With eq. (8) the lattice functions, expressed in Fourier
one can deduce the relations
Considering the common phase-space ellipse \( (x, dx/ds) \)
components of the guide field, are given by
Furthermore we note the relation between the action
variable \( j \) and the "emittance of a particle" \( \varepsilon \) :

The coefficients \( a \) and \( b_{2k} \) have to be determined by

where \( e \) is the periodicity of the non-linearity.

The p harmonic of \( F(m) \) drives the resonances

The function \( F(m)(\phi) \) is the coefficient of a term of
\( m \)-th degree and argument \( \phi \); \( m \) can take the values

We introduce action and angle variables

The non-resonant oscillating parts of \( n \)-th degree in the
Hamiltonian can be removed, using a transformation,
generated by a function of the form

The behaviour of the "analytical calculated betatron-
function for the PAMPUS lattice" is shown in figure 1.
Comparing this with the result of a matrix-code one
sees that there is a very good correspondence.

Figure 1. Behaviour of radial betatronfunction for
PAMPUS with \( Q=2.10 \)

Non-linear uncoupled orbit motion.

Non-linear fields often give rise to resonances,
which may limit the stability region. Considering the
one-dimensional, radial motion, the Hamiltonian becomes

This Hamiltonian can be rewritten as:

where \( j \) and \( k \) are positive integers \( j+k \); \( n=j+k \) is
the degree of the Hamiltonian. Furthermore the phase

The function \( F(m)(\phi) \) drives the resonances

The Hamiltonian can now be written as:

The new Hamiltonian, containing resonant terms, now
becomes:

Further we note the relation between the action
variable \( J \) and the "emittance of a particle" \( \varepsilon \) :

Considering the common phase-space ellipse \( (x, dx/ds) \)
one can deduce the relations

With eq. (5) the lattice functions, expressed in Fourier
components of the guide field, are given by

and furthermore

\[ \beta_1 = \beta_{1x} \]
The stability region can be predicted from the invariant \( K(\xi, \psi) \). The criterion for the fixed points (\( d\xi/d\xi = 0 \), \( \psi/d\psi = 0 \)) leads to the condition

\[
\frac{\tilde{\xi}}{\tilde{\xi}_{\text{p}}^-} = \mp (\frac{P}{N} \frac{R}{N}) (\frac{m_2}{m_1} \frac{m_3}{m_2})
\]

where \( \tilde{\xi} \) is the action variable belonging to the unstable fixed points.

A criterion for stability is that the beam lies entirely inside the stable region in the (\( \xi, \psi \)) phase plane. Representing the beam by a circle (correct when non-linearities are present) one gets the requirement

\[
\frac{\tilde{\xi}}{\tilde{\xi}_{\text{p}}^-} \geq c(\frac{Q}{R})_{c^0}
\]

where \( c \) is a constant depending on the form of the stable region and so depending on \( n \) \( n_1 = 2, 0 \) and \( c_{1.21} = 1.21 \). Eqs. (20) and (21) lead, for a given excitation term, to a minimum distance to the resonance line.

We will illustrate this for the third degree resonance 30 p, N excited by sextupole fields. The minimum distance is given by

\[
\frac{\tilde{\xi}}{\tilde{\xi}_{\text{p}}^-} = 3 \mathcal{F}_3(\xi_c, \psi_c, \beta_e, \gamma_e)
\]


The sextupoles needed for chromaticity correction in PAMPUS 4 (N=6) can for example excite the resonance 30 to 16. For a particle with a 10 \( \sigma \) amplitude, the minimum distance is \( \tilde{\xi}/\tilde{\xi}_{\text{p}}^- = 0.036 \).

Furthermore we note the \( \xi \) and \( \psi \) independent Hamiltonian of eq. (13) for \( m_1 = 0, \ m_2 = 0 \).

This condition can only occur for even degree \( n \). In this case there is a change of the tune, depending on the amplitude. Considering the fourth degree Hamiltonian function the tune-shift is given by

\[
\frac{\tilde{\xi}}{\tilde{\xi}_{\text{p}}^-} = 2 \mathcal{F}_4(\xi_c, \psi_c, \beta_e, \gamma_e)
\]


The first term gives rise to an inherent tune shift and the second term is the contribution of octupole fields. The needed octupole fields to provide a certain tune shift (spread) can be determined. As an example for PAMPUS 4 = 3.25 the octupole field should be \( 3B^2/3B^2 \) in order to get a tune shift of 10 \( \sigma \) for a particle with a 10 \( \sigma \) amplitude.

Non-linear coupled motion

Two transverse motions may be coupled in the presence of non-linearities. We will consider here the influences of sextupoles and octupoles and the Hamiltonian of interest is

\[
H = \frac{1}{2} p_x^2 + \frac{1}{2} (c-n) p_z^2 + \frac{1}{2} z^2 + \Sigma a_j(\theta) x_j^2
\]

where \( x, \xi, \beta, \gamma \) and \( p_x, p_z \) are relative variables according to eq. (1).

The \( \beta \)-dependence in the linear Hamiltonian is removed by a transformation, generated by the function (see also 2)

\[
G_2(x, \psi, \beta, \gamma) = (R/2B) x \{ \tan(\psi) \} + a_2 \}
\]

where \( x, \psi, \beta, \gamma \) are the betatron function, \( a_2(\theta) \).

