A FORMULATION OF TRANSVERSELY COUPLED BETATRON STOCHASTIC COOLING OF COASTING BEAMS

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Abstract

A formulation of stochastic cooling with coupled degrees of freedom, with emphasis on the tensorial signal suppression phenomenon for transversely coupled betatron stochastic cooling of coasting beams, is provided. The relevant expressions for the damping and diffusion coefficients and the elements of the signal suppression tensor are given in terms of the general cooling interaction harmonics. The formulation is necessarily abstract in order to provide a general framework of quantitative analysis and demands exact knowledge of the nature of the coupling interaction for it to be useful in any particular case.

1. Introduction

The pick-up in a stochastic cooling feedback loop generally derives signals from the motion of a particle in all three dimensions and the kicker affects all three degrees of freedom by producing fluctuating three-dimensional electromagnetic fields. The coupling between the degrees of freedom induced by the PU-amplifier kicker feedback loop will produce particle fluxes in all three dimensions in action I-space and the resulting transport equation for the time-evolution of the beam distribution will involve a third-rank "Diffusion Tensor" and a "Cooling Flux Vector" of dimension three. In particular, the coupling will produce cross-correlations among the cross-moments like (I_x, I_y, I_z), (I_x, J_y, J_z),... etc. on top of the mean diagonal moments like (I_x), (I_y) and (J_z), even when such cross-moments were initially absent in the beam before cooling started. The pick-up and kicker impedances (absorbed in P and K respectively), the transfer function of the amplifier, which contain the relevant amplitude-dependent harmonic frequencies (\omega_0 + \nu_0 \omega_k + \nu_0 \omega_k + \eta \cdot \omega_k), and the kickers themselves produce the relevant signal suppression tensor in an essentially coupled way. In the next sections, we provide expressions for damping and diffusion coefficients ignoring signal suppression.

2. Fokker-Planck coefficients without signal suppression

As discussed in an accompanying paper\(^1\), the cooling dynamics can be described by the following action and phase equations of motion:

\[
\dot{\psi}_i = \sum_{j=1}^{N} \mathbb{G}(I_{i,j};I_{j},\psi_j) \quad (i = j \text{ included})
\]

\[
\dot{\psi}_i = \omega_i + \sum_{j=1}^{N} \mathbb{H}(I_{i,j};\psi_{i,j})
\]

where \(\omega_i = (\omega_{x,i},\omega_{z,i},\omega_{B,i})\) are oscillation frequencies corresponding to the three dimensions with phases \(\phi_{x,i},\phi_{z,i},\phi_{B,i}\) respectively. In addition, we have the general Fourier series representation of the interaction in harmonics of the periodic angle variable \(\psi\) as

\[
\mathbb{G}(I_{i,j};I_{j},\psi_j) = \sum_{n=1}^{N} \mathbb{G}_{n,i}^{ij}(I_{i,j}) \hat{g}_{n}^{ij}(\psi_i + n;\psi_j)
\]

and similarly for the phase interaction \(\mathbb{H}\). Note that \(\mathbb{G}(\hat{G})\) and \(\mathbb{H}(\hat{H})\) are vector quantities now with a = x, z, B and are functions of the full three-dimensional action variables of the 'kicker' and 'kicked' particles, j and i, are thus capable of describing coupled degrees of freedom. The nature of the harmonics will determine the specific functional dependences of the various components \(\mathbb{G}^{x},\mathbb{G}^{z},\mathbb{G}^{B}\) and \(\hat{g}_{n,x},\hat{g}_{n,z},\hat{g}_{n,B}\) on their arguments. For a general set of harmonics \((n_x, n_z)\), the above describes, in addition, non-linear pick-ups and kickers which detect and affect higher harmonics \((n_x^2, n_z^2) > 1\) of betatron motion as well as the fundamental \(n_x n_z = 1\) corresponding to the dipole moment signals only. The third component of the harmonics \(n_B\) is simply the revolution harmonic 't' for coasting continuous beams and the transverse harmonic 'n' for bunched beams. For continuous coasting beams, the dependence of the interaction harmonics on particles j and i, i.e. on the 'kicker' and 'kicked' particles separate in most physical cases. The functional dependence on i and j can then be factored with good accuracy\(^2\) as

\[
\mathbb{G}_{ij}^{Bj} (I_{i,j}) = K_{ij}^{B} (I_{i}) \cdot P_{Bj} (I_{j})
\]

