MAXWELL-LORENTZ EQUATIONS IN GENERAL FRENET-SERRET COORDINATES

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Abstract

We consider the trajectory of a charged particle in an arbitrary external magnetic field. A local orthogonal coordinate system is given by the tangential, curvature, and torsion vectors. We write down Maxwell's equations in this coordinate system. The resulting partial differential equations for the magnetic fields fix conditions among its local multipole components, which can be viewed as ageneralization of the usual multipol expansion of the fields of magnetic elements.

MOTIVATION

The problem at hand came about while implementing a generalized co-moving coordinate system in TraFiC4 [1], which, in the generalized case, is curvilinear and non-orthogonal.

The usual approach to describing the field content of magnetic elements in accelerator physics is the expansion in multipoles. This is based on the fact that the magnetic field in vacuum can be derived from a potential obeying Laplace's equation; assuming a symmetry along one axis, it reduces to the two-dimensional Laplace equation, which is solved by analytic functions of a complex variable.

Strictly speaking, this approach is only admissible in straight section, where the co-moving coordinate system is cartesian. In curved sections, one has to use curvilinear coordinates, and the Laplace operator changes its shape, leading to a different set of solutions.

Furthermore, the transformation to curvilinear changes the equation of motion, introducing inertial forces.

In this paper, we write down the Maxwell-Lorentz equations for the case of an external purely magnetic field in a coordinate system co-moving on the orbit induced by the external field. We describe that orbit in terms of its local curvature and torsion; the Laplace and Lorentz equations are given to all orders in this frame, Laplace's equation is solved for two special cases.

FRENET-SERRET COORDINATES

We consider the orbit particle in the usual description of an accelerator. Ignoring energy changes, its trajectory is completely determined by its initial conditions and the external magnetic field.

We use the arc length s to parametrize its trajectory $\vec{r}(s)$. Then, we define the usual local dreibein by the Frenet field frame:

$$\begin{aligned}
\vec{E}_s &:= \vec{t}(s) = \vec{r}'(s) \\
\vec{E}_x &:= \vec{n}(s) = \frac{1}{k}\vec{t}'(s) \\
\vec{E}_y &:= \vec{b}(s) = \vec{t}(s) \times \vec{n}(s)
\end{aligned}$$

where k is the curvature. We also introduce the torsion

$$w = \frac{1}{k^2}\det(\vec{r}', \vec{r}'', \vec{r}''')$$

We express all quantities in the coordinate system spanned by $\vec{E}_i$, i.e. a vector $\vec{R}$ is decomposed as $\vec{R}(s,x,y) = \vec{r}(s) + x\vec{n}(s) + y\vec{b}(s)$.

The magnetic field is scaled such that

$$\vec{t}' = \vec{t} \times \vec{B}$$

i. e., we absorb charge and energy into the field.

After some algebra, we find

$$\begin{aligned}
\bar{B}_y &= -k \\
\bar{B}_s &= -w \qquad (1) \\
\bar{B}_x &= 0
\end{aligned}$$

where the index labels components with respect to $\vec{E}_i$ and barred quantities are values on the trajectory.

THE EQUATIONS OF MOTION

We now look at a particle with trajectory $\vec{x}(t) = x(t)\vec{n}(s(t)) + y(t)\vec{b}(s(t)) + \vec{r}(s(t))$, where the coordinate s is implicitly defined by $\vec{t}(s(t)) \cdot (\vec{r}(s(t)) - \vec{x}(t)) = 0$. The equations of motion for the coordinates x, y, s read:

$$\begin{aligned}
\dot{x} &= v_x + yw\dot{s} \\
\dot{y} &= v_y - xw\dot{s} \\
\dot{s} &= \frac{v_s}{1-kx}
\end{aligned}$$

and

$$\begin{aligned}
\ddot{x} &= a_x + x(\dot{s}^2(w^2+k^2)) + 2\dot{y}w\dot{s} + yw\ddot{s} + (yw' - k)\dot{s}^2 \\
\ddot{y} &= a_y - 2\dot{x}w\dot{s} + yw^2\dot{s}^2 - xw'\dot{s}^2 - xw\ddot{s} \\
\ddot{s} &= \frac{a_s + 2\dot{x}k\dot{s} + (xk' - ywk)\dot{s}^2}{1-kx}
\end{aligned}$$

where v_i and a_i are the components of the particle's velocity and external acceleration.

