INVESTIGATION OF GRADIENT CORRECTIONS FOR THE AGS

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I. Introduction

A non-superperiod symmetric gradient perturbation results in an instability in the vicinity of integer and half-integer \( v \) values. By studying the integral form of the equation for the perturbed motion, conditions to any order in the perturbation are found such that the resulting particle motion at these critical \( v \) values is cyclic (periodic for some integer number of revolutions) for arbitrary initial values of the coordinates of the particle trajectory. Such a cyclic orbit implies stable motion provided the deviations away from this orbit due to higher order terms in the perturbation are bounded. The nature of these corrected orbits and the cumulative effects on the motion over many particle revolutions are investigated to second order for the AGS.

II. Integral Form of the Equation of Motion

The equation of motion of a particle in an alternating gradient field is

\[
d^2y(s) + \left[ K(s) + k(s) \right] y(s) = 0,
\]

where \( y(s) \) is the displacement of the particle normal to the superperiod reference orbit and \( s \) is the distance along this reference orbit from an arbitrary initial point. Here \( K(s) \) is the gradient forcing function determining the betatron motion for the unperturbed superperiod symmetric machine while \( k(s) \) represents the perturbing gradient forcing function:

\[
k(s) = \text{perturbing gradient at point } s \text{ magnetic rigidity of the particle}.
\]

Equation (1) is simplified by introducing the normalized coordinates \( \eta(\psi) \) and \( \tilde{\eta}(\psi) \) defined by the matrix transformation

\[
\begin{bmatrix}
\tilde{\eta}(\psi) \\
\eta(\psi)
\end{bmatrix} =
\begin{bmatrix}
\delta^{-1}_o(s) & 0 \\
\delta^{-1}_o(s) \omega_o(s) & \delta^{-1}_o(s)
\end{bmatrix}
\begin{bmatrix}
y(s) \\
dy(s)/ds
\end{bmatrix}.
\]

The phase variable

\[
\psi(s) = \int_0^s \frac{dt}{p_0(\tau)};
\]

and the betatron parameters \( (\omega_o, \delta_o) \) are those of the unperturbed system. In terms of these normalized coordinates, the equation of motion reduces to

\[
\frac{d^2\eta}{d\psi^2} + \eta + q(\psi) \eta = 0,
\]

where

\[
q(\psi) = \delta^{-2}_o(s) k(s).
\]

Equation (5) and the initial conditions \((\eta_0, \dot{\eta}_0)\) at \( \psi = 0 \) are equivalent to the integral equations

\[
\eta(\psi) = \eta_0 \cos \psi + \dot{\eta}_0 \sin \psi - \int_0^\psi d\tau q(\tau) \eta(\tau) \sin(\psi - \tau),
\]

and

\[
\dot{\eta}(\psi) = -\eta_0 \sin \psi + \dot{\eta}_0 \cos \psi - \int_0^\psi d\tau q(\tau) \eta(\tau) \cos(\psi - \tau).
\]

We introduce the complex variable

\[
\zeta(\psi) = \eta(\psi) + i \dot{\eta}(\psi)
\]

and combine Eqs. (7) and (8), obtaining the single integral equation

\[
e^{i\psi} \zeta(\psi) = \zeta_0 + \frac{i}{2} \int_0^\psi d\tau q(\tau) \left[ \zeta(\tau) + \zeta^*(\tau) \right] e^{i\tau},
\]

where \( \zeta_0 \) is the initial value

\[
\zeta_0 = \eta_0 + i \dot{\eta}_0.
\]

In Eq. (10) the variable \( \psi \) changes continuously, increasing by \( 2\pi \nu \) (\( \nu \) is the number of betatron oscillations in one machine circumference) for each particle revolution. To specify a physical point in the machine we use the phase, \( \psi^0 \), of the particle on an initial cycle consisting of \( j \) revolutions. Thus this phase has the restriction

\[
0 \leq \psi^0 < 2\pi \nu j.
\]

On successive cycles the particle phase variable, \( \psi \), takes on the values given by the relation

\[
\psi = \psi^0 + 2\pi \nu j k \quad ; \quad k = 0, 1, 2, \ldots.
\]

