

SMITH-PURCELL RADIATION FROM A CHARGE MOVING ABOVE A FINITE-LENGTH GRATING*

A. S. Kesar, S. Korbly, R. J. Temkin, MIT/PSFC, Cambridge, MA, U.S.A.
 M. Hess, IUCF, Bloomington, IN, U.S.A.

Abstract

Smith-Purcell radiation (SPR), generated by an electron beam traveling above a grating, is characterized by a broad range of frequencies in which the radiated wavelength depends on the angle of observation according to the SPR resonance relationship. A rigorous theoretical model of SPR from a three-dimensional bunch of relativistic electrons passing above a grating of finite length is presented by an electric-field integral equation method. The finite-length grating results are compared with an infinitely-long grating assumption in which periodic boundary conditions are rigorously applied and with a model based on the image charge approximation. While the SPR resonance relationship is the same in all three formalisms, significant errors up to an order of magnitude in the strength of the radiated energy are introduced by the two approximations. Numerical examples are calculated for an ~ 18 MeV bunch traveling above different finite length gratings with a period of 2.5 mm.

INTRODUCTION

Smith-Purcell radiation (SPR) [1], typically formed by a charge passing above a periodic grating, is characterized by a broad spectrum of frequencies. The resonance relationship correlates the n th harmonic of the radiated wavelength λ to the spatial observation angles θ and ϕ by

$$\lambda = \frac{D_g}{n}(\beta^{-1} - \sin \theta \sin \phi), \quad (1)$$

where D_g is the grating period and $\beta = v_x/c = (1 - \gamma^{-2})^{1/2}$ is the relativistic bunch velocity as shown in Fig. 1. The components of the wave number $k = \omega/c$ are $k_x = k \sin \phi \sin \theta$, $k_y = k \cos \phi$, and $k_z = k \sin \phi \cos \theta$, where ω is the angular frequency.

The SPR, generalized as Čerenkov radiation, is caused by diffraction of the charge free-space evanescent waves from the grating [2]. Under the condition of an infinitely long, periodic grating an exact model was derived by van den Berg in which integral equations having periodic Green's functions are excited by the charge wake fields [3].

Based on the image-charge approximation, an induced surface current model was presented by Brownell *et al.* for

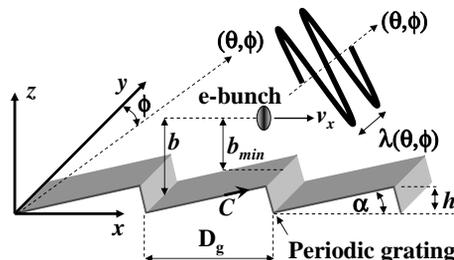


Figure 1: The SPR scheme. An electron bunch is traveling at an axial velocity v_x above an echelle grating of period D_g .

an arbitrary shaped grating [4]. This model has the advantage of $\sim 10^4$ shorter calculation time compared to the integral equation method.

Time- and frequency-domain models of SPR by a two-dimensional bunch moving above a finite-length grating were derived in [5]. A very good agreement was obtained between these models. It was shown that the finite length of the grating has to be taken into account in most experiments and that in the limit of an infinitely long grating the results are consistent with van den Berg's line of charge model [6].

Exact calculation of the radiated energy is important for terahertz generation as well as for bunch-length diagnostics. Bunch-lengths of 1.0 ± 0.1 ps and 0.6 ± 0.1 ps were evaluated at different accelerator operating parameters by measuring the SPR patterns from 15 MeV bunches [7, 8]. On a relative scale, the measured radiation patterns agree with the calculations using the electric field integral equation (EFIE) model in [5]. The measured bunch lengths agree very well with independent measurements by a circularly polarized deflector [9, 10].

In this paper we extend the two-dimensional EFIE model in [5] to the general case of a three-dimensional bunch moving above a finite-length grating and compare the radiated energy per groove calculated by the finite-length grating, van den Berg, and the image-charge models.