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Putting in the Lorentz equation with magnetic fields B_i, we have

$$\begin{aligned}
a_x &= (\dot{y} + xw\dot{s})vB_s - \dot{s}(1-kx)vB_y \\
a_y &= (1-kx)vB_x - (\dot{x} - yw\dot{s})vB_s \\
a_s &= (\dot{x} - yw\dot{s})vB_y - (\dot{y} + xw\dot{s})vB_x
\end{aligned}$$

where $v = v_s^{\text{Orbit}}$. The expressions for the second derivatives of the coordinates contain centrifugal (proportional to x, y) and Coriolis (proportional to $\dot{x}, \dot{y}, \dot{s}$) and higher-order terms. They reduced to small quantities ($O(v - v_s), O(kx)$) by the Lorentz force for the values given in (1).

THE OFF-ORBIT MAGNETIC FIELD

We are now interested in the external components of the acceleration. They are given by the external magnetic field; they can, however, not be arbitrary as the magnetic field has to fulfill Maxwell's equations. To find the conditions for b_x, b_y, b_s, we write down the metric tensor for the curvilinear coordinate system defined above:

$$g_{ik} = \begin{pmatrix} h^2 + w^2(x^2+y^2) & -wy & wx \\ -wy & 1 & 0 \\ wx & 0 & 1 \end{pmatrix}_{ik}$$

(where $x_1 = s, x_2 = x, x_3 = y$ and $h = 1 - k(s)x$) with

$$\det g = (1-kx)^2$$

The inverse is

$$g^{ik} = h^{-2} \begin{pmatrix} 1 & wy & -wx \\ wy & h^2 + w^2y^2 & -w^2xy \\ -wx & -w^2xy & h^2 + w^2x^2 \end{pmatrix}^{ik}$$

The coordinate region we are interested is free of currents, so the magnetic field is the gradient of a potential Φ with $\Delta\Phi = 0$. We express Laplace's operator in curvilinear coordinates; we find the following Laplace operators for the cases of vanishing torsion and constant curvature; constant curvature and torsion; and arbitrary curvature and torsion, resp.:

$$\begin{aligned}
\Delta_{k,0} &= h^{-2}\partial_s^2 + \partial_x^2 + \partial_y^2 - k/h\partial_x \\
\Delta_{k(s),0} &= \Delta_{k,0} + xh^{-3}k'(s)\partial_s \\
\Delta_{k,w} &= \Delta_{k,0} + wh^{-3}\left(2[ky + h(y\partial_x - x\partial_y)]\partial_s + \right. \\
&\quad \left. w[-y\partial_y + h(x^2\partial_x^2 - 2xy\partial_x\partial_y + y^2\partial_y^2)]\right) \\
\Delta_{k(s),w(s)} &= \Delta_{k,w(s)} + xh^{-3}k'(s)\partial_s
\end{aligned} \tag{2}$$

Note that the s derivative of the torsion does not enter the equations.

The magnetic field is

$$B^i = g^{ik}\partial_k\Phi$$

so Φ has to fulfill the conditions

$$\begin{aligned}
\partial_s\Phi &= -w \\
\partial_x\Phi &= 0 \\
\partial_y\Phi &= -k
\end{aligned}$$

at the origin of the local system.

Obviously, the usual analyticity property of the solutions is lost for nonvanishing curvature.

Constant Curvature

Let us solve the constant curvature case first. We notice that the partial differential equation separates, i. e.

$$\Psi(s,x,y) = \psi_s(s)\psi_x(x)\psi_y(y)$$

We notice that $\psi(s) = const$ because of the vanishing torsion of the orbit. $\psi_y(y)$ has to be a function fulfilling

$$\psi_y''(y) = \mu\psi_y(y)$$

and

$$(1-kx)\psi_x''(x) - \mu(1-kx)\psi_x(x) - \psi_x'(x) = 0$$

which is solved by Bessel functions

$$\psi_x = c_1 J_0\left(\sqrt{-\frac{\mu}{k^2}}(1-kx)\right) + c_2 Y_0\left(\sqrt{-\frac{\mu}{k^2}}(1-kx)\right)$$

so

$$\begin{aligned}
\Psi(s,x,y) = \int \mathrm{d}\mu (c_{1,y}(\mu)e^{-\mu y} + c_{2,y}(\mu)e^{\mu y}) \\
\left[c_{1,x}(\mu)J_0\left(\sqrt{-\frac{\mu}{k^2}}(1-kx)\right)\right. \\
\left. + c_{2,x}(\mu)Y_0\left(\sqrt{-\frac{\mu}{k^2}}(1-kx)\right)\right]
\end{aligned}$$

This, of course, is just the well-known radial solution of Laplace's equation in cylinder coordinates [2].

