PARTICLE TRAJECTORY PERTURBATIONS IN BEAM HANDLING SYSTEMS

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Abstract

This report discusses three types of perturbations on particle trajectories in beam handling systems due to space charge forces: (1) Owing to radial space charge forces the beam will expand in diameter in drift spaces along the beam orbit. (2) Owing to longitudinal space charge forces energy dispersion will increase along the beam trajectory when the beam is bunched. (3) While transiting cross magnetic fields particle trajectories on the inner and outer edge of monoenergetic beam are not similar: as a result there appears to be additional energy dispersion in the beam.

Radial Dispersion

The calculation of the radial divergence of a beam due to its own space charge is most directly calculated in the moving frame (with respect to which the axial motion of the beam is stationary). Assuming a dc beam of uniform charge density over the cross-section the relativistic equation of motion of an ion on the beam envelope is given, by Gauss' law, as

\[ \frac{\partial^2 r}{\partial t^2} = \frac{e p}{m c^2} \frac{1}{r} \left[ 1 - \left( \frac{\partial^2 r}{\partial t^2} \right)^2 \right]^{1/2} \]

(1)

where \( p \) is the linear charge density and \( r \) is the radius of the beam. Since under motion in this frame

\[ \gamma = \frac{1}{\sqrt{1 - \frac{1}{c^2} \left( \frac{dr}{dt} \right)^2}} \]

we have, differentiating and substituting in Eq (1),

\[ \frac{d^2 r}{d t^2} = \frac{e p}{m c^2 \gamma} \left[ 1 - \left( \frac{d^2 r}{d t^2} \right)^2 \right]^{1/2} \]

(3)

This will be recognized as a 'radius of curvature' type of differential equation, which may be integrated, with the boundary conditions \( r = a, \frac{dr}{d t} = 0 \), to give

\[ \left( \frac{c}{d t} \right)^2 = \frac{e p}{m c^2 \gamma} \left[ \frac{\ln \left( \frac{r}{a} \right)}{1} \right]^{1/2} \]

(4)

It is rather tedious to integrate this expression, which may be done either successively (by parts) or numerically. However, since we will wish to transform the solution into the frame of an observer moving along the axis of the beam at a constant velocity (corresponding to the beam velocity \( v = \frac{c}{\gamma} \)), we may write since then \( t = \frac{z}{v} \) in the laboratory frame

\[ \left( \frac{d^2 r}{d \tau^2} \right) = \frac{e p}{m c^2 \gamma \nu_0} \left[ \frac{\ln \left( \frac{r}{a} \right)}{1} \right]^{1/2} - 1 \]

(5)

Note that the linear charge density and mass of the ions transform in the same way, we have transformed the moving charge density into a beam current (since \( 1 - \gamma v^2 \)), and have put \( r = \nu_0 a \) (the impedance of free space). Note also that \( z \) (measured in the laboratory frame) is to be transformed from the moving frame.

In order to integrate this expression we use the substitutions \( x = r/a, \ y = z/a, \ k = \frac{e I \gamma / m e c^2}{\sqrt{x^2 - 1}} \) and \( w^2 = \left( k l m + x \right)^2 \). Then Eq(4) may be put in the form

\[ \frac{e p}{m e c^2} \frac{d^2 x}{d \tau^2} = \frac{d y}{d \tau^2} - \frac{d y}{\nu_0^2 - 1} \]

(6)

Expanding the LHS in series the expression may be integrated term by term,

\[ \frac{1}{\nu_0^2} \sum_{n=0}^{\infty} \frac{x^2}{(n+2)!} \frac{d^2 w}{d x^2} = \sum_{n=0}^{\infty} \frac{w^2}{(n+2)!} \frac{d^2 w}{d x^2} \]

(7)

For most practical cases Eq (5) can be approximated closely by the equation

\[ \frac{d^2 r}{d \tau^2} = \frac{e I \gamma}{m c^2 \nu_0^2 (x^2 - 1)^{3/2}} \ln \left( \frac{r}{a} \right) \]

(8)

This expression cannot be integrated in terms of elementary (tabulated) functions; however a substitution of variable will result in a convenient form. Putting \( x = \beta \left( r/a \right) \)

\[ \frac{e I \gamma}{m c^2 \nu_0^2 (x^2 - 1)^{3/2}} \ln \left( \frac{r}{a} \right) = \frac{1}{\sqrt{x}} \frac{d \chi}{d x} = \frac{1}{\sqrt{x}} \sum \frac{\chi^2 (n+1)^2}{n! (2 n + 1)} \]

(9)

Eqs (7) and (9) are shown in a universal diagram in Figs (1) and (2).

