ELLIPTICAL AND CIRCULAR CURRENT SHEETS TO PRODUCE A PRESCRIBED INTERNAL FIELD*

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Summary

Any desired two-dimensional magnetic field, which is compatible with Maxwell's equations in empty space, can be produced inside a given elliptic or circular cylinder by a distribution of currents flowing along the elements of the cylinder. Formulas are given for the required current distribution, the resulting external field, and for the total field energy in the internal and external regions.

It is further possible to eliminate the external or stray field completely while producing the prescribed inner field by adding a second confocal elliptic (or coaxial circular) current sheet. Formulas are given for the required current distributions.

To facilitate their use the formulas are here presented unencumbered by derivations but in a form which makes the correspondence between the elliptic and circular cases, as well as the specialization to median plane fields, appear as simply as possible.

Prescribed Field

The components of a two-dimensional magnetic field parallel to the x,y plane are real functions of the coordinates, $H_x(x,y)$ and $H_y(x,y)$. In empty space without currents Maxwell's equations,

$$\frac{\partial H_y}{\partial x} = \frac{\partial H_x}{\partial y} \quad \text{and} \quad \frac{\partial H_y}{\partial y} = -\frac{\partial H_x}{\partial x}$$

constitute Cauchy-Riemann equations which show that the complex combination of the components

$$H \equiv H_y(x,y) + i\, H_x(x,y) \qquad (1)$$

is an analytic function of the complex variable $z = x + iy$. We can therefore represent the most general non-singular field in the vicinity of the origin as a power series in z:

$$H(z) = H_1 + H_2 z + H_3 z^2 + \ldots = \sum_{n=1}^{\infty} H_n z^{n-1} \ . \qquad (2)$$

The complex coefficients, H_n , completely specify H(z). In particular, H_1 specifies the dipole component, H_2 the quadrupole, H_3 the sextupole, and, in general, H_n the 2n-pole component.

The x-axis represents a "median plane" when $H_x = 0$ for $y = 0$, that is, when H(z) is real for $z = x$. Hence all the coefficients, H_n , are real for a median plane field.

Elliptical or Circular Cylinder Current Sheet

To produce any prescribed field (2) within a given elliptic cylinder by currents flowing along the elements of the cylinder, we may use the following version of earlier results.[1]

Let a normal section of the cylinder be the ellipse

$$z = a\cos\theta + ib\sin\theta \qquad (3)$$

where a and b are the given semiaxes, and θ is a parameter which goes from 0 to 2π around the ellipse. For $a > b$ we define the real quantities:

$$c^2 = a^2 - b^2$$

$$r = \frac{a+b}{2} \qquad (4)$$

$$k = \frac{c}{2}$$

The foci of the ellipse lie at $z = \pm c$. Transition to the case of a circular cylinder of radius r implies $a \to b \to r$, $c \to 0$, $k \to 0$, and

$$z = r\, e^{i\theta} \ . \qquad (3')$$

From the coefficients, H_n , of the prescribed field we compute the complex values of

$$F_m = \sum_{n=m}^{\infty} D_{mn}\, k^{n-m}\, H_n \qquad (5)$$

using the D_{mn} values given in Table I. By means

TABLE I
Values of D_{mn} in Equation (5)

n =	1	2	3	4	5	6	7	8
m = 1	1	0	1	0	2	0	5	0
2		1	0	2	0	5	0	14
3			1	0	3	0	9	0
4				1	0	4	0	14
5					1	0	5	0
6						1	0	6
7							1	0
8								1

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of the recursion $D_{mn} = D_{m-1,n-1} + D_{m+1,n-1}$ together with $D_{mn} = 0$ for $m < 1$ and for $m > n$, Table I can be extended indefinitely. Note that $D_{mm} = 1$ and $D_{mn} = 0$ for $m + n$ = odd. Thus, for example,

$$F_1 = H_1 + k^2H_3 + 2k^4H_5 + 5k^6H_7 + \ldots$$

$$F_2 = H_2 + 2k^2H_4 + 5k^4H_6 + 14k^6H_8 + \ldots$$

etc.

and, for the circular case, $k = 0$,

$$F_m = H_m \quad . \tag{5'}$$

Let dI be the upward current in the elements of the cylinder (3) lying between the parameter values θ and $\theta + d\theta$. Then the current distribution required to produce the field (2) within the cylinder is given by

$$\frac{dI}{d\theta} = -\frac{1}{4\pi}\sum_{m=1}^{\infty} r^m \left(F_m e^{im\theta} + F_m^* e^{-im\theta} \right) \tag{6}$$

where F_m^* is the complex conjugate of F_m. For the case of a median plane field, $F_m = F_m^*$, and (6) becomes

$$\frac{dI}{d\theta} = -\frac{1}{2\pi}\sum_{m=1}^{\infty} r^m F_m \cos m\theta$$

as is well known, at least for the circular case, (5′).[2]