Constant Curvature and Torsion

This case corresponds to an infinitely extending helical orbit. The differential equation is given by line 2 of (2). We make a separation Ansatz $\psi(s,x,y) = -ws + \varphi(x,y)$. The resulting differential equation for φ reads

$$\begin{aligned}
&\left(\partial_x^2 + \partial_y^2 - k/h\partial_x + \right. \\
&\quad w^2h^{-3}\left(2[h(x\partial_y - y\partial_x) - ky] + \right. \\
&\quad \left.\left. h(x^2\partial_x^2 - 2xy\partial_x\partial_y + y^2\partial_y^2)\right)\right)\varphi(x,y) = 0
\end{aligned}$$

This equation is not solvable in closed form (the coordinate system is not one of the ones in which the Laplace equation is known to be separable[2]), we can, however, obtain a recursion relation for coefficients in a power series

$\varphi(x,y) = \sum_{m,n} a_{m,n} x^m y^n$. After some tedious work, one obtains

$$\begin{gathered} a_{m-3,n+2}k(w^2+k^2)(n+1)(n+2)+ \\ -a_{m-2,2+n}(w^2+3k^2)(n+1)(n+2)+ \\ -a_{m-1,n}k\left(-k^2(m-1)^2+w^2(m-1)(2n+1)\right)+ \\ 3\,a_{m-1,2+n}k\left(n^2+3\,n+2\right)+ \\ a_{m,n}\left(k^2(-3m^2+2m)+w^2(m+n+2nm)\right)+ \\ -a_{m,2+n}(n+2)(n+1)+ \\ a_{1+m,-2+n}kw^2\left(m^2-1\right)+ \\ a_{1+m,n}k(3m+1)(m+1)+ \\ -a_{2+m,-2+n}w^2(m+1)(m+2)+ \\ -a_{2+m,n}(m+1)(m+2) = -2w^2k\delta_{m,0}\delta_{n,1} \end{gathered} \quad (3)$$

where the rhs term comes from the separation Ansatz. The inital conditions are $a_{0,0} = 0$ (a global gauge fixing), $a_{1,0} = 0$, $a_{0,1} = -k$. Note that there are special cases of the recursion relation for $m \leq 3$ or $n \leq 3$, they are (using the initial conditions)

$$\begin{gathered} a_{2,0}+a_{0,2} = \\ ka_{1,1}-2a_{2,1}-6a_{0,3}+w^2k = \\ -2a_{1,2}+8ka_{2,0}-6a_{3,0}+6ka_{0,2} = \\ 4w^2a_{1,1}+18ka_{0,3}+8ka_{2,1}-6a_{3,1}-2k^2a_{1,1}-6a_{1,3} = \\ -6k^2a_{0,2}+6ka_{1,2}-12a_{4,0}-2w^2a_{0,2}- \\ 10k^2a_{2,0}+2w^2a_{2,0}-2a_{2,2}+21ka_{3,0} = \\ -3kw^2a_{1,1}-18k^2a_{0,3}+21ka_{3,1}-6w^2a_{0,3}- \\ 12a_{4,1}+7w^2a_{2,1}-10k^2a_{2,1}-6a_{2,3}+k^3a_{1,1}+18ka_{1,3} = 0 \end{gathered}$$

