quantity we call the nonlinear field energy. The nonlinear field energy depends on the shape of the charge distribution and corresponds to the residual field energy possessed by beams with nonuniform charge distributions. The equation can be integrated for the case of an rms matched beam to yield a formula for space-charge-induced emittance growth that we have tested numerically for a variety of initial distributions. The results provide a framework for discussing the scaling of rms emittance growth and an explanation for the well-established lower limit on output emittance.

Introduction

A quantitative understanding of emittance growth induced by space charge is of great importance for the design of high-current linear accelerators and beam transport systems. The first systematic numerical study of emittance growth for linac beams was done by Chasman, and more recent studies were reported by Jameson and Mills. An interest in beam instabilities of periodic transport channels was stimulated by heavy-ion fusion requirements, and led to work recently reported by Hofmann, Laslett, Smith, and Haber. The mechanism of equipartitioning and emittance growth in linacs was studied by Hofmann and Bozil and also by Jameson. Reviews of this work have been presented by Jameson and Hofmann.

In spite of these efforts, basic phenomena observed both in numerical studies and in unpublished experimental data have remained unexplained, and no clear explanation of many of the important effects has emerged. Until recently, little attention has been given to a comparison of rms emittance growth for different initial distributions. Recently, numerical studies were published by Struckmeier, Klubude, and Reiser for a continuous beam in a periodic channel, which led them to observe that some distributions experience a rapid initial emittance growth that increases with beam intensity. These authors hypothesized that the observed emittance growth was associated with a homogenization of the charge density and resulted in a conversion of space-charge field energy into transverse particle energy. They suggested that transverse energy conservation could be used to obtain a useful formula for emittance growth. Two formulas were presented, which were interpreted as upper and lower bounds on the emittance growth.

The idea of space-charge field energy as a useful quantity has been suggested before. In 1970, Gluckstern, et al. used electric-field energy comparisons to investigate the relative stability of different stationary distributions. Electric-field energy relationships were considered in more detail by Lapostolle, in his important work in 1971, where, together with B. Min, he derived a generalized rms envelope equation. Thus, stimulated by the suggestions of Struckmeier, Klubude, and Reiser, we have attempted to extend the initial work of Lapostolle and Sacherer, to investigate further the relationship between space-charge field energy and rms emittance for continuous beams in continuous focusing channels.

RMS Emittance and Nonlinear Field Energy

We consider the problem of a continuous, azimuthally symmetric beam that propagates in the +z-direction at constant velocity v. We assume that the beam is confined radially by a continuous, linear, external focusing force and that the paraxial approximation is valid in which all particles have the same longitudinal velocity \( v \gg v_t \), where \( v_t \) is the transverse velocity component. We consider the characteristics of a steady-state solution, where the charge density, current density, and fields have no explicit dependence on time. We allow for initial phase-space distributions that are not necessarily stationary, and in general, we expect that the charge density \( n(r,z) \) will evolve from an initial state at \( z = 0 \) to some final state, which may be stationary or independent of \( z \).

Maxwell's equations can be written using nonzero self-field components \( F_x, F_y, \) and \( \partial_\phi \), and current density components \( j_x \) and \( j_y \). A solution is easily obtained in the approximation that \( \partial \phi / \partial z \ll \rho / e_0 \), where \( e_0 \) is the free-space permittivity, which leads to

\[
R_y(r,z) = - F_y(r,z) / c^2, \quad (1)
\]

and

\[
j_x(r,z) = - e_0 \partial \phi(r,z) / \partial z. \quad (2)
\]

The component \( F_z(r,z) \) is not ignored, but it is determined by the requirement that \( \int F_z \, ds = 0 \), by a boundary condition such as a perfectly conducting pipe that encloses the beam.

The second moments of the charge distribution in \( x, x' \) phase space are \( x^2, x'^2 \) and \( x x' \). It can be shown that the derivatives of the moments are functions of the moments and of the values of \( F_x \), \( F_y \), and \( x^2 F_x, x' F_y \), where \( F_x \) is the \( x \)-component of the total force, the sum of the external plus the self-force. The definition of rms emittance \( e \) in \( x, x' \) phase space, given by Lapostolle, can be shown to correspond to the total emittance of a continuous equivalent uniform beam (same second moments as the real beam) and is given by

\[
e = 4\left(x^2 - x'^2\right)^{1/2}. \quad (3)
\]