In general, beams are not in free-space, as assumed above and are, at least, enclosed in a pipe or metallic vacuum envelope. In this case we may use the artifice employed earlier, ie. solve the diffusion problem in a frame of reference moving with the beam and transform the solution to the laboratory frame.

Assuming the beam is in equilibrium inside a grounded metal cylinder, the potential distribution inside the cylinder may be determined from Laplace's equation for the region between the beam (of diameter \( 2b \)) and the cylinder (of diameter \( 2a \)) and by Poisson's equation inside the beam.

The solutions must match at the common (beam) boundary. Thus,

\[ \nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} = 0 \quad (0 \leq r \leq a) \]

which, with the boundary conditions \( V = 0, \ r = a \) and \( V = V_0, \ r = b \) has the solution

\[ V = V_0 \ln \left( \frac{r}{a} \right) / \ln \left( \frac{b}{a} \right) \]

Within the beam

\[ \nabla^2 V = \frac{1}{r} \frac{\partial^2 V}{\partial r} + \frac{\partial V}{\partial r} = - \frac{e}{c} \quad (0 \leq r \leq b) \]

which, assuming uniform density in the cross-section of the beam and the boundary conditions \( V = V_0, \ r = b \) and \( \frac{d V}{d r} = 0, \ r = 0 \) has the solution.

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\[
V = V_0 + \frac{\rho_i}{4 \pi \varepsilon_0 E} (\beta^2 - r^2) \tag{11}
\]
where we have also exchanged the space charge density \( \rho \) for the linear charge density \( \rho = \pi B \rho \). The well-known 'potential depression' with the beam \( V = V_0 + \rho_i / 4 \pi \varepsilon_0 E_i \) is observed; however, since \( \rho = \rho \), the depression for relativistic beams is negligible.

Of especial importance, however, is the observation that the radial electric field intensity at the edge of the beam
\[
E_r = -\frac{\partial V}{\partial r} = \frac{\rho_i}{2 \pi \varepsilon_0 E} = \frac{E_0}{2 \pi \varepsilon_0 E_i V} \tag{12}
\]
is precisely the same as that observed without the beam enclosure, Eq. (11) and hence the space charge divergence is exactly as that given in Eq. (7) or (9). This result has been assumed by numerous writers who have employed the free-space fields of a beam to determine its space charge expansion within an enclosure.

The beam model assumed above has the defect that the current density is probably not uniform in the cross-section; this presumably can, in principle, be allowed for since
\[
I(\theta) - \int_0^\rho \beta_i(r) v(r) 2\pi r dr \tag{13}
\]
if the functions \( \beta_i(r) \) and \( v(r) \) were known.

In many relativistic beams, due to the method of acceleration the beam has an rf microstructure, ie., it is "bunched". In this case the radial expansion of the beam presents a somewhat more complex problem in analysis. The method of Inman and Muray, due to a method originally described by G. Darwin, involves the series expansion of Wiechert-Lienard retard potentials for the charge ensemble; this method is hopelessly inadequate for relativistic particles. On the other hand, Haimson and Mecklenburg have obtained solutions for a plethora of possible bunch shapes by the laborious, and not very elegant, method of dissecting the bunch into a collection of elementary volumes, 'tracking' them and reconstituting the entire bunch from time to time in its orbit.

The method used here follows that originally suggested by Leboutet. We assume, for simplicity, a spherical packet of \( N \) electrons, having a velocity \( v = \beta c \). This packet will not be spherical in the laboratory frame. Then a particle at the edge of the bunch experiences a radial force and its motion is given by
\[
\frac{d^2r}{dt^2} = \frac{N e^2}{4 \pi \varepsilon_0 E_0 m} \left( \frac{1}{t} - \frac{1}{t_0} \right) \tag{14}
\]
Integrating, with the boundary conditions \( dr/dt = 0 \), \( r = r_0 \), \( t = 0 \)
\[
\frac{dr}{dt} = \frac{N e^2}{2 \pi \varepsilon_0 E_0 m} \left( \frac{1}{t} - \frac{1}{t_0} \right) \tag{15}
\]
Re-integrating,
\[
\sqrt{\left( \frac{r}{r_0} \right)^2 - \left( \frac{r}{t} \right)^2} + \arccos \frac{r}{r_0} = \frac{\sqrt{N e^2}}{2 \pi \varepsilon_0 E_0 m} \frac{t}{t_0} \tag{16}
\]
Using the Lorentz transform, to pass to the laboratory frame,
\[
\tan \theta = \frac{E_r}{E_0} \tag{17}
\]
where \( \theta \) is the angle between \( r \) and the direction of motion measured in the moving frame (stationary frame of the bunch). This angle in the laboratory frame, \( \theta \), is related to that in the moving frame by the expression
\[
\sin \theta = \frac{E_r}{E_0} \tag{18}
\]
where \( \theta \), the velocity of the particle in the moving frame (stationary frame of the bunch) is known from Eq. (15).