In this equation, the second term equals the phase change corresponding to \( j \) cycles, each made up of \( k \) revolutions. Introducing Eq. (13) into Eq. (10) we obtain

\[
e^{i(\psi^0 + 2\pi \nu j k)} \left[ \zeta(\psi^0 + 2\pi \nu j k) - \zeta_0 + \frac{i}{2} \int_0^{\psi^0} d\tau q(\tau) \right] \times \left[ \zeta(\tau) + \zeta^*(\tau) \right] e^{i\tau}
\]

\[
\times \left[ \zeta(\tau) + \zeta^*(\tau) \right] e^{i\tau} \quad ; \quad k \geq 1.
\]
Using the identity

\[ e^{i(\varphi + 2\pi j\nu)} = e^{i\varphi} \]

\[ \int_{0}^{2\pi j\nu} q(\tau) \left[ c(\tau) + c^*(\tau) \right] e^{i\tau} \]

\[ = -j \int_{0}^{2\pi j\nu} q(\tau) \left[ c(\tau + 2\pi j\nu) + c^*(\tau + 2\pi j\nu) \right] e^{i(\tau + 2\pi j\nu)} \]

\[ + \int_{0}^{2\pi j\nu} q(\tau) \left[ c(\tau + 2\pi j(\tau + 1)) + c^*(\tau + 2\pi j(\tau + 1)) \right] e^{i(\tau + 2\pi j(\tau + 1))} \]

\[ + \int_{0}^{2\pi j\nu} q(\tau) \left[ c(\tau + 2\pi j\nu) + c^*(\tau + 2\pi j\nu) \right] e^{i(\tau + 2\pi j\nu)} \]

\[ = 2\pi j\nu \int_{0}^{2\pi j\nu} q(\tau) \left[ c(\tau + 2\pi j\nu) + c^*(\tau + 2\pi j\nu) \right] e^{i(\tau + 2\pi j\nu)} \]

\[ + \frac{1}{2} \int_{0}^{2\pi j\nu} q(\tau) \left[ c(\tau + 2\pi j\nu) + c^*(\tau + 2\pi j\nu) \right] e^{i(\tau + 2\pi j\nu)} \]

\[ = 0 \quad (15) \]

we have

\[ e^{i(\varphi + 2\pi j\nu)} \] \[ \int_{0}^{2\pi j\nu} q(\tau) \left[ c(\tau) + c^*(\tau) \right] e^{i\tau} \]

\[ = - e^{i\varphi} \int_{0}^{2\pi j\nu} q(\tau) \left[ c(\tau) + c^*(\tau) \right] e^{i\tau} \]

\[ + j \int_{0}^{2\pi j\nu} q(\tau) \left[ c(\tau) + c^*(\tau) \right] e^{i\tau} \]

\[ + j \int_{0}^{2\pi j\nu} q(\tau) \left[ c(\tau) + c^*(\tau) \right] e^{i\tau} \]

\[ = \left( e^{i\varphi} + 2 \right) \int_{0}^{2\pi j\nu} q(\tau) \left[ c(\tau) + c^*(\tau) \right] e^{i\tau} \]

\[ + 2\pi j\nu q(\tau) \int_{0}^{2\pi j\nu} q(\tau) \left[ c(\tau) + c^*(\tau) \right] e^{i\tau} \]

\[ + \frac{1}{2} \int_{0}^{2\pi j\nu} q(\tau) \left[ c(\tau) + c^*(\tau) \right] e^{i\tau} \]

\[ = 0 \quad (16) \]

III. The Cyclic Orbit for a Gradient Perturbation

In this section we develop the idea of a cyclic orbit, that is, one which is periodic over \( j \) revolutions for arbitrary initial conditions. The unperturbed orbit is cyclic when the \( \nu \) value satisfies

\[ \nu j = p ; \quad p \text{ integer} \quad (17) \]

We impose this condition in the case of the perturbed orbit to ensure that this orbit also be cyclic in the limit of zero perturbing gradient. Equation (17) together with the condition of periodicity over a cycle:

\[ \xi(\varphi + 2\pi p) = \xi(\varphi) \quad (18) \]

permits us to write Eq. (16) as

\[ e^{i\varphi} \int_{0}^{2\pi j\nu} q(\tau) \left[ c(\tau) + c^*(\tau) \right] e^{i\tau} = 0 \quad (19) \]

This equation represents a cyclic solution only if the term proportional to \( \ell \) vanishes. With this requirement:

\[ \int_{0}^{2\pi j\nu} q(\tau) \left[ c(\tau) + c^*(\tau) \right] e^{i\tau} = 0 \quad (20) \]

the cyclic orbit for the arbitrary initial value \( \xi_0 \) is given by the integral equation defined over one cycle

\[ e^{i\varphi} \xi(\varphi) = \xi_0 - \frac{1}{2} \int_{0}^{2\pi j\nu} q(\tau) \left[ c(\tau) + c^*(\tau) \right] e^{i\tau}; \]

\[ 0 \leq \varphi \leq 2\pi p \quad (21) \]