For bunched beams, the corresponding factorization is

\[
\mathbb{G}_{ij}^{Bj} (I_{i,j}) = \sum_{n=1}^{\infty} \hat{g}_{n}^{ij} (I_{i},I_{j})
\]

where the sum over the revolution harmonics 't' reflects the correlated Schottky harmonic structure inherently implied by the azimuthally bunched nature of the beam, as observed in an accompanying paper. The interaction harmonics depend on the particle orbits in the beam, the pick-up and kicker impedances (absorbed in P and K respectively), the transfer function of the amplifier, which contain the relevant amplitude-dependent harmonic frequencies \((\omega_0 + \nu_0 \omega_k + \nu_0 \omega_k + \eta \cdot \omega_k)\) for continuous beams and \(\omega_0 + \nu_0 \omega_k + \nu_0 \omega_k + \eta_0 \omega_k + \omega_0 = \omega_0 + \nu_0 \omega_k + \eta \cdot \omega_k\) for bunched beams) and on the nature of the coupling between degrees of freedom induced by cooling. It is important to remember that all the above representations are valid for non-overlapping revolution and betatron bands only for the bunched beam case.

According to the general theory of cooling outlined in other papers\(^1\),\(^2\), the time evolution of the phase averaged one-particle distribution \(f(I;t)\) is given by the basic Fokker-Planck equation, valid up to two-body correlations, as

\[
\frac{\partial f(I;t)}{\partial t} = \sum_{\alpha} \frac{3}{\mathbb{G}_{\alpha}} \cdot \left[ \mathbb{F}^\alpha(I) \cdot \nabla f(I;t) \right]
\]

\[
+ \frac{1}{3} \sum_{\alpha,\beta} \sum_{3} \frac{3}{\mathbb{G}_{\alpha,\beta}} \cdot \left[ \mathbb{D}_{\alpha,\beta} (I ; t) \right] \cdot \frac{\partial}{\partial I} f(I;t)
\]

where the "Cooling Flux vector" \(\mathbb{F}^\alpha(I)\) and the "Diffusion Tensor" \(\mathbb{D}_{\alpha,\beta} (I; t)\) are given in terms of the general interaction harmonics \(\mathbb{G}\) as

\[
\mathbb{F}^\alpha (I) = \sum_{n} \mathbb{G}_{n}^{\alpha} (I) \cdot \hat{f}(n;\psi_I) \hat{g}_{n}^{\alpha} (I)
\]

\[
\mathbb{D}_{\alpha,\beta} (I; t) = (2\pi) \cdot N \sum_{n} \sum_{\beta'} \frac{1}{2} \hat{f}(n;I^\alpha) \mathbb{G}_{n}^{\alpha} (I; \alpha) \times \mathbb{G}_{n}^{\beta} (I; \beta') \cdot \hat{g}_{n}^{\alpha} (I; \alpha) \hat{g}_{n}^{\beta} (I; \beta')
\]
The phase harmonics do not appear owing to rapid-phase averaging and the use of the Hamiltonian flow condition of the fluctuation cooling dynamics without the self-action ($j = i$) term as discussed elsewhere.

Further simplification occurs for coasting beams, by using the separated variable representation (3) when

$$P_n^i(I) = \sum_{n} K_n^i(I) \cdot P_n^i(I)$$

$$D_n^i(I) = \int dI' \sum_{n} K_n^i(I) \cdot K_n^i(I') \int dI'' dI'''$$

The $S$-functions contain the full Schottky band overlap contribution for coasting beams, but contain only the synchrotron band-overlap contribution for bunched beams, with no revolution band overlap.

### 3. Signal suppression tensor for continuous beams

We derive the signal suppression tensor of rank two for the transversely coupled betatron cooling of continuous beams from first order linearized Vlasov equation for the perturbed distribution function $f \equiv f(I, \phi, \omega, z; t)$ given by

$$\frac{df}{dt} = \omega \cdot \frac{df}{dt} + \frac{df}{dt} + \frac{df}{dt} = 0$$

where $I \equiv (I_x, I_z)$, $\phi \equiv (\phi_x, \phi_z)$, $\omega \equiv (\omega_x, \omega_z)$, $f_0 \equiv f_0(I, \omega, z, t)$ the stationary zero-order distribution and $\omega$ = 0 for no longitudinal cooling. Our model for the cooling interaction is

$$f(I, \phi, \omega, z; t) = \delta(\omega - \omega_0) \int d\phi' d\omega' d\phi' d\omega' \times G(I, \phi, \omega, \phi', \omega', I' = t' - t) \cdot f(I', \phi', \omega', t')$$

where

$$\delta' = \int_{-\pi}^{\pi} \delta(\theta - \theta_0) = \int_{-\pi}^{\pi} \delta(\theta - \theta_K = -2\pi n)$$

is the periodic delta function and $\theta_0$ and $\theta_K$ are the pickup and kicker azimuths. Substituting (10) in (11), Fourier series decomposing in harmonics $(m, n)$ of the angle variables ($\phi, \omega$) and Fourier transforming to frequency $w$, we obtain:

$$f_0(I, \phi, \omega, z; t) = f_0(I, \omega) = \left(\frac{1}{2\pi}\right) \int d\phi d\omega \cdot \frac{\delta(\omega - \omega_0) \int d\phi d\omega \cdot e^{i\omega t} K_0^\phi(I, \phi; \omega, \omega_t; t) \int d\phi' d\omega' \cdot G(I, \phi, \omega, \phi', \omega', I' = t' - t) \cdot f(I', \phi', \omega', t')$$

where $K_0^\phi(I, \omega)$ is an arbitrary excitation, to be identified with the coherent Schottky signal excitation in the absence of kicker induced modulations. Using the separated variable representation (3), we can write $G$, in a more general notation as:

$$G_n^m(I, \phi; \omega; t) = K_n^\phi(I, \phi; \omega; t) \cdot P_n^m(I, \phi; \omega; t)$$

where $K$ is the matrix

$$K_n^\phi(I, \phi; \omega; t) = \begin{bmatrix} K_0^\phi(I, \phi; \omega; t) & 0 \\ 0 & K_n^\phi(I, \phi; \omega; t) \end{bmatrix}$$

and $P_n^m$ the vector

$$P_n^m(I, \phi; \omega; t) = \begin{bmatrix} P_n^m(I, \phi; \omega; t) \\ P_n^m(I, \phi; \omega; t) \end{bmatrix}$$

Using (13), (14) and (15) in (12) and defining a collectively screened vector signal by

$$X(\omega) = \int d\omega' \sum_{n} e^{i\omega \omega'} P_n^m(I, \phi; \omega; t)$$

we obtain the effect of signal suppression as

$$\xi(\omega) = \frac{\sum_{m} e^{i\omega \omega'} P_n^m(I, \phi; \omega; t)}{i[n^2 - n^2 + \omega^2]}$$

where

$$\tilde{P}_n^m(\omega; t) = \int d\omega' P_n^m(I, \phi; \omega, \omega'; t)$$

is the incoherent Schottky signal contribution.

The feedback through the beam, subjected to a betatron cooling interaction that couples both the transverse degrees of freedom is thus described in general by the signal suppression tensor $\xi(\omega)$ given by (19) at any given frequency $\omega$, resulting in a distortion of signals as

$$X(\omega) = \xi(\omega) \cdot X(\omega)$$

The inversion of this matrix naturally implies a tensorial character of the response and a single scalar signal suppression factor does not exist in general, except under special circumstances. Explicitly, the components of $\xi$ are given by

$$\xi_{nm}(\omega) = \delta_{nm} + \frac{1}{2\pi} \sum_{\alpha} e^{i\alpha} \int d\alpha \cdot e^{i\alpha \omega} \int d\phi d\omega \cdot \frac{-i\xi_{nm}(\omega)}{i[n^2 - n^2 + \omega^2]} \cdot P_n^m(I, \phi; \omega; t)$$

where $\alpha$ and $\omega$ are the pickup and kicker azimuths, and $\xi_{nm}(\omega)$ is the incoherent Schottky signal contribution.

In the situation of non-overlapping Schottky bands, betatron bands decouple and no sum over $n$ is present so that (18) becomes

$$\xi_{nm}(\omega) = \delta_{nm} + \frac{1}{2\pi} \int d\alpha \cdot e^{i\alpha \omega} \int d\phi d\omega \cdot X(\omega)$$

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where $\alpha$ and $\omega$ are the pickup and kicker azimuths, and $\xi_{nm}(\omega)$ is the incoherent Schottky signal contribution.

If, in addition, the signals are predominantly 'longitudinal' in character, i.e. if

$$X(\omega) = \alpha X(\omega) \quad \text{and} \quad X(\omega) = \alpha X(\omega)$$

it is easily verified that this results in an isotropic scalar signal suppression as

$$X(\omega) = \alpha X(\omega)$$

where

$$\alpha = \int d\omega P_n^m(I, \phi; \omega; t)$$

and $P_n^m(\omega; t)$ is the Fourier transform of $P_n^m(I, \phi; \omega; t)$.
where

\[ \varepsilon = \left[ \begin{array}{c} \varepsilon_{\alpha} \\ \varepsilon_{\beta} \end{array} \right] = \frac{1}{|\beta|^2} \left[ \begin{array}{c} \beta \cdot \varepsilon_{\alpha} \\ \beta \cdot \varepsilon_{\beta} \end{array} \right] \]

\[ = \left[ \begin{array}{c} \varepsilon_{\alpha \alpha} \beta^2 + (\varepsilon_{\alpha \beta} + \varepsilon_{\beta \alpha}) \alpha \beta + \varepsilon_{\beta \beta} \beta^2 \\ (\alpha^2 + \beta^2) \end{array} \right] \]