The rms emittance represents a convenient measure of the macroscopic or effective emittance of a beam subjected to nonlinear forces that arise either from external or self-fields. If Eq. (3) is differentiated and if the external force is assumed to be linear, it can be shown that

\[
\partial e^2 / \partial z = \frac{32}{n m v^2} \left(x^2 F_x - x x' F_{xx} \right), \quad (4)
\]

where \( n \) is the mass, and \( \gamma = (1 - \gamma^2 c^2)^{-1} \). For beam particles with charge \( e \), \( F_x = e E_y / \gamma^2 \) is the \( x \)-component of the self-force, including both electric and magnetic terms, which have opposite signs. Sacherer derived \( F_{xx} \) for an arbitrary charge distribution with elliptical symmetry, and his result can be written
nonuniform charge distributions. Because nonuniform beams have nonlinear self-fields, we call $U$ the nonlinear field energy. Equation (10) implies that a self-electric field energy possessed by beams with a similar equation was obtained by Lee, Yu, and Barletta for a beam with no external focusing. The special role of the uniform distribution can be explained by its associated linear self-force, which causes no rms emittance growth. Thus, $U$ is the residual self-electric field energy possessed by beams with nonuniform charge distributions. Because nonuniform beams have nonlinear self-fields, we call $U$ the nonlinear field energy. Equation (10) implies that a decrease in nonlinear field energy $U$ corresponds to an increase in rms emittance. Both the electric- and magnetic-field contributions are contained in Eq. (10) by including the factor $\gamma^3$ in the definition of $K$ ($\gamma^2$ accounts for the magnetic field and $\gamma$ accounts for relativistic mass). Although the total electromagnetic stored energy is the sum of electric plus magnetic terms, it is the difference that is related to changes in transverse rms emittance. This is because the transverse electric and magnetic self-force terms have opposite signs. Therefore, rms emittance growth, induced by space charge, is inherently a nonrelativistic effect; it is most important when $\gamma$ is near unity.

From Eq. (9), $W_0$ is the self-electric field energy per unit length within the beam boundary of an equivalent uniform beam. The quantity $U/W_0$ is dimensionless, and its properties can easily be demonstrated by considering the example of a power-law charge distribution. We find that $U/W_0$ is zero for a uniform charge distribution and is positive both for peaked and hollow distributions, increasing as the distribution becomes more nonuniform. Furthermore, $U/W_0$ is independent of both beam current and rms beam size, and is a function only of the shape of the distribution. Thus, Eq. (10) shows that rms emittance changes are associated with three separate factors: rms beam size, perveance, and changes in shape of the charge distribution. Values of $U/W_0$ are given for example distributions in Table I. These distributions are listed in order from most peaked to most hollow.

\[
\begin{align*}
\text{Distribution Function} & \quad \text{Charge Density } \rho(r) & \quad \frac{U}{W_0} \\
\text{Gaussian} & \quad \exp\left(-\frac{r^2}{a}\right) & \quad 0.154 \\
\text{Waterbag} & \quad 1 - \left(\frac{r}{R}\right)^2 & \quad 0.0224 \\
\text{Uniform} & \quad 1 & \quad 0.000 \\
\text{Hollow} (n = 2) & \quad \frac{r^2}{2} & \quad 0.0754 \\
\text{Hollow} (n = 10) & \quad \frac{r^10}{10} & \quad 0.245 \\
\end{align*}
\]

Equation (10) can be easily integrated, if we assume an rms-matched beam with constant $X$. The result relating between rms emittance and nonlinear field energy can be expressed as

\[
\varepsilon = \left[ 1 - \frac{U - U_1}{2W_0} \right]^{1/2}
\]

where $\varepsilon$ is the fractional energy stored in the beam; $U$ is the initial energy; $U_1$ is the energy stored in a field-free beam; and $W_0$ is the total energy stored in the beam. In Eq. (13), the rms emittance is expressed as the product of two factors, one, $(U - U_1)/\mu x$, related to the change in the shape of the distribution, and one related to the initial betatron tune ratio $\varepsilon_1$. It is easy to show that the tune ratio is

\[
\varepsilon_1 = \frac{U - U_1}{2W_0} \left( \frac{\varepsilon_1^2}{\varepsilon_1^2} - 1 \right)^{1/2}
\]

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\]
We have made numerical studies using 1000 beam particles per calculation to investigate the effects described above using a computer code written especially for this problem. We have studied azimuthally symmetric beams with different rms-matched initial distributions. At each time step, the particles were given transverse deflections based on both the external- and self-forces. The radial self-forces were calculated from Gauss' law, assuming infinitely long cylindrical charge distributions. The objectives were to study the evolution of the beam distribution, \( U_{\text{per}} / W_0 \) and \( \varepsilon \) for different initial distributions and different tune ratios \( \omega_1 / \omega_0 \). We show two examples that illustrate the main results.