If we iterate these coupled equations once, we obtain the cyclic orbit to first order in the perturbing strength \( q \).

\[ e^{i\varphi} \xi(\varphi) - \xi_0 - \frac{1}{2} \int_{0}^{2\pi j\nu} q(\tau) \xi(\varphi); \quad \frac{1}{2} \int_{0}^{2\pi j\nu} q(\tau) e^{i2\tau}; \]

\[ 0 \leq \varphi \leq 2\pi p \quad (22) \]

and the subsidiary conditions

\[ \int_{0}^{2\pi j\nu} q(\tau) = R_0(p) = 0 \quad (23) \]

\[ \int_{0}^{2\pi j\nu} q(\tau) e^{i2\tau} = R_2(p) = 0 \quad (24) \]

Since the integer \( p \) equals \( j\nu \), and \( q(\tau) \) is periodic with period \( 2\pi\nu \), these conditions simplify to

\[ \int_{0}^{2\pi j\nu} q(\tau) = 0 \quad (25) \]

\[ \int_{0}^{2\pi j\nu} q(\tau) e^{i2\tau} = 0 \quad (26) \]

Thus to obtain a cyclic orbit for any value of \( j \) not equal to 1 or 2, it is only necessary to satisfy Eq. (25), i.e. make the average value of the \( q(\tau) \) function around the machine circumference equal to zero. On the other hand, for \( j \) equal to 1 or 2, that is, at the critical integral or half-integral \( \nu \) values, Eq. (26) does not vanish identically but yields the further condition on the gradient function

\[ \int_{0}^{2\pi j\nu} q(\tau) e^{i2\tau} = 0 \quad (27) \]

We can therefore achieve a stable machine at integral or half-integral \( \nu \) values by adding gradient corrections such that the total perturbation satisfies Eqs. (25) and (27). When such corrections are added, a particle at a critical \( \nu \) value will follow a cyclic orbit, Eq. (22), for all initial values of its coordinates.

In the above paragraph we have discussed first order corrections. Higher order corrections, that is, added gradients which are capable of giving a cyclic orbit to higher order in the perturbation, can be obtained by continuing the iteration process. Thus, the second order cyclic orbit is
\[ e^{i\frac{\pi}{2}} \zeta' \psi' = \zeta - \frac{1}{2} \zeta \int_0^{2\pi} d\tau q(\tau) - \frac{i}{2} \zeta' \int_0^{2\pi} d\tau q(\tau) e^{i2\tau} \\
- \frac{\zeta_0}{4} \left\{ \int_0^{2\pi} d\tau q(\tau) \int_0^{2\pi} d\sigma q(\sigma) - \int_0^{2\pi} d\tau \int_0^{2\pi} d\sigma q(\tau)q(\sigma) e^{i2(\tau-\sigma)} \right\} \]

\[ + \frac{\zeta_0}{4} \int_0^{2\pi} d\tau \int_0^{2\pi} d\sigma q(\tau)q(\sigma) \left( e^{i2\tau} - e^{i2\sigma} \right), \]  

(28)

while the additional restrictions on the total \( q \) function are given by
\[ \int_0^{2\pi} d\tau q(\tau) e^{i2\tau} \int_0^{2\pi} d\sigma q(\sigma) e^{-i2\sigma} = R_{2,-2}(p) = 0, \]  

(29)

\[ \int_0^{2\pi} d\tau q(\tau) \int_0^{2\pi} d\sigma q(\sigma) \left[ e^{i2\tau} - e^{i2\sigma} \right] = R_{2,0}(p) - R_{0,2}(p) = 0, \]  

(30)

where
\[ R_{m,n}(p) = \int_0^{2\pi} d\tau q(\tau) e^{im\tau} \int_0^{2\pi} d\sigma q(\sigma) e^{in\sigma}, \]  

(31)

IV. Deviations from the Cyclic Orbit and Stability

In this section we investigate the motion of a particle away from the first order cyclic orbit due to second order effects in the perturbation. The question that arises is whether the deviations from the cyclic orbit remain bounded over many cycles of the particle motion. In short, is the motion stable?