and, for $p > 0$

$$\begin{gathered} w^2pa_{0,1+p}-kw^2a_{1,-1+p}+ka_{1,1+p}+w^2a_{0,1+p}- \\ 6a_{0,3+p}-5a_{0,3+p}p-a_{0,3+p}p^2-2w^2a_{2,-1+p}-2a_{2,1+p} = 0 \\ 3w^2pa_{1,1+p}+18ka_{0,3+p}+15ka_{0,3+p}p+3ka_{0,3+p}p^2+ \\ 8ka_{2,1+p}-6a_{1,3+p}-5a_{1,3+p}p-a_{1,3+p}p^2- \\ 6a_{3,1+p}+4w^2a_{1,1+p}-6w^2a_{3,-1+p}-2k^2a_{1,1+p} = 0 \\ -2kw^2pa_{1,1+p}-5w^2a_{0,3+p}p-w^2a_{0,3+p}p^2+ \\ 3ka_{1,3+p}p^2+15ka_{1,3+p}p-15k^2a_{0,3+p}p-3k^2a_{0,3+p}p^2+ \\ 5w^2pa_{2,1+p}-3kw^2a_{1,1+p}+3kw^2a_{3,-1+p}-a_{2,3+p}p^2- \\ 5a_{2,3+p}p-6w^2a_{0,3+p}-10k^2a_{2,1+p}+21ka_{3,1+p}+ \\ 18ka_{1,3+p}+7w^2a_{2,1+p}+k^3a_{1,1+p}- \\ 18k^2a_{0,3+p}-6a_{2,3+p}-12w^2a_{4,-1+p}-12a_{4,1+p} = 0 \end{gathered}$$

and

$$\begin{gathered} -kw^2pa_{1+p,0}+k^3a_{1+p,0}p^2+w^2a_{2+p,0}p+ \\ 16ka_{3+p,0}p+3ka_{3+p,0}p^2-kw^2a_{1+p,0}+2k^3pa_{1+p,0}+ \\ 2kw^2a_{-1+p,2}-3p^2k^2a_{2+p,0}-a_{4+p,0}p^2-7a_{4+p,0}p- \\ 2w^2a_{p,2}+6ka_{1+p,2}-10k^2a_{2+p,0}-6k^2a_{p,2}-12a_{4+p,0}+ \\ 2w^2a_{2+p,0}+k^3a_{1+p,0}+2k^3a_{-1+p,2}+21ka_{3+p,0}- \\ 11k^2pa_{2+p,0}-2a_{2+p,2} = 0 \\ -3kw^2pa_{1+p,1}+16ka_{3+p,1}p-3kw^2a_{1+p,1}+2k^3pa_{1+p,1}- \\ 11k^2pa_{2+p,1}-3p^2k^2a_{2+p,1}+6kw^2a_{-1+p,3}+k^3a_{1+p,1}p^2+ \\ 3w^2a_{2+p,1}p+3ka_{3+p,1}p^2-a_{4+p,1}p^2-7a_{4+p,1}p+ \\ 18ka_{1+p,3}-6w^2a_{p,3}+6k^3a_{-1+p,3}-10k^2a_{2+p,1}+ \\ k^3a_{1+p,1}-12a_{4+p,1}-18k^2a_{p,3}+ \\ 21ka_{3+p,1}+7w^2a_{2+p,1}-6a_{2+p,3} = 0 \\ -3kw^2a_{1,1}-18k^2a_{0,3}+21ka_{3,1}-6w^2a_{0,3}-12a_{4,1}+ \\ 7w^2a_{2,1}-10k^2a_{2,1}-6a_{2,3}+ \\ k^3a_{1,1}+18ka_{1,3} = 0 \end{gathered}$$

(3) can be viewed as a generalization of the multipole Ansatz for helical orbits. Of course, setting $w = k = 0$ we obtain the recursion relation valid for harmonic functions in x, y, namely $(n+1)(n+2)a_{m,2+n} + (m+1)(m+2)a_{2+m,n} = 0$
. Obviously, the complexity of the power series prescription limits its usefulness.

Planar Undulator

Here, we have vanishing torsion and a periodic curvature $k(s) = \kappa \exp(-i\lambda s)$. We substitute $k(s)$ for s and obtain, after separating off the harmonic y dependence as $\partial_y^2\psi = -\mu\psi$, where $\mu = 0$ due to the boundary condition $w = 0$

$$(-\lambda^2 k(\partial_k + hk\partial_k^2) + h^2\partial_x h\partial_x)\psi(k,x) = 0 \quad (4)$$

Again, a solution is only possible in terms of a power series in terms of x and k, similiar as above. We obtain the recursion relation

$$\begin{gathered} a_{m-1,n-3}(m-1)^2-a_{m-1,n-1}\lambda^2(n-1)(n-2)- \\ a_{m,n-2}m(3m-1)+\lambda^2a_{m,n}n^2+ \\ a_{m+1,n-1}(3m+1)(m+1)- \\ a_{m+2,n}(m+1)(m+2) = 0 \end{gathered} \quad (5)$$

, as well as special cases for small m, n and for initial conditions, which will not be given here for lack of space.

REFERENCES

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