A FUNDAMENTAL THEOREM ON PARTICLE ACCELERATION

Ming Xie, Lawrence Berkeley National Laboratory, Berkeley, CA 94720, USA

Abstract

A fundamental theorem on particle acceleration is derived from the reciprocity principle of electromagnetism and a rigorous proof of the theorem is presented. The theorem establishes a relation between acceleration and radiation, which is particularly useful for insightful understanding of and practical calculation about the first order acceleration in which energy gain of the accelerated particle is linearly proportional to the accelerating field.

INTRODUCTION

Near Field and Far Field.– Electromagnetic fields may be separated into two classes, near fields and far fields. By this separation, we define all radiation fields as far fields for they are capable of carrying electromagnetic energy far away from their sources, and all the rest, therefore, as near fields. In all conventional accelerators near fields have been used for particle acceleration. However, for laser driven particle acceleration, which is characterized by the use of electromagnetic fields with exceedingly smaller wavelengths, it is becoming imperative to make far fields our primary choice.

Linear Acceleration and Nonlinear Acceleration.– Energy coupling between a far field and a charged particle, and hence the mechanisms of laser acceleration, may be separated into two classes, first order or linear accelerations and nonlinear accelerations. By this separation, we define all acceleration processes in which energy gain of the accelerated particle is linearly proportional to the accelerating field as linear accelerations, and all the rest, therefore, as nonlinear accelerations. In all conventional accelerators linear accelerations have been used and proved effective and practical. For this reason, significant efforts of research on laser acceleration in the past have been dedicated to the understanding and implementation of linear accelerations with far fields. Through this collective experience a few rules of thumb on linear acceleration have been accumulated [1] and later summarized by Palmer [2].

Energy Conservation for Linear acceleration.– According to Palmer, linear acceleration is possible only if a particle would radiate in the absence of an accelerating field since the energy gain of the particle in the presence of an accelerating field, by energy conservation, should be proportional to the interference or cross-term between the accelerating field and the field radiated by the particle. Palmer argued by pointing out that all known mechanisms of linear acceleration are in one way or another based on inverse processes of radiation. From this point of view, it appears evident that linear acceleration is not possible in a field free vacuum since a particle moving at a constant velocity in vacuum would not radiate.

Objective of This Article.– Although the point of view of Palmer is physically insightful and the argument empirically compelling, a rigorous proof in a general and useful form, despite earnest effort by Zolotorev et al. [3], remains elusive. Earlier, I pointed out [4] that the apparently intimate relationship between acceleration and radiation can be established fundamentally and rigorously from the reciprocity principle of electromagnetism. In this article, I present the proof.

Reciprocity Principle.– The reciprocity principle of electromagnetism is rooted in a symmetry in the Maxwell's equations, a symmetry between two different solutions. The principle has been formulated into reciprocity theorems in many different forms for a wide range of applications. Of these the most well-known are perhaps the theorem derived by Lorentz and another one often attributed to Rayleigh-Carson. An extensive references on reciprocity theorems in electromagnetism can be found in [5].

Unique Situation of Particle Acceleration.– Yet, despite the great variety of reciprocity theorems in existence, none is applicable to the situation of particle acceleration that we are about to consider, for in which we have to deal with a current source of a point charge which in time spans over an infinite region of space. This intrinsic difference and its influence on the ways we establish a new reciprocity theorem will become self-evident later on.

Conclusions and Acknowledgments.– A comprehensive user instructions with examples on the reciprocity theorem proved here will be published elsewhere. Stimulating discussions with Max Zolotorev are acknowledged. This work was supported by the U.S. Department of Energy under contract No.DE-AC03-76SF00098.

