ANALYTICAL STUDY OF ENVELOPE MODES FOR A FULLY DEPRESSED BEAM IN SOLENOIDAL AND QUADRUPOLE PERIODIC TRANSPORT CHANNELS

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Abstract

We present an analysis of envelope perturbations evolving in the limit of a fully space-charge depressed (zero emittance) beam in periodic, thin-lens focusing channels. Both periodic solenoidal and FODO quadrupole focusing channels are analyzed. The phase advance and growth rate of normal mode perturbations are analytically calculated as a function of the undepressed particle phase advance to characterize the evolution of envelope perturbations.

INTRODUCTION

The KV envelope equations for a fully depressed coasting beam with elliptical edge radii \( r_x = 2\sqrt{\langle x^2 \rangle}, r_y = 2\sqrt{\langle y^2 \rangle} \) aligned along the transverse \( x \) and \( y \) axes are [2, 3]

\[
r_{xy}(s) + \kappa_j r_j(s) - \frac{2Q}{r_x(s) + r_y(s)} = 0,
\]

where \( j \) ranges over \( x \) and \( y \), \( Q \) is the dimensionless beam perveance, and \( s \) is the axial coordinate. The equations (1) apply directly to a beam in a quadrupole focusing channel with \( \kappa_x = -\kappa_y \), but for solenoidal focusing one has to assume zero beam canonical angular momentum with \( \kappa_x = \kappa_y \) and interpret all results in a rotating Larmor frame[2, App. A]. The equations can be written in terms of scaled sum and difference coordinates \( R_{\pm} = (r_x \pm r_y)/(2\sqrt{2Q}) \) as

\[
2R''_+(s) + 2\kappa_x(s)R_-(s) - \frac{1}{R_+(s)} = 0, \tag{2a}
\]

\[
2R''_-(s) + 2\kappa_x(s)R_+(s) = 0, \tag{2b}
\]

for solenoidal focusing, and

\[
2R''_+(s) + 2\kappa_x(s)R_-(s) - \frac{1}{R_+(s)} = 0, \tag{2b}
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2R''_-(s) + 2\kappa_x(s)R_+(s) = 0
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for quadrupole focusing. In free drift regions \( \kappa_x(s) = \kappa_y(s) = 0 \), and the equations can be integrated by using constancy of envelope Hamiltonian

\[
R^2_+(s) - \ln R_+(s) = \text{const}
\]

\[
\ln \frac{R_+(0)}{R_+(s)} = \left( \frac{e^\kappa_0}{\sqrt{\pi R_+(0)}} \right)^2
\]

\[
R_-(s) = R_-(0) + sR'_-(0)
\]

where \( \text{erfi}(z) = \frac{2}{\sqrt{\pi}} \text{erf}(iz)/i \) is the imaginary error function.

Without loss of generality[2, Sec. II E], we assume that the length of the free drift interval between the two adjacent thin lenses is 2 as in Fig. 1. By symmetry we need only to consider the envelope evolution of the beam between two neighboring lenses only. We take the first lens to be at axial location \( s = -1 \) and the second one to be at \( s = 1 \). We also assume that in alternating gradient channel the second lens (at \( s = 1 \) is focusing in \( x \). Then for both thin lens solenoids and quadrupoles we take near \( s = 1 \)

\[
\kappa_x(s) = \frac{1}{s}(s-1),
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for solenoidal focusing, and

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2R''_+(s) + 2\kappa_x(s)R_-(s) - \frac{1}{R_+(s)} = 0, \tag{2b}
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\[
\kappa_x(s) = \frac{1}{s}(s-1),
\]
where \( f = \text{const} \) is the thin lens focal length and \( \delta(s) \) is the Dirac delta-function. The focal length \( f \) can be related to the undepressed particle phase advance over one lattice period \( \sigma_0 \) as [2, Sec. II D]

\[
\frac{1}{f} = \begin{cases} 
2 \sin^2 \sigma_0/2, & \text{soloidal focusing}, \\
\sin \frac{\sigma_0}{2}, & \text{quadrupole focusing}. 
\end{cases}
\]  

(6)

We analyze the perturbations of the envelope coordinate vector \( \mathbf{R}(s) \) by computing the Jacobian matrix \( \mathbf{M}(0,2) \) where \( \mathbf{M}(s_1,s_2) = \partial \mathbf{R}(s_2)/\partial \mathbf{R}(s_1) \) and derivatives are evaluated for a matched envelope. Since \( \mathbf{M}(0|2) \) is simplectic, then the first-order perturbations are stable if and only if all eigenvalues of \( \mathbf{M} \) lie on the unit circle \( |z| = 1 \).

