The kick can be expressed in terms of the "tune shift" and through equally spaced extremely non-linear δ-function kicks. The positron performing a stable linear one-dimensional transverse oscillation perturbed by a series of beam colliding head-on with a strong bunched electron noise as an integral part. Noise as an integral part of the formulation. Non-linear dynamics is fascinating because a purely deterministic system can exhibit apparently stochastic behaviors, thus leading to topics such as singular mapping curves, Arnold diffusion, KAM theorem, overlapping resonances etc. But, so far, the relevance of these inherent features of non-linear dynamics to the observed behaviors of colliding beams has not been established. This prompted some investigators to suggest other approaches to an explanation of the beam-beam effects. Moreover, on the conceptual level there should at the very outset introduce statistical phenomena. The exact temporal evolution of the non-linear motion of a particle, stochastic as it may be, is nevertheless time-reversible, whereas the beam-beam behavior exhibits irreversible diffusion-like characteristics. One should at the very outset introduce statistical averages into the equations of motion - a process similar to the transition from the Liouville equation to the Liouville equation.

The increment of the action invariant \( W = \) due to a kick \( \Delta x' \) is

\[
\Delta W = \beta (\Delta x')^2 + 2 (\alpha x + \phi x') \Delta x.' \tag{1}
\]

The kicked can be expressed in terms of the "tune shift" \( \xi \) through

\[
\xi = \frac{\beta}{\Delta W} \Delta x' \quad \text{or} \quad \Delta x' = \frac{4\pi \xi}{\beta}. \tag{11}
\]

Partial Quantitative Formulation

The equation for the y-motion of the positron is

\[
y'' + k(s)y = -\frac{2V}{\alpha} \delta(s) \tag{1}
\]

where the force term on the right-hand-side is

\[
\delta(s) = 0.0079
\]

where the last number corresponds to \( \xi = 0.01 \), an easily attainable value on all electron colliders. This is a very large number indeed, giving an e-fold increase in \( W \) in only \( \frac{\Delta W}{W} = 127 \) kicks. The only reason that the positron motion can be stable is because these strong kicks are not random but periodic, and all evils are concentrated into resonances. On-resonance the effects of the kicks add coherently and the oscillation amplitude grows proportionally to the number of kicks. With random kicks the amplitude still grows as the square-root of the number of kicks. Off-resonance the effects of the kicks cancel systematically to give zero amplitude growth. The off-resonance cancellation is essential for the survival of the beam and is very exacting, especially for the very high order resonances. Any irregularity will upset the delicate cancellation. These time-domain descriptions are illustrated in Fig. 1.

For colliding beams both the non-linearity and the harmonics of the kicks extend to extremely high orders. The tune-space is covered dense by resonances (density of rational numbers), and the oscillation tune sits in a continuum of high order resonances even when all strong low order resonances are avoided. In fact, since the electron bunches are not exactly identical from collision to collision the kicks are not exactly periodic and all resonances have some "spreads" or "widths". This situation is equivalently described by assigning a natural "width" to the tune. This description avoids the possibility of confusing the "spread" of a resonance due to inexact periodicity of the kicks with the usual resonance width in non-linear dynamics. This description further suggests that the (small) portion of the kick-spectrum which is flat and equal in height to the part lying within the "width" of the tune will constitute a random series of (small) kicks in the time domain and cause the amplitude to grow. This is because a "white" spectrum in the frequency domain corresponds to a series of random signals in the time domain. The "natural width" is rather small, but the ever present hardware noise will contribute to the resonance spread and make the "total width" substantial. As described, the ultimate effect of noises, natural (beam) or external (hardware), is to take "strength" off from the resonances and smear it in between resonances to form the "white" spectrum of a set of random kicks.

\[
\frac{\Delta W}{W} \approx 0.0079
\]
expressible in closed form for round bi-Gaussian electron beam bunches and is given by

$$\delta(s) = \frac{r_0 N_0}{\gamma_p \sigma^2} \int_0^\infty dt \frac{1}{2(t^2 + \sigma^2)} \left( -\frac{y^2}{2\sigma^2} \right) \epsilon(s)$$

where

- $\sigma$ = Gaussian standard deviation
- $\gamma_p = \frac{E}{m c^2}$ of positron
- $r_0 = \frac{e^2}{m c^2}$ = classical radius of positron
- $N_0$ = total number of electrons.