PROOF OF THE THEOREM

Formulation of Reciprocity Relations.– Consider two independent solutions for a given system, field $\{\mathbf{E}_a, \mathbf{H}_a\}$ generated by source $\{\mathbf{J}_a, \rho_a\}$ and field $\{\mathbf{E}_p, \mathbf{H}_p\}$ generated by source $\{\mathbf{J}_p, \rho_p\}$, each satisfying the Maxwell's equations, respectively

$$\nabla \times \mathbf{E}_a = -\frac{\partial \mathbf{B}_a}{\partial t}\ , \quad \nabla \times \mathbf{H}_a = \frac{\partial \mathbf{D}_a}{\partial t} + \mathbf{J}_a \qquad (1)$$

$$\nabla \times \mathbf{E}_p = -\frac{\partial \mathbf{B}_p}{\partial t}\ , \quad \nabla \times \mathbf{H}_p = \frac{\partial \mathbf{D}_p}{\partial t} + \mathbf{J}_p \qquad (2)$$

Using identity $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{B}$, a relation between the two solutions in differential form can be obtained from Eq.(1) and Eq.(2)

$$\begin{aligned}\nabla\cdot(\mathbf{E}_a\times\mathbf{H}_p + \mathbf{E}_p\times\mathbf{H}_a)&\\ + P_{ap} + \mathbf{E}_a\cdot\mathbf{J}_p + \mathbf{E}_p\cdot\mathbf{J}_a &= 0\end{aligned} \tag{3}$$

where $P_{ap} = P_e + P_m$

$$P_e = \mathbf{E}_a\cdot\frac{\partial\mathbf{D}_p}{\partial t} + \mathbf{E}_p\cdot\frac{\partial\mathbf{D}_a}{\partial t} \tag{4}$$

$$P_m = \mathbf{H}_a\cdot\frac{\partial\mathbf{B}_p}{\partial t} + \mathbf{H}_p\cdot\frac{\partial\mathbf{B}_a}{\partial t} \tag{5}$$

Integrating Eq.(3) over a volume V from which $\mathbf{J}_a$ is excluded and over time, we obtain an integral relation

$$\begin{aligned}&\int_{-\infty}^{\infty}dt\int_S dS\hat{\mathbf{n}}\cdot(\mathbf{E}_a\times\mathbf{H}_p + \mathbf{E}_p\times\mathbf{H}_a)\\ &+\int_{-\infty}^{\infty}dt\int_V dV P_{ap} + \int_{-\infty}^{\infty}dt\int_V dV\mathbf{E}_a\cdot\mathbf{J}_p = 0\end{aligned} \tag{6}$$

where S is a surface enclosing V with a unit normal vector $\hat{\mathbf{n}}$ pointing outward from the enclosed region.

Four Conditions Defining the System.– In order to describe a situation that is generally applicable to particle acceleration, we shall make the following specifications and assumptions about the system. (I): Take $\mathbf{J}_p$ as the current density of a charged particle

$$\mathbf{J}_p = q\mathbf{v}_p(t)\delta[\mathbf{r} - \mathbf{r}_p(t)] \tag{7}$$

where $\mathbf{r}_p$ and $\mathbf{v}_p$ are the position and velocity of the particle along a trajectory which, by our assumption of the two solutions being independent, describes the motion unperturbed by the electromagnetic accelerating fields $\{\mathbf{E}_a, \mathbf{H}_a\}$. (II): Separate the unperturbed trajectory into three segments: for $t = \{-\infty, t_1\}$, $\mathbf{v}_p(t) = \mathbf{v}_1$; for $t = \{t_1, t_2\}$, $\mathbf{v}_p(t)$ may vary; and for $t = \{t_2, \infty\}$, $\mathbf{v}_p(t) = \mathbf{v}_2$; where $\mathbf{v}_1$ and $\mathbf{v}_2$ are constant velocities with which the particle enters and leaves a spatial region V_s within which the particle may interact with a passive environment of the system and radiate spontaneously as a result. (III): Assume that the interaction between the particle and the accelerating fields is confined in a finite region V_f beyond which the magnitude of the fields $\{\mathbf{E}_a, \mathbf{H}_a\}$ scales inversely proportional to the distance measured from within V_f. (IV): Define an interaction volume by $V_{int} = \max(V_s, V_f)$ and take the integration volume V to be sufficiently larger than V_{int}. Under these four conditions, Eq.(6) can be transformed into a transparent and convenient form in four steps.