In calculating \( \mathbf{M}(0|2) \), we henceforth denote \( \mathcal{F}(s+\delta) \equiv \lim_{\delta \rightarrow 0} \mathcal{F}(s+\delta) \) to represent the discontinuous action of the thin lenses on the beam envelope functions. To exploit lattice symmetries, we split the interval \((0,2)\) into three parts \((0,1-0), \,(1-0,1+0)\) and \((1+0,2)\), and calculate \( \mathbf{M}(0,2) \) as \( \mathbf{M}(0|2) = \mathbf{M}(1+0|2)\mathbf{M}(1-0|1+0)\mathbf{M}(0|1-0) \). By symmetry, \( \mathbf{M}(1+0|2) = \mathbf{M}(0|1-0) \). Thus,

\[
\mathbf{M}(0|2) = \mathbf{M}_f (-1+0)^{-1} \mathbf{M}_s \mathbf{M}_f (1-0),
\]  

(7)

where \( \mathbf{M}_s = \mathbf{M}(1-0|1+0) \) is the “singular Jacobian” associated with the thin lens focusing kick, and \( \mathbf{M}_f (s) = \mathbf{M}(0|s) \) for \( |s| < 1 \) is the “free drift Jacobian” associated with the half-drift.

To evaluate \( \mathbf{M}_s \), we consider the action of the thin lens according to Eqs. (2) and (5). We obtain

\[
\mathbf{M}_s = \begin{bmatrix} 1 & 0 & 0 & 0 \\
-\frac{1}{2} & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -\frac{1}{2} & 1 
\end{bmatrix}, \quad \mathbf{M}_s = \begin{bmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & -\frac{1}{2} & 0 \\
0 & 0 & -1 & 0 \\
\frac{1}{2} & 0 & 0 & -1 
\end{bmatrix}.
\]  

(8)

for solenoidal and quadrupole channels respectively.

To evaluate \( \mathbf{M}_f (s) \), the free expansion solutions in Eqs. (4) and the matched beam symmetry condition \( R'_{\pm}(0) = 0 \) are employed to evaluate Jacobian elements:

\[
\mathbf{M}_f (s) = \begin{bmatrix} \frac{R_+(s) - sR'_+(s)}{R_+(0)} & 2R_+(0)R'_+(s) & 0 & 0 \\
-\frac{2R_+(0)R'_+(s)}{R_+(s)} & \frac{R_+(0)}{R_+(s)} & 0 & 0 \\
0 & 0 & 1 & s \\
0 & 0 & 0 & 1 
\end{bmatrix}.
\]  

(9)

To complete the evaluation of \( \mathbf{M}_f (1-0) \), we find relations of the elements to \( \sigma_0 \) by deriving equations connecting \( R_+(1-0) \equiv R_+(1), \, R'_+(1-0), \) and \( R_+(0) \) to these quantities for the matched beam envelope. By symmetry, for a periodic, matched envelope

\[
R'_+(1-0) = -R'_+(1+0),
\]  

(10)

For solenoids, Eqs. (2a) and (5) can be integrated once about \( s = 1 \) to obtain

\[
R_+(1+0) = R_+(1-0) - \frac{1}{f} R_+(1).
\]

Combining these constraints with the matching conditions (10), we get

\[
R'_+(1-0) = \frac{1}{2f} R_+(1).
\]  

(11)

Similarly, using Eqs. (2b) and (5) for alternating gradient focusing and matched beam symmetries (10), we obtain

\[
R'_+(1-0) = \frac{1}{2f} R_+(1).
\]  

(12)

The solenoidal and quadrupole matching conditions in Eq. (12) for \( R_+ \) can be expressed as

\[
\hat{k} R_+(1) = 2R'_+(1-0),
\]  

(13)

where \( \hat{k} = \frac{1}{f} = 1 - \cos \sigma_0 \), solenoidal focusing, and \( \hat{k} = \frac{1}{2f} = \frac{1}{1 - \cos \sigma_0} \), quadrupole focusing.

Applying Eqs. (3) between \( s = 0 \) and \( s = 1 - 0 \) with the matched beam condition \( R'_+(0) = 0 \) leads to

\[
R_+(1) = R_+(0)e^{R'_+(1-0)}.
\]  

(14)

Using Eqs. (13) and (14) in Eq. (4) then yields

\[
\hat{k} = 2 \sqrt{\pi} e^{-R'_+(1-0)} R'_+(1-0) \text{erfi} R'_+(1-0).
\]  

(15)

FIG. 2: Phase advances \( \sigma_{\pm} \) and growth factors \( \gamma_{\pm} \) for the breathing and quadrupole modes for a thin-lens solenoidal focusing channel and a fully depressed beam. Continuous focusing model predictions for \( \sigma_{\pm} \) are superimposed (dashed curves).
Equations (13)–(15) provide the needed constraints to relate the elements of \( \mathbf{M}_f(1) \) to \( \sigma_0 \). Elements of \( \mathbf{M}_f(-1) \) can be calculated from these constraints using the matched beam symmetries
\[
R_+(-1) = R_+(1), \quad R'_+(-1+0) = -R'_+(1-0). \quad (16)
\]