(21)

where \( \varepsilon_{\alpha \beta} \), etc. are given by (22) with sum over \( \alpha \) omitted. In this special case the modified damping and diffusion coefficients, (8) and (9) appearing in the Fokker-Planck equation (5) are simply obtained by dividing \( \dot{\mathbf{P}} \) by \( |\varepsilon(\beta)|^2 \). In the general situation, according to prescriptions outlined elsewhere, the diffusion coefficient is modified and obtained from

\[ \dot{D}_{\alpha \beta} = \frac{\dot{R}_{\alpha \beta}(\beta)}{\varepsilon(\beta)} \mid_{\beta = 0} = \int_{-\infty}^{+\infty} d\tau \dot{R}_{\alpha \beta}(\tau) \]

(28)

where \( \dot{R}_{\alpha \beta}(\tau) \) is the time stationary averaged autocorrelation function of the sampled "signal-suppressed" signal

\[ s(t) = \sum_{n=-N}^{N} \delta \left[ \theta(t) - \theta_{\nu} - 2\pi n \right] X(t) \]

(29)

and is given by:

\[ R_{\alpha \beta}(t, t') = \frac{2}{(2\pi)^2} \sum_{n} \sum_{\nu} \int d\Omega \hat{\omega} e^{i(n+m)\Omega} e^{i(n+m)\Omega} \]

\[ \times e^{-i(n+m)\theta_{\nu}} e^{i(n+m)\theta_{\nu}} \]

\[ \sum_{\nu} \sum_{\nu} \delta^{\nu \nu} \langle \sum_{m} \hat{\omega}^{\nu \nu} \langle \sum_{m} \hat{\omega}^{\nu \nu} \langle \sum_{m} \hat{\omega}^{\nu \nu} \rangle \}

(30)

and a similar modification for \( \dot{F}_{\alpha \beta} \) involving elements \( \dot{D}_{\alpha \beta} \) only once.

4. Discussion

The detailed nature of the stochastic cooling in action \( \mathbf{x} \)-space in the situation of coupling is complicated as is evident from the above and depends on the particular case under study. In general, the Schottky noise from one degree of freedom will heat the other degree of freedom and vice-versa. The region in action-space \( (I_x, I_z) \) where coupling induces simultaneous cooling of both degrees of freedom is expected to be typically small. If we neglect the thermal noise from the amplifiers and Schottky heating from other particles in the same degree of freedom, then the equilibrium size in any one dimension is determined by the noise from the other coupled degree of freedom and the coherent cooling rate in that particular dimension. Qualitatively, we can approximate the cooling equation in any one dimension as

\[ \frac{d \{ I \}}{d\tau} = -\gamma_{\alpha} (I, I_x, J) \] \[ + \dot{\phi}_{\alpha} \}

\[ \times \{ I, I_x, J \} \] \[ (\alpha = x, z) \]

(31)

where we have assumed a constant diffusion speed \( \dot{\phi}_{\alpha} \) due to noise from the other degree of freedom \( \beta \neq \alpha \), which is only true if the diffusion coefficient can be linearized in the action variables. In general, non-linear functional dependences of transport coefficients on \( I \) relate any moment to all the higher moments of \( I \) and equations for single moments like (31) do not exist. Under these approximations, Fig. 1 illustrates one possible situation of movement in action-space with two-dimensional trajectories satisfying

\[ \frac{d \{ I_x \}}{d\tau} = \frac{d \{ I_z \}}{d\tau} \]

(32)

when \( J = 0 \). The dashed curves a and b correspond to \( v_\alpha = 0 \) and \( v_\beta = 0 \) determined by \( F_{\alpha \beta}(I_x, I_z, J) = 0 \), for equilibrium situation in each dimension separately. For any fixed cooling system, simultaneous cooling of \( I_x \) and \( I_z \) can occur only in the region \( R \) bounded by these two curves a and b. The region in \( I_x, I_z \) space containing amplitudes that can be eventually captured is bounded by curves c and d, beyond which amplitudes grow in both dimensions. In this region of captured amplitudes, strong signal in any dimension (large amplitude) leads to heating in the other dimension at first, but eventually entering region \( R \) where both dimensions are cooled. The region OPEFG is inaccessible to the amplitudes, E being the stable equilibrium position determined by the interplay between the noise-heating terms from both dimensions. Trajectory movements around the point D suggest that it is an unstable equilibrium, with the lifetime of the beam at D with \( \gamma_{\alpha} = \gamma_{\beta} = 0 \) being infinitely short. While it is highly doubtful whether an asymptotic equilibrium distribution, corresponding to the point E in this trivialized example, exists for the beam in the general non-linear situation, qualitative analyses along these lines might prove meaningful and useful in practical situations of design of, and estimates for cooling systems. Such analyses have been briefly outlined by Derbenev and Skrinsky3 in the context of electron cooling previously.

References

1. S. Chattopadhyay, Theory of bunched beam stochastic cooling, these proceedings.