The Hamiltonian can now be written as \( K(J_p, \psi, \beta, \gamma, \theta) \). Keeping in this Hamiltonian only the non-linear terms and applying the moving-coordinate transformation of the type as given in eq. (16), which is now generated by the function \( \xi \):

\[
G_3(J_1, x_1, x_2, \beta, \gamma) = -J_1 \{ \psi - (m_1/m_2) x_2 \} - x_1 \}
\]

with \( J_1 = J_1 - (m_1/m_2) J_2, J_2 = J_2, \psi = \psi + (m_1/m_2) \beta \) and \( \beta \) the Fourier component that drives the resonance.

The Hamiltonian now becomes

\[
K = \mathcal{F}_3(J_1, \psi, \beta, \gamma) - \mathcal{F}_4(p_c, \xi_c, \beta_c, \gamma_c)
\]

where \( \mathcal{F}_3 = 3 \mathcal{F}_3(\xi_c, \psi_c, \beta_c, \gamma_c) \)

\[ F_1, F_2, F_3 \]

\[ p_c, \xi_c, \beta_c, \gamma_c \]

The study of phase plane trajectories is somewhat simplified by the transformation

\[
X = \sqrt{2} x \cos \psi, \quad \gamma = \sqrt{2} x \sin \psi
\]

and the new Hamiltonian is called \( K \).

The fixed points in the \((x, \gamma)\) phase plane are now given by

\[
\frac{dx}{d\psi} = 0, \quad \frac{d\gamma}{d\psi} = 0
\]

where we omitted the subscripts at \( F, J, \) and \( p_c \).

It follows from eq. (30) that the flowlines and the position of the fixed points are related to the value of \( J = J_1 - J_2 \) and \( 0 \leq J \leq |F| \).

An example is shown in fig. 3: a machine with a = 22.8 \( \sigma \) and \( J = 78.8 \sigma \) should satisfy the condition \( 0 \leq |F| \leq 10.6 \sigma \)
The study of this resonance goes in the same way as the previous resonance. The excitation term is now
\[ p_1(x_0, y_0) = (2/6) \left( x_0^2/3 + y_0^2/3 \right) \]  
and furthermore \( \delta Q = 1/2 \left( Q_x + 2Q_y - p_N \right) \), \( J = J_x Z \).  

Trajectories in \((x,y)\) phase plane are given in figure 4.

\[ Q_x + 2Q_y = p_N : \text{normal sextupoles.} \]

**Figure 3.** \(\delta Q/|F|\) curves in the \(J_x, J_y\) diagram for the resonance \(Q_x + 2Q_y = p_N\).

**Figure 4.** \((x,y)\) phase plane for PAMPUS for the resonance \(Q_x + 2Q_y = p_N\); \(\delta Q = 0.075\) and \(|F| = 10\). The region inside the dashed circle \((b,c,d)\) indicate the unphysical region \(J_x < 0\).

In the same way as done in the previous section we can calculate the minimum \(\delta Q/|F|\) value for a given \(J_x Z\). The result is given in fig.5: PAMPUS should satisfy

\[ \delta Q/|F| > 1.1 \times 10^{-3} \]

Results of this phase-plane treatment are compared with results, obtained by using the theory of Guignard.

The minimum distance given by Guignard is strongly related to the equation which holds for the fixed points and substituting \(J_2 f.p. = J_z^{(2)} \) (see also 6)

\[ \text{Table 1. } \delta Q = \frac{1}{2} (Q_x + 2Q_y - p_N) \text{ values for PAMPUS.} \]

| Resonance | \( |F| \) | \( \delta Q = 0.075 \) (our method) | \( \delta Q = 0.075 \) (Guignard) |
|-----------|------|------------------------------|-------------------------------|
| \( Q_x + 2Q_y = 8 \) | 0.14 | 0.00050 | 0.00039 |
| \( Q_x + 2Q_y = 16 \) | 40.1 | 0.014 | 0.034 |

Figure 5 \(\delta Q/|F|\) curves as function of \(J_x Z\). Results of this phase-plane treatment are compared with results, obtained by using the theory of Guignard.

This resonance leads to a periodic exchange of energy between the two transverse planes. This exchange can be determined by using extreme values \( \pm 1 \) for \( \cos 2\theta_z \) in \( \delta Q \) (eq.27) : \( \alpha = J_2 \min f.J_2 \max \) and

\[ \alpha^2 \left[ \frac{|\delta Q| + 2|F|J_1}{2|F|^2J_2 \max} \right] = \left[ \frac{|\delta Q| - 2|F|J_1}{2|F|^2J_2 \max} \right]^2 \]

The upper sign holds for \( \delta Q > 0 \), the lower sign for \( \delta Q < 0 \).

**Final remarks.**

The analytical expressions give a good behaviour of the lattice functions. Furthermore the description of non-linear resonances in a one-dimensional phase-space gives a very good insight in the influence of the exciting non-linear fields.

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In the same way as done in the previous section we can calculate the minimum \(\delta Q/|F|\) value for a given \(J_x Z\). The result is given in fig.5: PAMPUS should satisfy

\[ \delta Q/|F| > 1.1 \times 10^{-3} \]