Let the unperturbed system be at a critical \( \nu \) value, and suppose that a non-superperiod symmetric gradient perturbation has been corrected to first order. The coordinates of a particle at \( \Phi = 0 \) after one cycle are found by setting \( \Phi = 2\pi \nu \) in Eq. (10). Thus,
\[ \zeta(2\pi\nu) - \zeta_0 - \frac{1}{2} \int_0^{2\pi} d\tau q(\tau) \left[ \zeta(\tau) + \zeta^*(\tau) \right] e^{i\tau}, \]  

(32)

Since Eq. (10) is valid at all points within the cycle, we use it to obtain the iterated expansion for \( \zeta(2\pi\nu) \) to second order:
\[ \zeta(2\pi\nu) = \zeta_0 \left[ 1 - \frac{1}{2} R_0(p) - \frac{1}{4} R_{0,0}(p) + \frac{1}{4} R_{2,0}(p) \right] + \frac{i}{2} \zeta_0 \left[ \frac{1}{2} R_2(p) - \frac{1}{4} R_{0,2}(p) + \frac{1}{4} R_{2,0}(p) \right], \]  

(33)

where the integrals (R) over a cycle are defined in Eqs. (23), (24), and (31). In terms of the coordinates \( \eta \) and \( \eta' \) [see Eq. (9)] and specializing to the case
\[ 2\nu = p, \quad p = \text{odd integer} \]  

we can write Eq. (33) in the matrix form
\[ \begin{bmatrix} \eta(2\pi\nu) \\ \eta'(2\pi\nu) \end{bmatrix} = M \begin{bmatrix} \eta_0 \\ \eta'_0 \end{bmatrix}, \]  

(35)

where
\[ M_{11} = 1 + \text{Im} J_2 - \frac{1}{2} \left[ J_0^2 - |J_2|^2 + A_3 \right], \]  

(36)

\[ M_{12} = J_0 - \text{Re} J_2 - \frac{1}{2} \left[ A_1 + A_2 \right], \]  

(37)

\[ M_{21} = -J_0 - \text{Re} J_2 + \frac{1}{2} \left[ A_1 - A_2 \right], \]  

and
\[ M_{22} = 1 - \text{Re} J_0 - \frac{1}{2} \left[ J_0^2 - |J_2|^2 - A_3 \right]. \]  

(39)

Here we have introduced the notation of Ref. 2 wherein:
\[ J_0 = \int_0^{2\pi} d\tau q(\tau), \quad J_2 = \int_0^{2\pi} d\tau q(\tau) e^{i2\tau} = \text{Re} J_2 + i \text{Im} J_2, \]  

(40)

\[ A_1 = \int_0^{2\pi} d\tau q(\tau) \int_0^{2\pi} d\sigma q(\sigma) \sin 2(\tau - \sigma), \]  

(42)

\[ A_2 = -\int_0^{2\pi} d\tau q(\tau) \int_0^{2\pi} d\sigma q(\sigma) \left[ \sin 2\tau - \sin 2\sigma \right], \]  

(43)

and
\[ A_3 = -\int_0^{2\pi} d\tau q(\tau) \int_0^{2\pi} d\sigma q(\sigma) \left[ \cos 2\tau - \cos 2\sigma \right]. \]  

(44)

Since the matrix \( M \) is unimodular, we can write the phase-space invariant characterizing the motion at \( \Phi = 0 \) as
\[ I = e_{\mu\nu} M_{\nu\sigma} \eta_\sigma, \quad \mu, \nu, \sigma = 1, 2 \]  

(45)

where \([\eta_1, \eta_2] = [\eta, \eta'], \) and \( e \) is the antisymmetric matrix with \( e_{12} = 1. \) To first order this gives
\[ I_1 = (J_0 - \text{Re} J_2) \eta - 2(\text{Im} J_2) \eta' - (J_0 - \text{Re} J_2) \eta'. \]  

(46)

For stable motion the phase plane trajectory must be an ellipse, that is,
\[ |J_2|^2 - J_0^2 < 0. \]  

(47)

If \( J_2 = 0 \) and \( J_0 \neq 0, \) the motion is stable. How-
ever, if both \( J_2 \) and \( J_0 \) are zero, the first order invariant vanishes identically, and the motion away from the resultant cyclic orbit may be stable or unstable. Under these circumstances the stability is determined by the second order invariant, \( I_2 \), given by

\[
I_2 = \frac{1}{2} \left( A_1 - A_2 \right) \frac{\Theta^2}{A_3} + A_3 \frac{\Pi^2}{\left( A_1 + A_2 \right)^2} + \frac{1}{2} \left( A_1 + A_2 \right) \frac{\varphi^2}{A_3}. \tag{48}
\]

Thus, the deviations from the cyclic orbit on successive cycles are stable when

\[
A_3 - \frac{A_1^2}{A_2} < 0,
\tag{49}
\]

but unstable when

\[
A_3 - \frac{A_1^2}{A_2} > 0.
\tag{50}
\]

In summary, to establish stability at a critical \( \nu \) value when the conditions for a cyclic orbit pertain, it is necessary to satisfy the second order condition on the perturbation, Eq. (49).