Step 1: Relating the Third Term to Energy Gain.– Given Eq.(7), the third term in Eq.(6) becomes

$$\int_{-\infty}^{\infty}dt\int_V dV\mathbf{E}_a\cdot\mathbf{J}_p = q\int_{T_1}^{T_2}dt\mathbf{E}_a[\mathbf{r}_p(t), t]\cdot\mathbf{v}_p(t) \tag{8}$$

where T_1 and T_2 are the times at which the particle enters and leaves the volume V, respectively. Under the conditions (III) and (IV) that the accelerating fields diminish with sufficiently large V, Eq.(8) approaches the value of the accumulated energy gain or loss of the particle along the entire unperturbed trajectory

$$\Delta W_p \equiv q\int_{-\infty}^{\infty}dt\mathbf{E}_a[\mathbf{r}_p(t), t]\cdot\mathbf{v}_p(t)$$

Step 2: Eliminating the Second Term.– The second term in Eq.(6) can be eliminated altogether. Upon rewriting P_e and P_m in Eq.(4) and Eq.(5) as

$$P_e = \frac{\partial(\mathbf{E}_p\cdot\mathbf{D}_a)}{\partial t} + \mathbf{E}_a\cdot\frac{\partial\mathbf{D}_p}{\partial t} - \frac{\partial\mathbf{E}_p}{\partial t}\cdot\mathbf{D}_a \tag{9}$$

$$P_m = \frac{\partial(\mathbf{H}_p\cdot\mathbf{B}_a)}{\partial t} + \mathbf{H}_a\cdot\frac{\partial\mathbf{B}_p}{\partial t} - \frac{\partial\mathbf{H}_p}{\partial t}\cdot\mathbf{B}_a \tag{10}$$

it is noticed that the first terms on the RHS of Eq.(9) and Eq.(10) vanish after time integration, since $\mathbf{E}_p\cdot\mathbf{D}_a = 0$ and $\mathbf{H}_p\cdot\mathbf{B}_a = 0$ within V at $t = \pm\infty$, long before and after the particle enters and leaves the region. In addition, the second terms would cancel the third terms if the constitutive relations take the following form

$$\mathbf{D}(\mathbf{r}, t) = \epsilon(\mathbf{r})\mathbf{E}(\mathbf{r}, t), \quad \mathbf{B}(\mathbf{r}, t) = \mu(\mathbf{r})\mathbf{H}(\mathbf{r}, t)$$

which hold in vacuum and in isotropic but nondispersive medium. For more general situations with dispersive medium, consider the relations in frequency domain

$$\mathbf{D}_\omega(\mathbf{r}) = \epsilon(\mathbf{r}, \omega)\mathbf{E}_\omega(\mathbf{r}), \quad \mathbf{B}_\omega(\mathbf{r}) = \mu(\mathbf{r}, \omega)\mathbf{H}_\omega(\mathbf{r})$$

where the Fourier transform is defined by

$$F_\omega(\mathbf{r}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dte^{i\omega t}F(\mathbf{r}, t)$$

Then, it follows

$$\int_{-\infty}^{\infty}dtP_e(t) = i\int_{-\infty}^{\infty}d\omega\omega(\epsilon^* - \epsilon)\mathbf{E}_{\omega a}\cdot\mathbf{E}_{\omega p}^*$$

$$\int_{-\infty}^{\infty}dtP_m(t) = i\int_{-\infty}^{\infty}d\omega\omega(\mu^* - \mu)\mathbf{H}_{\omega a}\cdot\mathbf{H}_{\omega p}^*$$

For lossless medium, we have $\epsilon^* = \epsilon$ and $\mu^* = \mu$, thus

$$\int_{-\infty}^{\infty}dt\int_V dVP_{ap} = 0$$

Step 3: Removing a Singularity in the First Term.– The first term in Eq.(6) depends on the values of two sets of fields on the surface S far away from the interaction region V_{int}. The fields due to the particle have two parts, $\{\mathbf{E}_p, \mathbf{H}_p\} = \{\mathbf{E}_r, \mathbf{H}_r\} + \{\mathbf{E}_c, \mathbf{H}_c\}$, i.e., the fields spontaneously radiated within V_s by the particle interacting with the passive system, and the Coulomb fields of a point charge. Our goal is to prove that at any time t

$$\int_S dS\hat{\mathbf{n}}\cdot(\mathbf{E}_a\times\mathbf{H}_c + \mathbf{E}_c\times\mathbf{H}_a) = 0 \tag{11}$$

