COMMENTS ON STABLE MOTIONS
IN NONLINEAR COUPLED RESONANCES

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Summary

It is shown that, of all nonlinear coupled resonances of the form $m V_1 + n V_2 = k$ where $m$, $n$ and $k$ are positive integers and $m + n \geq 3$, those with $m$ or $n = 1$ exhibit a different property compared to others in their stable regions of phase space. The difference explains the paradoxical result obtained by Sturrock and Guignard that there are points of arbitrarily small amplitudes which lie outside the stable region.

I.

Consider a coupled resonance of the form

$$ (2p) V_1 + (2q) V_2 = n + \epsilon \tag{1} $$

where $(2p)$, $(2q)$ and $n$ are positive integers. It is further assumed that $p < q$ so that $p = 1/2, 1, 3/2, ..., q = 1, 3/2, 2, ...$.

In the tune diagram, it is convenient to define the point on the resonance line that is nearest to the point $(V_1, U_2)$.

$$ (2p) V_1^0 + (2q) V_2^0 = n \tag{2} $$

$$ v_1 = V_1 + e_1, \quad v_2 = V_2 + e_2 \tag{3} $$

$$ e_1 = \epsilon (2p)/(2p)^2 + (2q)^2, \quad e_2 = e_1 (q/p) \tag{4} $$

The distance from the point $(V_1, U_2)$ to the resonance line is

$$ \Delta = [e_1^2 + e_2^2]^{1/2} = |\epsilon|/((2p)^2 + (2q)^2)^{1/2}. \tag{5} $$

When one retains only the resonance-deriving term, the Hamiltonian in terms of the action-angle variables $(I_1, a)$ can be written in the form

$$ H = (e_1/2)(2I_1) + (e_2/2)(2I_2) + D \cdot \cos(\phi)(2I_1)^p(2I_2)^q \tag{6} $$

where $\phi = (2p) a_1 + (2q) a_2 + \delta$. (7)

The amplitude $D$ and the phase $\delta$ of the driving term can be expressed in terms of the machine parameters and the nonlinear force which is driving the resonance. By writing equations of motion for $I_1$ and $I_2$, one can easily verify that the quantity

$$ C = (2I_1)/(2p) - (2I_2)/(2q) \tag{8} $$

is an invariant, that is, $dC/dq = 0$ with the independent variable $\delta$.

Analogous to the concept of fixed points in the two-dimensional phase space, one can define "fixed lines" in the four-dimensional space $(I_1, a_1, I_2, a_2)$ from the following three conditions:

$$ \frac{dI_1}{d\phi} = \frac{dI_2}{d\phi} = \frac{da_1}{d\phi} = \frac{da_2}{d\phi} = 0. \tag{9} $$

Conditions for $I_1$ and $I_2$ are satisfied (excluding the trivial solution $I_1 = I_2 = 0$) by taking $\sin(\phi) = 0$.

II.

Since the Hamiltonian, Eq. (6), is independent of the variable $\theta$, it is an invariant. From two invariants $H$ and $C$, Eq. (8), one can construct two invariant expressions $\Phi_1$ and $\Phi_2$.

$$ \Phi_1 = H + \epsilon (2p) C \cdot (2q)^2/[(2p)^2 + (2q)^2] \tag{10} $$

$$ = (e/2)(2I_1)/(2p) + D \cdot \cos(\phi)(2I_1)^p(2I_2)^q \tag{11} $$

$$ \Phi_2 = H - (e/2) C \cdot (2p)^2/[(2p)^2 + (2q)^2] \tag{12} $$

$$ = (e/2)(2I_2)/(2q) + D \cdot \cos(\phi)(2I_1)^p(2I_2)^q \tag{13} $$

One can further simplify the form of two invariants by the normalization

$$ \lambda \equiv A \cdot [e/|e|] \cdot \Phi_1, \quad \mu = A \cdot [e/|e|] \cdot \Phi_2 \tag{14} $$

where
The corresponding normalization of two action variables is
\[ u^2 = (2L_1) \cdot (2D/|e|) \cdot 1/s \cdot (2p)^{p/2} \cdot (2q)^{q/2}, \]
\[ v^2 = (2L_2) \cdot (2D/|e|) \cdot 1/s \cdot (2p)^{p/2} \cdot (2q)^{p/2}, \]
where \( p' = 1-p \) and \( q' = 1-q \). The final form of two invariants is
\[ \lambda = u^2 + v^2 p - x q, \]
\[ \mu = u^2 p - x q. \]

For physically meaningful solutions, both \( u \) and \( v \) must be positive (or zero) and \( |w| \) must be less than or equal to unity. One can eliminate the variable \( v \) using the relation
\[ v^2 = u^2 - (\lambda - \mu), \]
and the problem is reduced to finding the amplitude \( u \) such that the absolute value of
\[ w = \frac{\lambda - u^2}{2p (u^2 - \lambda + \mu)} \]
is less than or equal to unity. The motion is stable if this condition restricts the value of \( u \) within a finite range. In the \((\lambda, u)\) space, there are three regions with different characteristics:

1. First quadrant, \( \lambda > 0 \) and \( u > 0 \). See Fig. 1A. The function \( w(u) \) has one minimum point. If the minimum point is below \(-1\), the motion is stable (curve S). If the point is above \(-1\) (curve U), \( u \) can take any value and the motion is unstable. The limiting case is the curve L.