Inserting these transformations in Eq. (16) for two cases (where we have also put the drift distance in place of the time, \( t = z/v = (z/cX/\mu_c - 1) \)) we have:

(1) for the transverse expansion \( (\theta = \pi/2) \)
\[
\sqrt{\left( \frac{r}{r_0} \right)^2 - \left( \frac{r}{t} \right)^2} + \arccosh \frac{r}{r_0} = \frac{\sqrt{N e^2}}{2 \pi \varepsilon_0 E_0 m} \frac{r}{t_0} \tag{19}
\]
(2) for the longitudinal expansion \( (\theta = 0) \)
\[
\sqrt{\left( \frac{r}{r_0} \right)^2 - \left( \frac{r}{t} \right)^2} + \arccosh \frac{r}{r_0} = \frac{\sqrt{N e^2}}{2 \pi \varepsilon_0 E_0 m} \frac{r}{t_0} \tag{20}
\]
Note that at \( Z = 0 \) \( r = r_0 = r_0 \) (the diameter of the spherical packet in the moving frame), but \( r = r_0 = r_0 \) so that, in the laboratory frame the bunch looks disc-like.

When the bunch is inside a pipe the problem appears even more complex; however, as an approximation an approach to a solution can be gotten by assuming a disc-shaped packet containing a total charge \( q = 1/1 \) of diameter \( 2b \). The potential of such a disc can be shown to be given by
\[
\Psi = \frac{1}{2 \pi \varepsilon_0} \sum \frac{J_i(B_{ab}) J_i(B_{ab})}{A_{ab} J_i^2(B_{ab})} \tag{21}
\]
where \( 2a \) is the pipe diameter. The radial electric field intensity at the edge of the disc is then
\[
E_r = \frac{1}{2 \pi \varepsilon_0} \sum \frac{J_i(B_{ab}) J_i(B_{ab})}{A_{ab} J_i^2(B_{ab})} \tag{22}
\]
and the motion of a particle on the edge of the disc (assumed stationary) is given by
\[
\frac{d^2r}{dt^2} = \frac{e q}{m \pi \varepsilon_0 a} \sum \frac{J_i(B_{ab}) J_i(B_{ab})}{A_{ab} J_i^2(B_{ab})} \tag{23}
\]
The solution of this equation, with the boundary conditions \( r = r_0 \), \( t = 0 \), \( dr/dt = 0 \)
\[
\frac{d^2r}{dt^2} = \frac{2 e q}{m \pi \varepsilon_0 a} \sum \frac{J_i(B_{ab}) J_i(B_{ab})}{A_{ab} J_i^2(B_{ab})} \tag{24}
\]
Re-integration of this expression would seem to be formidable, and this appearance is not deceptive. Transforming the above expression into the laboratory frame and using the drift distance, rather than time, we have (where \( \lambda \) is the bunch spacing)
which can, of course, easily be integrated by machine, using the additional equation
\[ r = r_0 + \frac{dr}{dz} \Delta z \] This function can evidently be normalized into a universal beam spread curve with \( r_0/d \) as a parameter.

As an alternate calculation, the dispersion of a disc-shaped ion bunch in free-space is interesting. The potential of a thin circular disc of radius \( a \) and of homogeneous charge density is
\[ V = \frac{2Ne}{4\pi \varepsilon_0} \left[ \frac{1}{2r} - \frac{1}{8} \left( \frac{a}{r} \right)^2 (\cos \theta) + \frac{1}{16} \left( \frac{a}{r} \right)^4 (\cos \theta)^2 \right] \]
Hence the radial electric field on the edge of the disc is \( \theta = \pi/2 \), \( r \to a \)
\[ E_r = \frac{2Ne}{4\pi \varepsilon_0} \] (27)
and the motion of particles on the edge of the disc
\[ \frac{d^2r}{dt^2} = \frac{2Ne^2}{m_0r^2} \] (28)
the solution of this equation is, of course, the same as that of Eq (14) within a constant.