As the Coulomb fields are short ranged, $\{\mathbf{E}_c, \mathbf{H}_c\}$ have significant value on the surface only in the region near the points where the particle traverses the surface. Hence, without loss of generality, we define a local Cartesian coordinate system with its origin chosen at the point where the trajectory of the particle moving along the z-axis with a constant velocity $\mathbf{v}$ intersects the surface in the xy-plane.

The second term in Eq.(11) can be expressed as

$$\int_S dS\mathbf{E}_c \cdot (\mathbf{H}_a \times \hat{\mathbf{n}}) \tag{12}$$

where $\mathbf{H}_a \times \hat{\mathbf{n}} = H_a^s\ \hat{\mathbf{s}}$, H_a^s is the tangential component of the magnetic field and $\hat{\mathbf{s}}$ is a unit vector on the surface. Since the accelerating fields in the far zone must form a local plane wave which is either linearly or elliptically polarized, we may choose $\hat{\mathbf{s}} = \hat{\mathbf{x}}$ for the linearly polarized case, and treat $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ components in $\hat{\mathbf{s}}$ separately but similarly for the elliptically polarized case.

Assuming, for simplicity, that the charge passes through the origin at $t = 0$ and noting that $\epsilon = \epsilon_0$ and $\mu = \mu_0$ beyond V_s, as required by the condition (II) that no radiation, including Cherenkov and transition radiation, occurs beyond V_s, the Coulomb fields of a moving charge in vacuum are given explicitly by [6]

$$\mathbf{E}_c = \frac{q\gamma[x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + (z - vt)\hat{\mathbf{z}}]}{4\pi\epsilon_0[x^2 + y^2 + \gamma^2(z - vt)^2]^{3/2}}, \quad \mathbf{H}_c = \epsilon_0\mathbf{v}\times\mathbf{E}_c$$

where γ is the Lorentz factor.

It is noted that the surface integral Eq.(12) becomes singular at the moment the point charge passes through the origin on the surface. To remove this singularity, we separate the integral into two parts, $S = S_1 + S_2$, and assume that the area of S_1 is so small that over which H_a^s is a constant. As a result, the principal value of the integral over S_1 exists and vanishes in the following sense

$$\int_{S_1} dSH_a^s\mathbf{E}_c \cdot \hat{\mathbf{x}} = H_a^s \int_{-a}^{a} dx \int_{-b}^{b} dy\mathbf{E}_c \cdot \hat{\mathbf{x}} = 0$$

Having removed the singularity over S_1, it is easy to see

$$\int_{S_2} dSH_a^s\mathbf{E}_c \cdot \hat{\mathbf{x}} \sim \frac{1}{R} \to 0$$

as required by the conditions (III) and (IV), where R is the distance measured from within V_f. Similarly, it can be shown that the first term in Eq.(11) vanishes as well.

Step 4: Separating Incoming and Outgoing Waves.– The first term in Eq.(6) can be further simplified noting that the accelerating fields constrained by the condition (III) must have a focal point within V_f. Hence on a remote surface we may separate the fields by $\{\mathbf{E}_a, \mathbf{H}_a\} = \{\mathbf{E}_i, \mathbf{H}_i\} + \{\mathbf{E}_o, \mathbf{H}_o\}$ with the incoming and outgoing local plane waves satisfying $\mathbf{E}_i = Z_0(\hat{\mathbf{r}} \times \mathbf{H}_i)$, $\mathbf{H}_i = -(\hat{\mathbf{r}}\times\mathbf{E}_i)/Z_0$ and $\mathbf{E}_o = -Z_0(\hat{\mathbf{r}}\times\mathbf{H}_o)$, $\mathbf{H}_o = (\hat{\mathbf{r}}\times\mathbf{E}_o)/Z_0$, respectively, where $\hat{\mathbf{r}}$ is a unit vector pointing outward from a reference point within V_f, and $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the vacuum impedance. Noting that the radiation fields by definition are outgoing waves and using identity $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$, we have