2. \( \lambda = 0 \) and \( u > 0 \). See Fig. 1B for \( p = 1/2 \). There is no stable motion for other values of \( p \).

3. Second, third and fourth quadrants, \((\lambda < 0, u > 0)\) or \((\lambda > 0, u < 0)\). Note that \( \mu = 0 \) is excluded. The function \( w(u) \) can have one maximum and one minimum point. See Fig. 1C. It will be shown below that there is no stable motion of this class unless \( p = 1/2 \).

The maximum or minimum points are solutions of the condition \( dw(u)/du = 0 \) which takes the form, with \( x = u^2, \)
\[ (2s - p)x^2 - 2[(s+p)\lambda + (1-p)\mu]x + (2p)\lambda(\lambda-\mu) = 0 \]
and the solutions are
\[ u^2_M = 1/(2s) \cdot [(s+p)\lambda + (1-p)\mu \pm \sqrt{M}], \]
\[ M = (s-p)^2 \cdot \lambda^2 + (s+p)^2 \cdot \mu^2 + 2[(s+p)\lambda + (1-p)\mu] \lambda \mu. \]
The corresponding values of \( v^2 \) are
\[ v^2_M = u^2_M - (\lambda - \mu) \]
\[ = (1/2s) \cdot [-((s+p)\lambda + (2s+p-1)\mu \cdot \sqrt{M}]. \]
Since \( u^2_M \) must be real, \( M \) must be either 0 or positive.
(1) \( p = 1, \lambda > 0 \) (fourth quadrant) which is already excluded.

(2) \( p > 1, \mu < [(s+\rho)/(p-1)]\lambda \).

The coefficient in front of \( \lambda \) is always larger than 3 and the condition is in contradiction with the condition (31).

(3) \( p = 1/2 \),

\[
\mu > -2q/\lambda. \tag{35}
\]

This condition as well as the condition (31) are satisfied in the second quadrant \( \lambda < 0 \) and \( \mu > 0 \).

By evaluating \( \xi \) and \( \eta \) in Eq. (30) for \( p = 1/2 \),

\[
(\xi, \eta) = (6s + 1) \pm 4\sqrt{s(2s + 1)}, \tag{36}
\]

one can see that

\[
\eta < (2q) < \xi. \tag{37}
\]

In conclusion, stable motions are possible if

(1) \( \lambda > 0 \) and \( u > 0 \) for \( p \neq 1/2 \).

(2) \( \lambda > 0 \) and \( u > 0 \) or \( \lambda < 0 \) and

\[
\mu > -6s+1 + 4\sqrt{s(2s+1)}T/\lambda \text{ for } p = 1/2. \tag{39}
\]

The resonance studied by Sturrock1 corresponds to \( p = 1/2 \) and \( q = 1 \). He missed the region \( \lambda < 0 \) and \( u \gtrsim 8\lambda \) which is shown in Fig. 1C. If \( \lambda \) is limited to positive values, one finds from Eq. (21) with \( w = -1 \),

\[
u^2(p-1)v^2q < 1. \tag{38}
\]

For \( p = 1/2 \), this leads to the exclusion of points near the origin as stated by Sturrock. In order to find the stable region in the phase space or, equivalently, in \((u, v)\) space, one must solve \([dv(u)/du] = 0\) together with \( w(u) = -1 \) for \( u = u_M \), the maximum possible stable amplitude. Analytical solutions are possible for \((p = 1/2, q=1)^2\) and for \((pq=1)^4\). For \( v_1 + 2v_2 = n \), the limiting values \((v_{M}, v_{M})\) of Figs. 1A and 1B satisfy the relation

\[
2v^2 - 2u_M(1 - u_M) \tag{39}
\]

which is equivalent to the expression of bandwidth, Eq. (13), found by Guignard.2 if the case represented by Fig. 1C is included, one finds that the stable motion is confined in the region bounded by \( u = 0 \), \( v = 0 \), Eq. (39) and

\[
v = (1 + u)/2 \tag{40}
\]

as shown in Fig. 2.

References

1. P.A. Sturrock, Annals of Physics, 3, 113 (1958)
3. W.P. Lysenko, Particle Accelerators, 2, 1 (1973) See Figure 4.