For solenoidal focusing \( R_\pm \) are uncoupled, and \( \mathbf{M}(0|2) \) is of block diagonal form with \( \mathbf{M}(0|2) = \begin{bmatrix} \mathbf{M}_+^{(0|2)} \ \mathbf{0} \\ \mathbf{0} \ \mathbf{M}_-^{(0|2)} \end{bmatrix} \), where \( \mathbf{M}_\pm^{(0|2)} \) are \( 2 \times 2 \) symplectic matrices that can be independently analyzed for the stability of perturbations. We compute \( \mathbf{M}_\pm^{(0|2)} \) from Eq. (7):
\[
\mathbf{M}_+(0|2) = \begin{bmatrix}
\frac{R_+(1)+R_+′(1+0)}{0-2R_+(0)}, \\
\frac{R_+(0)}{2R_+(0)R_+(1)}
\end{bmatrix}^{-1}
\begin{bmatrix}
1 \quad 0 \\
-\frac{1}{2} \quad 1
\end{bmatrix}
\begin{bmatrix}
\frac{R_+(1)-R_+′(1-0)}{0-2R_+(0)}, \\
\frac{R_+(0)}{2R_+(0)R_+(1)}
\end{bmatrix}.
\]
\[
\mathbf{M}_-(0|2) = \begin{bmatrix}
\cos \sigma_0 - 4R_+^2(1-0)\cos^2(\frac{\sigma_0}{2}) \\
-1+\cos \sigma_0
\end{bmatrix}.
\]

Eigenvalues \( \lambda_\pm \) of the matrices \( \mathbf{M}_\pm^{(0|2)} \) are
\[
\lambda_+ = \cos \sigma_0 - 4R_+^2(1-0)\cos^2(\frac{\sigma_0}{2}) \pm 2i\cos(\frac{\sigma_0}{2}),
\]
\[
\lambda_- = \cos \sigma_0 \pm i\sin \sigma_0.
\]

Real-valued mode phase advances \( \sigma_\pm \) and growth factors \( \gamma_\pm \) per lattice period satisfy \( \lambda_\pm = \gamma_\pm e^{i\sigma_\pm} \). With proper branch selection[2] we get
\[
\begin{align*}
\sigma_+ &= \arg \lambda_+ \text{ with } + \text{ sign in Eq. (18)}, \\
\sigma_- &= \sigma_0, \quad (19)
\end{align*}
\]

and growth factors as
\[
\gamma_+ = \begin{cases}
1, & \text{stable,} \\
\sqrt{2 + 2[2\cos \sigma_0 - 4R_+^2(1-0)\cos^2(\frac{\sigma_0}{2})]^2 - 1}, & \text{unstable,}
\end{cases}
\gamma_- = 1.
\]

These solutions are plotted in Fig. 2 as a function of \( \sigma_0 \).
The extent of the band of instability (\( \gamma_+ \neq 0 \)) in \( \sigma_0 \) can be calculated from \( \gamma_+ \) directly as
\[
\sigma_0 \in \arg \left[[2\arccos\left(1 - 2\sqrt{\frac{\pi}{e} \text{ erfi}\left(\frac{1}{\sqrt{2}}\right)}\right), 0]\right] \approx [116.715^\circ, 180^\circ].
\]

The stability of quadrupole focusing can be investigated analogously except that we must work with the full \( 4 \times 4 \) Jacobian matrix \( \mathbf{M}(0|2) \). After multiplying out the matrices in Eq. (7) and calculating the eigenvalues using the constraints in Eqs. (12)–(15) yields
\[
\lambda = w - \frac{1}{2}k \pm i\sqrt{w + k - [1 + \frac{1}{2}k][k + 8R_+^2(1-0)]}, \quad (20)
\]
where \( w = \pm \sqrt{[1 + \frac{1}{2}k][1 + \frac{1}{2}k - 8R_+^2(1-0)]} \) and \( k \) is given by Eq. (13). These eigenvalues can be employed to calculate phase advances \( (\sigma_+ \text{ and } \sigma_-) \) and growth factors \( (\gamma_B \text{ and } \gamma_Q) \) of the breathing and quadrupole modes as
\[
\sigma_{B,Q} = 2\arg \lambda \text{ and } \gamma_{B,Q} = |\lambda|^2 \text{ (see Fig. 3). Using Eqs. (15) and Eq. (20) we find numerically that the instability band is located on the interval } \sigma_0 \in [121.055^\circ, 180^\circ].
\]

\( \text{REFERENCES} \)


\( \text{FIG. 3: Phase advance (}\sigma_Q\text{ and } \sigma_B\text{) and growth factors (}\gamma_Q\text{ and } \gamma_B\text{) for the breathing and quadrupole modes for a thinlens FODO quadrupole focusing channel and a fully depressed beam. Continuous focusing model predictions for } \sigma_\pm \text{ are superimposed (dashed curves).} \)