An entirely different class of phenomena, which will not be discussed here, is the wake fields of the charge pulse moving in a pipe, which has been treated by Morton, Neil and Sessler.\(^5\)

Energy Dispersion

Eq (14) indicates that a considerable electric field intensity exists on the periphery of the packet \( E_r = Ne/4\pi \varepsilon_0 r^2 \). (For example 10 10 electrons in a bunch of one centimeter diameter has a peripheral field of about one MeV/meter.)

The component of electric field in the direction of motion remains unchanged in the relativistic transform, so that this field is quite effective in causing energy spread in a drifting bunch.

In order to estimate the amount of energy dispersion we will assume a disc model for the packet. The potential of the disc in the pipe is given by Eq (21); therefore the electric field intensity on the face of the pipe is
\[ E_z = -\frac{\partial W}{\partial z} = \sum \frac{2(\beta \mu_b)}{\pi \varepsilon_0} \left[ J_r(\beta \mu_b) \right] \sum \frac{J_r(\mu_b)}{\beta \mu_b J_r(\mu_b)} \] (29)
where \( 2a \) is the pipe diameter, \( 2b \) the disc diameter and \( d \) the disc thickness. Averaging this field over the face of the disc (since the 'potential depression' can be shown to be negligible)
\[ E_z = \frac{2q}{\pi \varepsilon_0 B^2} \sum \frac{J_r(\beta \mu_b)}{\beta \mu_b J_r(\mu_b)} \] (30)
The summation is a constant, depending upon the ratios of beam/pipe diameter and disc thickness to pipe diameter. Setting \( q = 1/\lambda \) and \( \lambda \) the packet spacing,
\[ E_z = \frac{2q}{\pi \varepsilon_0 B^2} \sum \frac{J_r(\beta \mu_b)}{\beta \mu_b J_r(\mu_b)} \] (31)
which indicates that the longitudinal space charge forces are considerably reduced by the presence of the pipe.

Apparent Energy Dispersion

A modification of particle trajectories in a beam bending (or energy analyzing) magnet will occur due to the self-field of the beam at high currents, which causes an increase in the beam diameter and horizontal divergence of an otherwise mono-energetic beam. Moreover, in an electron-optical system, where the beam is focussed with a sector magnet, the focal point will be moved farther from the magnet face at high current.

It would appear, at first, that the space charge forces in the beam would be approximately cancelled by the self magnetic field of the beam. However, examination of the equations of motion (in cylindrical coordinates):
\[ \frac{d^2r}{dt^2} = \frac{e}{m_0} \left( E_r + r \phi B_r - r \phi B_r \right) \]
\[ \frac{d^2\phi}{dt^2} = \frac{e}{m_0} \left( E_r + r \phi B_r - r \phi B_r \right) + \gamma r \phi^2 \]
\[ \frac{d^2r}{dt^2} = -\frac{e}{m_0} B_{2r} \]
The applicable fields for the horizontal ray (or energy) diagram of peripheral electrons in a mono-energetic beam traversing a magnetic field are \( B_{2r} \) (due to the magnet), \( E_r = I/2\pi \varepsilon_0 v a^2 \) (due to space charge in a beam of radius \( a \), where \( I \) is the instantaneous current) and \( B_{2r} = dI/2\pi a \) (the magnetic field associated with the beam).

The equations of motion are, then:
\[ \frac{d^2r}{dt^2} = -\frac{e}{m_0} B_{2r} \]
\[ \frac{d^2\phi}{dt^2} = \frac{e}{m_0} \left( E_r + r \phi B_r \right) + \gamma r \phi^2 \]
\[ \frac{d^2r}{dt^2} = -\frac{e}{m_0} B_{2r} \]
It is because the magnetic field of the beam shows up twice in the equations of motion, but the radial electric field does only once, that there is no mutual cancellation. Integrating Eq (33c)
\[ \frac{d\phi}{dt} = -\frac{e}{m_0} B_{2r} \] (34)
which is the Larmor frequency; hence \( \omega = \omega_L \).
Inserting this result in the radial equation,
\[ r = \frac{eE_r}{2m_0} - \left( \frac{eB_r}{2m_0} \right)^2 \]
For the central ray (or without space charge)
\[ r = -r \omega_L^2 \]
which is the differential equation of simple harmonic motion. When considering the trajectory of a peripheral particle, the differential Eq (33b) has the form
\[
\frac{d^2r}{dt^2} = A - Br
\]

where \(A = eE_0/r_m\) and \(B = [e(B_{20} + B_{21})/2r_m]^2\).