FINITE-LENGTH GRATING EFIE

Assuming a metal grating as a reflector, the EFIE correlating the unknown surface current $\mathbf{J}(\mathbf{r})$ and the charge free-space incident electric field tangent to the surface of

* Work supported by the Department of Energy, High Energy Physics, under contract DE-FG02-91ER40648.

the grating $\mathbf{E}^i(\mathbf{r})$ is

$$\begin{aligned} \mathbf{E}^i(\mathbf{r}) &= \frac{jZ_0}{k} (\nabla\nabla + k^2) \int_C \int_{-W/2}^{W/2} \mathbf{J}(\mathbf{r}') \\ &\quad \times \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} dy' dc' \\ &\simeq \frac{jZ_0}{k} (\nabla\nabla + k^2) \int_C \int_{-\infty}^{\infty} \mathbf{J}(\mathbf{r}') \\ &\quad \times \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} dy' dc', \end{aligned} \quad (2)$$

where the grating groove lines are parallel to the y direction and are of width W , the free-space impedance is Z_0 , a frequency dependence $e^{j\omega t}$ is assumed, and the observation and source points are $\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z}$ and $\mathbf{r}' = x'\hat{x} + y'\hat{y} + z'\hat{z}$, respectively. The integration path along the grating profile at the xz plane is denoted by C , as described in Fig. 1. Following the bunch transverse decay length, the approximation in Eq. (2) is valid for a grating of a sufficient width $W/2 \gg \beta\gamma/k$ and $W/2 \gg b$ where b is the average height of the bunch above the grating.

Applying the spatial Fourier operator $\int_{-\infty}^{\infty} e^{jk_y y} dy$ on Eq. (2) results, for $k_y < k$, in two coupled one-dimension electric-field integral equations, solved along the grating profile

$$\begin{bmatrix} \tilde{E}_x^i(x, z, k_y) \\ \tilde{E}_y^i(x, z, k_y) \end{bmatrix} = \frac{jZ_0}{k} \int_C \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} \tilde{J}_x \\ \tilde{J}_y \end{bmatrix} \tilde{G} dc'. \quad (3)$$

For a charged bunch having a transverse and longitudinal distribution function $f(x_0, y_0, z_0)$ travelling above the grating, i.e. $z_0 > h$ for all of the particles in the bunch where h is the grating height as shown in Fig. 1, the spatial electric field incident on the grating is

$$\begin{aligned} \tilde{\mathbf{E}}^i(x, z, k_y) &= \frac{qZ_0}{2\beta} e^{-j(k/\beta)x + k_a z} \\ &\quad \times \left[\hat{y} \frac{jk_y}{k_a} - \hat{z} + j\hat{x} \frac{k/\beta\gamma}{\gamma k_a} \right] F(k, x_0, y_0, z_0), \end{aligned} \quad (4)$$

where $k_a = \sqrt{(k/\beta\gamma)^2 + k_y^2}$ and the bunch form factor, $F(k, x_0, y_0, z_0) = \iint \int e^{j(k/\beta)x_0 + jk_y y_0 - k_a z_0} f(x_0, y_0, z_0) dx_0 dy_0 dz_0$, affects the coupling of the wake to the grating and produces the cutoff frequency. Throughout this paper the bunch longitudinal distribution is assumed to be Gaussian with a full width at half maximum (FWHM) of σ_x and the transverse distribution is assumed as a δ function at $y_0 = 0$ and $z_0 = b$, namely, $F(k, x_0, y_0, z_0) = \exp(-k_a b - (k/\beta)^2 \sigma_x^2 / 16 \ln 2)$.

The spatial electric field components tangent to the grating profile and along the grating grooves are $\tilde{E}_x^i(x, z, k_y) = \tilde{E}_x^i \cos(\alpha) + \tilde{E}_z^i \sin(\alpha)$ and $\tilde{E}_y^i(x, z, k_y)$, respectively, the corresponding spatial components of the surface current are $\tilde{J}_c = \tilde{J}_x \cos(\alpha') + \tilde{J}_z \sin(\alpha')$ and \tilde{J}_y , respectively, where α and α' are the observation and source angles tangent to the grating profile, respectively. The two-dimensional free-space Green's function is $\tilde{G} = (1/4j)H_0^{(2)}[k_\perp \sqrt{(x-x')^2 + (z-z')^2}]$ where