External Field

With the current distribution (6) the field external to the elliptic cylinder is

$$H_{out}(z) = \left[-\sum_{m=1}^{\infty} f_m \left(\frac{2}{z + \sqrt{z^2 - c^2}} \right)^m \right] \Big/ \sqrt{z^2 - c^2} \tag{7}$$

where

$$f_m = k^{2m} F_m + r^{2m} F_m^* \quad . \tag{8}$$

For the circular case $c = 2k = 0$ and $F_m = H_m$, so that expression (7) reduces to

$$H_{out}(z) = -\sum_{m=1}^{\infty} \frac{r^{2m} H_m^*}{z^{m+1}} \quad . \tag{7'}$$

Field Energy

The energy E stored in space, per unit thickness of the two-dimensional field, can be evaluated in closed form by methods previously described.[3]

In units which make the energy density $(H_x^2 + H_y^2)/8\pi$ at any point, we find, for the regions inside and outside the general elliptical current sheet

$$E_{in} = \frac{1}{8}\sum_{m=1}^{\infty}\left[r^{2m} - \left(\frac{k^2}{r}\right)^{2m} \right] F_m F_m^* \Big/ m$$

$$= \frac{1}{8}\sum_{m=1}^{\infty}\left[(a+b)^{2m} - (a-b)^{2m} \right] F_m F_m^* \Big/ (2^{2m}\, m) \tag{9a}$$

and

$$E_{out} = \frac{1}{8}\sum_{m=1}^{\infty} f_m f_m^* / (r^{2m}\, m) \tag{9b}$$

where F_m and f_m are given by (5) and (8), respectively. For the circular case, $k = 0$, $f_m = r^{2m}F_m^*$ and $F_m = H_m$; hence

$$E_{in} = E_{out} = \frac{1}{8}\sum_{m=1}^{\infty} H_m H_m^* r^{2m} \Big/ m \quad . \tag{9'}$$

Two Cylinders with Zero External Field

It is possible to produce the prescribed field (2) within the inner of two confocal elliptic cylinders (or, of two coaxial circular cylinders) and, simultaneously, to cancel the field in the whole region outside of both cylinders.[4]

Denote quantities relating to the inner and outer cylinders by single and double primes, respectively. For confocal cylinders k is the same. With the F_m computed from the prescribed field as in (5) we find the two required current distributions in the form (6) by setting

$$F'_m = (r''^{2m} F_m + k^{2m} F_m^*) \,/\, \Delta_m$$
$$F''_m = -(r'^{2m} F_m + k^{2m} F_m^*) \,/\, \Delta_m \tag{10}$$

where $\Delta_m = r''^{2m} - r'^{2m}$.

For the interior field we have $F'_m + F''_m = F_m$ and for the respective exterior fields (7) we find from (8)

$$f'_m = -f''_m = \frac{k^{2m}(r''^{2m} + r'^{2m})F_m + (k^{4m} + r'^{2m} r''^{2m})F_m^*}{\Delta_m} \tag{11}$$

so that superposition yields zero for the field exterior to both cylinders. For coaxial circular cylinders, $k = 0$, we have simply

$$F'_m = r''^{2m} F_m \,/\, \Delta_m$$
$$F''_m = -r'^{2m} F_m \,/\, \Delta_m \tag{10'}$$

and

$$f'_m = -f''_m = r'^{2m} r''^{2m} F_m^* \,/\, \Delta_m \quad . \tag{11'}$$

The field between the cylinders can also be obtained by appropriate superposition. Note that for $r'' \to \infty$ all the double cylinder expressions reduce to the single cylinder case.

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Practical methods of constructing cylindrical current sheets are illustrated by the superconducting quadrupoles built at Brookhaven by Sampson and Britton.[5]

References

1. R.A. Beth, "Fields Produced by Elliptical Current Sheets," BNL Accelerator Dept. Internal Report AADD-112, July 6, 1966; see also AADD-102, 103 and 110.
2. For example, W.R. Smythe, "Static and Dynamic Electricity," p. 279 (McGraw-Hill, New York, 1950).
3. R.A. Beth, "Stored Energy and Inductance in Two-Dimensional Fields," BNL Accelerator Dept. Internal Report AADD-106, May 20, 1966; "Elliptical Current Sheets to Produce Constant Gradient Fields," BNL Accelerator Dept. Internal Report AADD-110, June 10, 1966.
4. R.A. Beth, "Combination of Cylindrical Current Sheets to Cancel External Field," BNL Accelerator Dept. Internal Report AADD-103, March 29, 1966; AADD-110, in Ref. 3; AADD-112, Ref. 1.
5. R.B. Britton and W.B. Sampson, "Superconducting Beam Handling Equipment," to be published in the Proceedings of this Conference.