Integrating twice
\[
\frac{dr}{dt} = 2Ar - Br^2 - 4(AR - BR)^2
\]

\[
BR - A = \cos B t
\]

where the boundary conditions are \(r = 2R_1,\ dr/dt = 0,\ t = 0\). The latter equation may be put in the form
\[
r = A + (2BR - A)\cos \sqrt{B} t
\]

For the central ray \(R = m_0\sqrt{2\gamma - 1}/eB_{20}\), so that Eq (40) may be put in the form
\[
r = \frac{E_r 4\gamma m_0}{(B_{20} + B_{21})^2 e}\left[2 \frac{m_0 c^2 \sqrt{2\gamma - 1}}{eB_{20}} \pm \alpha\right]
\]

\[
- \frac{E_r 4\gamma m_0}{(B_{20} + B_{21})^2 e} \cos \left(\frac{B_{20} + B_{21}}{2r_m}\right) t
\]

where \(R_r = R \pm \alpha = m_0 c^2 \sqrt{2\gamma - 1} \pm \alpha\), which depends on which side of the beam the particle is on.

The above treatment is rather cumbersome; hence an approximate solution would be desirable. The equation of motion of a particle in cylindrical coordinates and in terms of a deviation from the equilibrium orbit, or central ray \((X = r - R_0)\) is
\[
\frac{d^2X}{dt^2} + \omega_n^2 (1 - \delta_n) X = 0
\]

where \(\omega_n = eB_{20}/2r_m\) and \(\delta_n\) is an index of variation from the equilibrium field.

The perturbing force due to the self-fields of the beam (in the horizontal plane) we have shown in a previous analysis (on space charge defocusing) to be given by
\[
\delta_n = eE + evB = \frac{eE \ln X}{2\pi a^2} \frac{1 - \beta^2}{\beta}\]

where \(\eta = v/a\), \(\alpha\) is the radius of the beam and \(X\) is the particle displacement. Since this force is directly proportional to \(X\) its effect really is to change the effective value of the field index of the magnet. If \(\delta_n/\gamma\), the solution of the above differential equation is
\[
x = x_o \cos \omega_n \sqrt{1 - \delta_n} t + \frac{\omega_n}{\sqrt{1 - \delta_n}} \sin \omega_n \sqrt{1 - \delta_n} t
\]

where the boundary conditions are \(X = x_0,\ t = 0,\ and\ \theta = \theta_0\). The angle \(\theta = \pi x/\gamma a\) where \(\alpha\)

is the total bending angle, is introduced in order to eliminate the time, since \(a/\omega_n = t\) and, thus, \(\omega_n \sqrt{1 - \delta_n} t = \pi/a\). But now we must keep track of \(\theta\). Thus, we have, finally, at exit
\[
x = x_0 \cos \omega_n \sqrt{1 - \delta_n} \alpha + \frac{\omega_0}{\sqrt{1 - \delta_n}} \sin \omega_n \sqrt{1 - \delta_n} \alpha
\]

\[
\theta = x_0 \sqrt{1 - \delta_n} \sin \sqrt{1 - \delta_n} \alpha + \theta_0 \sqrt{1 - \delta_n} \alpha
\]

or in matrix form.

\[
\begin{bmatrix}
X \\
\theta
\end{bmatrix} = \begin{bmatrix}
\cos \sqrt{1 - \delta_n} \alpha & \sin \sqrt{1 - \delta_n} \alpha \\
\sqrt{1 - \delta_n} \sin \sqrt{1 - \delta_n} \alpha & \cos \sqrt{1 - \delta_n} \alpha
\end{bmatrix} \begin{bmatrix}
x_0 \\
\theta_0
\end{bmatrix}
\]

The procedure now may follow that given by Penner, to determine the effects on the system electron-optics. The reader is also referred to comments on this problem by Yadavalli.

References
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3. H. Leboutet, Annales de Radioelectricite, No. 32, 13 (1958)
7. W. R. Smythe, Static and Dynamic Electricity, p 177
8. Morse and Feshbach, Methods of Theoretical Physics, p 1259

Figure 1. Universal Beam Spreading Curve, Eq (7).
Figure 2. Universal Beam Spreading Curve, Eq (9).

Figure 3. Bunched Beam Spreading, Eqs (19), (20).