First, we present results for a space-charge-dominated beam with initial Gaussian distributions, both in position and divergence, truncated at two standard deviations and with an initial tune ratio \( \omega_1 / \omega_0 = 0.02 \). We have introduced a radial distribution parameter, defined as \( p = 1 - r / r_0 \), where \( r \) is the average radius and \( r_0 \) is the average radius for the equivalent (same second moments) uniform beam. For simple distributions, we find that \( p \) is positive for a peaked charge distribution, zero for a uniform beam, and negative for a hollow beam. However, the parameter \( p \) must be interpreted with caution, because in some cases the distributions are too complicated for such a simple characterization. Figures 1a and 1b show \( p \) and \( U_{\text{per}} / W_0 \) as a function of distance \( z / \lambda_p \) along the beamline, where \( \lambda_p \) is the distance the beam travels during one plasma period. The parameter \( p \) indicates that the beam distribution undergoes damped radial oscillations between peaked and hollow configurations at the plasma frequency, a result that is confirmed by a more detailed examination of the charge distributions. The quantity \( U_{\text{per}} / W_0 \) is maximum at both the extreme peaked and hollow configurations and therefore oscillates at twice the frequency of the parameter \( p \). The minimum value of \( U_{\text{per}} / W_0 \) during these oscillations generally is not the zero value expected for an exactly uniform charge distribution. Examination of the beam cross section (not shown) when \( U_{\text{per}} / W_0 \) is minimum reveals the formation of a predominantly uniform beam, plus a low-density halo. We believe that two effects contribute to the nonzero minimum values of \( U_{\text{per}} / W_0 \), the low-density halo, and a numerical error caused by statistical fluctuations in the charge distribution. Statistical fluctuations also appear to contribute to some of the details of the curves in Fig. 1, beyond about five plasma oscillations. Figure 1c shows the emittance ratio \( \varepsilon / \varepsilon_1 \) as a function of \( z / \lambda_p \). Two overlapping curves are shown, one obtained from Eq. (13) with an evaluation of second moments, and the other from evaluation of \( U \) and using Eq. (13). The two curves are in excellent agreement and show a rapid initial growth that occurs during the first quarter of the first plasma period. A closer examination shows that the two curves are not in exact agreement, because of a slight mismatch caused by the emittance growth, an effect that is not included in the derivation of Eq. (13).

A second numerical example is shown in Fig. 2 for a beam with an initial semi-Gaussian or thermal distribution, corresponding to uniform charge density, Gaussian distribution in velocity space (truncated at two standard deviations), and with an initial tune ratio \( \omega_1 / \omega_0 = 0.25 \). Figures 2a and 2b show \( p \) and \( U_{\text{per}} / W_0 \) versus \( z / \lambda_p \). The parameter \( p \) rises from its initial value of zero to remain positive throughout, indicating a peaked distribution. An examination of the charge distribution reveals that the initial hard edge of the beam evolved rapidly to one with a tail or soft edge. The quantity \( U_{\text{per}} / W_0 \) increases, which would imply...
an emittance decrease according to Eq. (13). Figure 2c shows the rms emittance ratio \( \varepsilon_f/\varepsilon_i \). Again, the two overlapping curves, one from Eq. (3) and the other from Eq. (13), are in excellent agreement, and both show a small decrease in the rms emittance. Thus, we conclude that not only can the rms emittance increase as the beam becomes more uniform, but also an rms emittance decrease is possible when the beam becomes less uniform. As in the previous example, damped oscillations are observed, but at present, we have no quantitative explanation of oscillation frequencies for beams that are not highly space-charge dominated.

Numerical studies have been made for other initial distributions and tune ratios and for distances up to 100 plasma lengths. Our interpretation of these results is that the initially rms-matched beam undergoes damped radial oscillations and evolves to a final state with a central core and sometimes with a low-density halo that contains a few percent of the beam. The final charge density and final \( U/w_0 \) depend on the initial charge density and initial \( U/w_0 \) and on the initial tune ratio. Therefore, Eq. (13) implies that the rms emittance growth rate for a given initial space-charge distribution is a function only of the initial tune ratio.

Equations (10) and (13) describe rms emittance variation caused by any change in \( U/w_0 \) (associated with a change in the shape of the charge distribution) regardless of the detailed mechanism. In particular, unstable oscillation modes of the charge density, which are a well-known property of the Kapchinskii-Vladimirskii (K-V) distribution below certain tune-ratio thresholds,\(^\text{14}\) can cause such changes in charge density. We have seen evidence for such instabilities in our numerical studies for an initial K-V beam. In such cases, we find that the charge density changes from the initial K-V uniform density to a nonuniform, slightly peaked configuration. As in the semi-Gaussian example described above, we observe an increase in \( U/w_0 \) and a small decrease in rms emittance.