$k_\perp = \sqrt{k^2 - k_y^2}$ and the operators acting on it are

$$\begin{aligned} \mathcal{A} &= k^2 \cos(\alpha - \alpha') + \cos \alpha \cos \alpha' \partial_x^2 \\ &\quad + \sin \alpha \sin \alpha' \partial_z^2 + \sin(\alpha + \alpha') \partial_{xz} \\ \mathcal{B} &= -jk_y (\cos \alpha \partial_x + \sin \alpha \partial_z) \\ \mathcal{C} &= -jk_y (\cos \alpha' \partial_x + \sin \alpha' \partial_z) \\ \mathcal{D} &= k_\perp^2. \end{aligned}$$

The special case of $k_y = 0$ results in $\tilde{E}_y^{inc} = 0$ and Eq. (3) is reduced to \tilde{J}_c as a function of \tilde{E}_c^{inc} , in agreement with the two-dimensional TE_y polarized SPR described in [5].

The unknown surface currents were solved by dividing the grating profile into N straight segments, approximating a piecewise constant current in each one, and solving a set of $2N$ linear equations. The far-field vector potential approximated for a large argument $\mathbf{r} \gg \mathbf{r}'$ is

$$\mathbf{A}^{far}(r, \theta, \phi) \simeq \frac{e^{-jkr}}{4\pi r} \int_C \tilde{\mathbf{J}}(x', z', k_y) e^{jk_x x' + jk_z z'} dc', \quad (5)$$

where $k_x = k \sin \phi \sin \theta$, $k_y = k \cos \phi$, $k_z = k \sin \phi \cos \theta$, and $\tilde{\mathbf{J}}(x', z', k_y) = \hat{x} \tilde{J}_x \cos \alpha' + \hat{y} \tilde{J}_y + \hat{z} \tilde{J}_z \sin \alpha'$. The angular distribution of the n th order average radiated energy per groove is given by Parseval's theorem,

$$E_{AV}(\theta, \phi) = \frac{Z_0 r^2}{N_g \pi} \int_{0.5\omega_n}^{1.5\omega_n} |\mathbf{k} \times \mathbf{A}^{far}|^2 d\omega, \quad (6)$$

where $\omega_n = 2\pi n c / D_g (\beta^{-1} - \sin \theta \sin \phi)$.

Table 1: Smith-Purcell radiation parameters

| | |
|-------------------------------------|-------------|
| Bunch charge q | 50 pC |
| Bunch relativistic factor γ | 36 |
| Height above the grating, b_{min} | 0.6 mm |
| Bunch length σ_x | 300 μ m |
| Grating period D_g | 2.5 mm |
| Blaze angle α | 30° |
| Number of periods, N_g | 20 |

NUMERICAL EXAMPLE

A numerical example of SPR from a finite-length echelle grating is presented in the following section. The bunch and grating parameters are as listed in Table 1, unless specified otherwise.

The average first-order radiated energy per groove by the 20-period grating is shown in Figs. 2(a) and 2(b) for angular ranges of $-60^\circ \leq \theta \leq 60^\circ$, $\phi = 90^\circ$ and $\theta = 0^\circ, 30^\circ \leq \phi \leq 90^\circ$, respectively, by a solid line. This energy is compared to the first-order radiated energy per groove calculated by van den Berg's model for an infinitely long grating [3] (dotted line) and by the surface current model which is based on the image-charge approximation [4] (dashed line).

It is seen in Fig. 2(a) that an order of magnitude error is introduced by either the infinite-length assumption or