The real electron beams are, however, not round but flat ribbons with $\sigma_x >> \sigma_y$ ($x$ = horizontal, $y$ = vertical). Hence the vertical kicks are larger and dictate the intensity limit. Introducing the vertical "tune shift"

$$\xi_y = \frac{1}{4\pi} \frac{r_{0 \gamma_y}}{\gamma_p \sigma_y \sigma_x}$$

and integrating Eq. (3) once to get the kick $\Delta y'$ we obtain

$$\Delta y' = -4\pi \xi_y \frac{\sigma_y}{\gamma_y} \left( \frac{\sigma_x}{\gamma_x} \right)$$

The random part of these kicks in the sense discussed earlier will contribute to the growth of $W_y$. From Eqs. (2) and (5) we obtain

$$\left\langle \Delta W_y \right\rangle = \beta_y \left\langle \Delta y' \right\rangle^2 = \xi_y^2 \frac{\sigma_y}{\gamma_y} \left( \frac{\sigma_x}{\gamma_x} \right)^2$$

where

$$F(u) \equiv 32\pi^2 \left[ \left( 1 - e^{-u^2} \right) \right]$$

We drop some of the subscripts $y$ as being understood and write

$$\frac{dW}{dt} = \frac{Q}{\beta_y} \frac{\sigma_x}{\gamma_x} - f \xi_y^2 \frac{\sigma_x}{\gamma_x}$$

where $f$ = frequency of kicks. The total motion of the positron is, then, given by

$$\frac{dW}{dt} = W + f \xi_y^2 \frac{\sigma_x}{\gamma_x}$$

where

- $Q$ = growth due to quantum fluctuation
- $\tau$ = synchrotron radiation damping time.

The function $F$ has a maximum at $u = 1.26$ or $y = 1.59 \sigma_y$. The maximum "tune shift" $\xi_{max}$ that can be obtained is given by the condition $dW/dt = 0$ at $F = F_{max}$, namely

$$\xi_{max}^2 \frac{\sigma_x}{\gamma_x} \frac{\sigma_y}{\gamma_y} = \frac{N_0}{\gamma_p}$$

The energy dependence of the quantities are

$$W = \sigma_y^2 \gamma_y, \quad \sigma_x = \sigma_y, \quad \gamma = \gamma_x$$

This gives

$$\xi_{max} \propto \varepsilon^{5/2}$$

The energy dependence of the maximum luminosity $L_{max}$ is related to that of $\xi_{max}$ by

$$L_{max} \propto \varepsilon^{7/2}$$

Fig. 2 and 3 show the fits to the measured data from SPEAR with

$$\xi_{max} = 0.01 \varepsilon^{5/2} \text{ and } L_{max} = 0.03 \varepsilon^{7}$$

($L_{max}$ in cm$^{-2}$s$^{-1}$ and $\varepsilon$ in GeV).

Discussions

1. Although the meaning was clearly stated no mathematical procedure has been developed to extract the "random part" in the definition, Eq. (7), of $F$. This involves contributions from both high order resonances and their "spreads" due to inexact periodicity or randomness of the kicks. It is possible that the mechanism proposed in Ref. 3 is appropriate. For the energy-dependence $F$, however, all one needs is that $\frac{dW}{dt}$ be proportional to $\xi_x^2$ and the proportionality factor be energy independent.

2. No explanation was given to the ultimate limit of $\xi_{max} \approx 0.05$. This limit is not statistical in nature and could well be given by the single particle non-linear dynamics. The conventional stochasticity limit due to overlapping of resonances is entirely consistent with the physical picture presented here.

References

1. See e.g. J. Moser, "Stable and Unstable Motion in Dynamical Systems", AIP Conf. Proc. No. 57, p. 222 (1979)

Fig. 1. Effects of kicks in the time domain (A) on resonance (B) random (C) off-resonance

Fig. 2. Maximum vertical tune shift versus energy in SPEAR.

Fig. 3. Maximum luminosity versus energy in SPEAR.