$$\int_S dS\hat{\mathbf{n}} \cdot (\mathbf{E}_i \times \mathbf{H}_r + \mathbf{E}_r \times \mathbf{H}_i) = 0$$

and finally, the surface integral is reduced to

$$\int_S dS\hat{\mathbf{n}}\cdot(\mathbf{E}_o\times\mathbf{H}_r+\mathbf{E}_r\times\mathbf{H}_o) = \frac{2}{Z_0}\int_S dS(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}})(\mathbf{E}_o\cdot\mathbf{E}_r)$$

Reciprocity Theorem on Particle Acceleration.– Collecting all steps proven above, we obtain

$$\begin{aligned} \Delta W_p &\equiv q\int_{-\infty}^{\infty} dt\mathbf{E}_a[\mathbf{r}_p(t), t] \cdot \mathbf{v}_p(t) \\ &= -\frac{2}{Z_0}\int_{-\infty}^{\infty} dt \int_S dS(\hat{\mathbf{n}} \cdot \hat{\mathbf{r}})(\mathbf{E}_o \cdot \mathbf{E}_r) \end{aligned} \tag{13}$$

The theorem states that the accumulated energy gain of a charged particle in the presence of an accelerating field along an unperturbed trajectory is equal to the overlapping integral in space and in time of the outgoing accelerating field with the field radiated by the particle in the far zone on a surface enclosing the interaction region. Q.E.D.

Theorem Expressed in Frequency Domain.– Although the reciprocity theorem is conditioned and proved in time domain, once established, however, it can be evaluated in frequency domain if it is convenient. Substituting into Eq.(13) the Fourier transforms of the fields, we have

$$\Delta W_p = \int_0^{\infty} d\omega\Delta W_{\omega p} \tag{14}$$

where for each frequency component

$$\begin{aligned} \Delta W_{\omega p} &\equiv q\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} dt\ Re\{\mathbf{E}_{\omega a}[\mathbf{r}_p(t)]e^{-i\omega t}\} \cdot \mathbf{v}_p(t) \\ &= -\frac{4}{Z_0}Re\int_S dS(\hat{\mathbf{n}} \cdot \hat{\mathbf{r}})(\mathbf{E}_{\omega o} \cdot \mathbf{E}^*_{\omega r}) \end{aligned}$$

Alternatively, Eq.(14) can also be derived following the similar steps of proof, but starting from the conjugated pairs of Maxwell's equations in frequency domain

$$\nabla \times \mathbf{E}_{\omega a} = i\omega\mathbf{B}_{\omega a}, \quad \nabla \times \mathbf{H}_{\omega a} = -i\omega\mathbf{D}_{\omega a} + \mathbf{J}_{\omega a}$$
$$\nabla \times \mathbf{E}^*_{\omega p} = -i\omega\mathbf{B}^*_{\omega p}, \quad \nabla \times \mathbf{H}^*_{\omega p} = i\omega\mathbf{D}^*_{\omega p} + \mathbf{J}^*_{\omega p}$$

Nevertheless, it is in time domain that we observe and enjoy nature in its unobstructed clarity and simplicity.

REFERENCES

[1] AIP Proc. for AAC Workshops in the past two decades.

[2] R. Palmer, AIP Conf. Proc. for AAC, **335** (1995) 90.

[3] M. Zolotorev, S. Chattopadhyay, K. McDonald, *A Maxwellian Perspective on Particle Acceleration*, unfinished note, (1999), (http://www.hep.princeton.edu/~mcdonald/accel/).

[4] M. Xie, PAC01 Conf. Proc. (2001) 3876.

[5] C. Baum and H. Kritikos, *Electrodynamic Symmetry*, Chapter 1 (Taylor & Francis, 1995).

[6] P. Lorrain and D. Corson, *Electromagnetic Fields and Waves*, 2nd edition, Chapter 6 (Freeman, 1970).