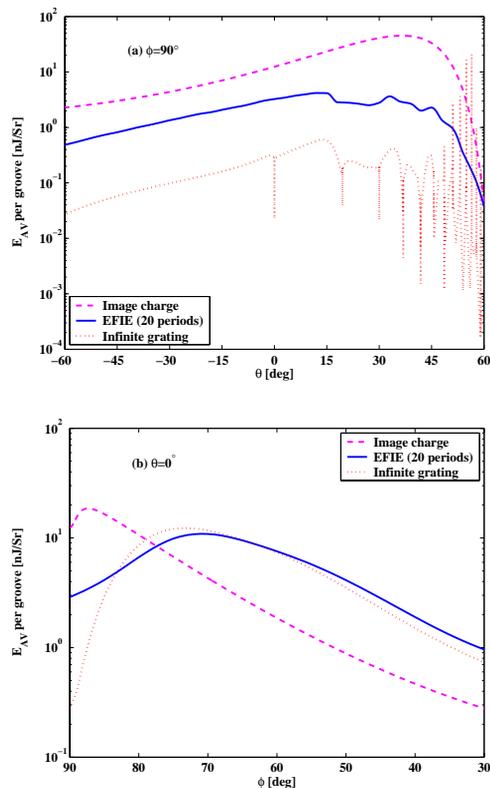


Figure 2: Comparison of average first-order radiated energy per groove by the 20-period grating (solid line) with the energy per groove by the image-charge model (dashed line) and by the infinitely long grating assumption (dotted line). The energies are plotted versus θ when $\phi = 90^\circ$ (a) and versus ϕ when $\theta = 0^\circ$ (b)

by the image-charge approximation compared to the finite-length EFIE calculation. Fig. 2(b) shows that for $\phi < \sim 60^\circ$ (i.e. $k_y/k > \sim 0.5$), the difference between the energy per groove emitted from the 20-period and the infinitely-long grating becomes smaller.

In order to demonstrate convergence to the special case of an infinitely long, periodic structure by van den Berg's model, the minimum number of grooves N_{min} to provide less than 10% difference between the radiated energy per groove by the finite-length and the infinitely long grating was calculated. It was found that N_{min} was strongly dependant on ϕ . For $\phi = 90^\circ$ ($k_y = 0$) it required more than 250 grooves, in agreement with the convergence described in [5]. However, for $\phi = 80, 70$, and 60 degrees, N_{min} was $\sim 100, 20$, and 10 , respectively.

DISCUSSION

The EFIE model in [5] was extended to the three-dimensional case of a bunch moving above a finite-length grating. The results were compared to those of an infinitely long grating in which periodic boundary conditions are assumed [3] and to those by the image charge approximation [4]. A considerable error in the strength of the radiated

energy per groove is introduced by either assumption, and especially for transverse angles $\phi \sim 90^\circ$ in which a maximum coupling occurs between the bunch wake and the radiated fields by the surface currents. The results by the EFIE model approach those by van den Berg [3] when N_g is increased, thus, indicating that the EFIE model is consistent with van den Berg's model for sufficiently long gratings.

The EFIE method could be used for improving the accuracy of SPR as a bunch-length diagnostic tool [7, 8]. Such improvement may be obtained by positioning a detector at an angle of maximum expected radiation. Calibrating the detector at a single wavelength corresponding to that observation angle may result in an alternative setup in which the radiation is measured on an absolute scale in order to determine the bunch length without any mechanical sweeping of observation angles as in [8]. Thus, this method may be simpler to implement and will provide a real time, nondestructive, measurement feature.

REFERENCES

- [1] S. J. Smith and E. M. Purcell, *Physical Review* 92 (1953) 1069.
- [2] G. Toraldo Di Francia, *Nuovo Cimento* 16 (1960) 61.
- [3] P. M. van den Berg, *Journal of the Optical Society of America* 63 (1973) 1588.
- [4] J. H. Brownell, J. Walsh, and G. Doucas, *Physical Review E* 57 (1998) 1075.
- [5] A. S. Kesar, M. Hess, S. E. Korbly, and R. J. Temkin, *Physical Review E* 71 (2005) 016501.
- [6] P. M. van den Berg, *Journal of the Optical Society of America* 63 (1973) 689.
- [7] S. E. Korbly, A. S. Kesar, J. R. Sirigiri, and R. J. Temkin, *Physical Review Letters* 94 (2005) 054803.
- [8] S. E. Korbly, A. S. Kesar, R. J. Temkin, and J. H. Brownell, *Physical Review ST-AB* (2005), submitted.
- [9] J. Haimson, B. Mecklenburg, G. Stowell, and B. Ishii, *AIP Conference Proceedings* 647 (2002) 810.
- [10] J. Haimson, *AIP Conference Proceedings* 737 (2004) 95.