**Final, Uniform Charge-Density Approximation**

In spite of the excellent agreement of Eq. (13) with the numerical simulation results, we are unable to predict the final rms emittance growth unless the final value of \( U/w_0 \) is known. In the extreme space-charge limit where \( c_i \) approaches zero, we expect that the beam will evolve towards a final stationary state with a uniform charge distribution to obtain complete shielding of the linear applied focusing force. As an approximation for all initial tune ratios, we will assume that in the final state, \( U/w_0 = 0 \), which corresponds to a uniform beam with no halo. This approximation should be good in the space-charge-dominated limit, where the emittance growth is largest; in the emittance-dominated limit, where the approximation could lead to a large error in emittance growth, the emittance growth itself is small. In all cases, the approximation will result in an overestimate of the emittance growth.

With this approximation, the final emittance for an rms-matched beam can be written directly from Eq. (13) as

\[
\frac{\varepsilon_f}{\varepsilon_i} = \left[ 1 + \frac{U_1}{2w_0} \left( \frac{c_i}{\omega_i} - 1 \right) \right]^{1/2}
\]

(14)

This equation corresponds to the upper limit emittance growth formula, derived by Struckmeier, Klabunde, and Boiser.\(^\text{10}\) If betatron frequencies are replaced by phase advances per quadrupole focusing period. We note that Eq. (14) depends on the properties of the initial beam, which can be easily calculated.

For a space-charge-dominated beam, Eq. (14) can be written as

\[
\frac{\varepsilon_f}{\varepsilon_i} = \frac{1}{\omega_i} + \left( \frac{Kv}{\omega_i} \right)^2 \left( \frac{U_1}{w_0} \right)
\]

(15)

This result implies that as \( c_i \) approaches zero, the final emittance approaches a minimum value that decreases with increased focusing force (larger \( \omega_0 \)), increases with perveance \( K \), and increases with initial nonuniformity as measured by \( U_1/w_0 \). This formula provides a quantitative description of the familiar lower limit on final emittance similar to that first reported in numerical studies by Chasman\(^\text{2} \) for linear accelerator beams.

Numerical results have been used to test Eqs. (14) and (15). As an example, Fig. 3 shows \( \varepsilon_f/\varepsilon_i \) versus \( \omega_i/\omega_0 \) for an initial Gaussian distribution, truncated at four standard deviations. Values from the particle simulations are plotted and are in good agreement with the curve from Eq. (14).

![Fig. 3. Final emittance ratio versus initial tune ratio for an initial Gaussian distribution, truncated at four standard deviations.](image-url)

The curve is generated from the approximate formula, Eq. (14), and the plus symbols show the results of the particle simulations after 100 plasma periods.

Therefore, we conclude that emittance growth can be approximately predicted in advance from a knowledge only of the initial tune ratio and the initial value of \( U/w_0 \). A comparison of Eq. (15) with numerical particle simulations is shown in Fig. 4, where \( \varepsilon_f \) versus \( \varepsilon_i \) is plotted for the same initial truncated Gaussian distribution at a value of \( kv/\omega_0 = 5.0 \times 10^{-9} \) m. The results from the simulation and the curve from Eq. (15) are, again, in close agreement.
The studies of Ref. 8 were made for continuous beams in periodic quadrupole channels, and the results described in this paper apply for continuous beams in continuous focusing channels. Because continuous focusing channels represent a smooth approximation representation of a periodic channel, we anticipate that our results will be approximately valid for periodic systems in a smooth approximation, if the betatron frequencies \( \omega_0 \) and \( \omega_1 \) are replaced by phase advances per focusing period \( a_0 \) and \( a_1 \). We plan to conduct further numerical studies to test the formulas for periodic systems.

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Conclusions

We have attempted to evaluate the ideas expressed by Struckmeier, Klabunde, and Reiser that emittance growth is associated with the conversion of field energy to particle energy. We find that transverse energy conservation is valid for continuous channels, using the approximations given in this paper. We have obtained a formula, Eq. (10), which shows that rms emittance changes can be related to three separate factors: rms beam size, perveance, and changes in a quantity we call the nonlinear field energy. This can be done by producing the beam with an initial uniform charge distribution.

\[
\text{whereas initially uniform distributions will become more nonuniform in a less space-charge-dominated situation, leading to a small rms emittance decrease. Equations (10) and (13) imply that rms emittance growth can be minimized by minimizing any decrease in the nonlinear field energy \( U \). This can be done by producing the beam with an initial uniform charge distribution.}
\]

References