V. Correction of a Point Gradient Perturbation in the AGS

We have shown in the previous sections that a corrected machine for which particle orbits are cyclic to first order at a critical value may, in fact, be either stable or unstable to second order. To illustrate this, we consider two examples both having \( J_0 = 0 \) and \( J_2 = 0 \) at \( \nu = 8.5 \), one stable to second order and the other unstable. In the first of these, see Table I, we have used four quadrupoles at C-17, F-17, I-17, and L-173 to correct a point perturbation at A-13. In Fig. 1 we see that the resulting phase space (\( \Theta, \Pi \)) trajectory at straight section L-20 (\( \varphi = 0 \)) is an ellipse, Eq. (48). Thus the motion is stable. Using an analytical expression for the phase function, \( \Phi(s) \), of the AGS, as given in this figure, we have also evaluated the particle coordinates on successive revolutions by means of a point-to-point matrix calculation. These exact coordinates for various revolution numbers are indicated by crosses (+) in the figure. As can be seen, the true motion is well described by the second order ellipse. In the second example (see Table II and Fig. 2), a gradient perturbation at A-17 is corrected to first order in the same manner as above (\( J_0 = J_2 = 0 \)), but we now find that the resulting motion is unstable to second order and the phase-space trajectory is a hyperbola. The difference between the second order theory and the exact matrix calculation in this unstable case arises because, in contrast to the stable case, the deviations from the cyclic orbit become large as the number of revolutions increases. Thus at 360 revolutions \( \varphi \) at L-20 has increased from its initial zero value to a value of 115 in. This corresponds to a maximum excursion in the machine of approximately 3.5 in, as compared to an initial excursion of 0.1 in.

In conclusion we can say that for small deviations from the cyclic orbit, the theory developed in this paper can predict the behavior of a particle in a machine corrected to a given order and either stable or unstable to the next order.

References

### TABLE I

Gradient perturbation at straight section A-13 corrected to first order
\((J_0 = J_2 = 0)\), stable to second order \((- A_1^2 + A_2^2 + A_3^2 = - 0.0004615 < 0)\);
\(\nu = 8.5000; H_0 = 6.907 \times 10^5\) gauss-in. \(\psi\) at \(L-20 = \psi(0) = 0\);
\(\beta(0) = R/\nu = 594.97\) in.

<table>
<thead>
<tr>
<th>Straight Section</th>
<th>(\beta), inches</th>
<th>(\psi), radians</th>
<th>Integrated Quad. Strength</th>
<th>(G = (k\beta) \theta) ((H\psi))</th>
<th>Re ((Ge^{12}\psi))</th>
<th>Im ((Ge^{12}\psi))</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-13</td>
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</table>

\(J_0 = \sum G_j = 0\)
\(A_1 = \sum G_j G_j \sin 2(\psi_j - \psi_j) = - 0.03183\)
\(J_2 = \sum G_0 e^{12\psi_j} = 0\)
\(A_2 = - \sum G_j G_j \left(\sin 2\psi_j - \sin 2\psi_j\right) = 0.01899\)
\(A_3 = - \sum G_j G_j \left(\cos 2\psi_j - \cos 2\psi_j\right) = 0.01383\)

Ellipse: \(-0.8543 \times 10^{-6} = -0.05081 \, \bar{\eta}^2 + 0.02765 \, \bar{\eta} \bar{\bar{\eta}} - 0.01284 \, \bar{\eta}^4\)

### TABLE II

Gradient perturbation at straight section A-17 corrected to first order
\((J_0 = J_2 = 0)\), unstable to second order \((- A_1^2 + A_2^2 + A_3^2 = + 0.0015230 > 0)\);
\(\nu = 8.5000; H_0 = 6.907 \times 10^5\) gauss-in. \(\psi\) at \(L-20 = \psi(0) = 0\);
\(\beta(0) = R/\nu = 594.97\) in.

<table>
<thead>
<tr>
<th>Straight Section</th>
<th>(\beta), inches</th>
<th>(\psi), radians</th>
<th>Integrated Quad. Strength</th>
<th>(G = (k\beta) \theta) ((H\psi))</th>
<th>Re ((Ge^{12}\psi))</th>
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\(J_0 = 0;\)
\(A_1 = 0.02892\);  
\(J_2 = 0;\)
\(A_2 = -0.04317\);  
\(A_3 = +0.02227\).

Hyperbola: \(+0.1211 \times 10^{-5} = 0.07208 \, \bar{\eta}^2 + 0.04454 \, \bar{\eta} \bar{\bar{\eta}} - 0.01425 \, \bar{\eta}^4\)