

# ACCELERATION AND SELF-FOCUSED PARTICLE BEAM DRIVERS \*

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## Abstract

Here it is shown that the Vlasov equation is an adequate model in case of high-intensity charged-particle beams.

Several instances are analyzed when it is possible to construct an integral basis of the operator, associated with the dynamic system under study. This is the case, in particular, for the two-dimensional dynamic systems, just such systems describing the longitudinal motion of a perturbed system. For systems of more general structure we advance a method of reduction of the quasilinear Vlasov equation to an integral Fredholm equation. The main cases are examined when it is possible to construct kernels of corresponding integral operators. In particular, a feasibility to employ Feier - Chesaro kernels is demonstrated. Using the universality (according to V.I.Zubov) of Maxwell equations the problem of a search for stabilizing and focusing fields is reduced to the construction of Toeplitz matrix.

Also conditions are analyzed, ensuring initiating of a continuous spectrum points within the spectrum of a dynamic system. Physically, this phenomenon is related to the chaotic motion of the particles. Also, the dispersion equation, expressed in terms of solutions to the Fredholm equation, is deduced.

**Keywords:** Vlasov-Maxwell equation, kernel operator, self-focused particle beams, asymptotic stability.

## INTRODUCTION

In this paper we propose a new approach for solving the Vlasov-Maxwell equations on the basis of optimal control theory. With the help of control algorithms solution of the nonlinear wave equations can be essentially simplified.

Many analytical studies and computer experiments are devoted to analysis of these system of equations. Our earlier works [1-5] contain fresh analytic results on this subject. In this paper we outline an approach to study some problems of beam acceleration and strong focusing. This approach appears as the development of previous results of the authors [1-5] and essentially use the universality of Maxwell equations and some methods of optimal control theory. It is based on the fundamental result of V.Zubov on the necessary and sufficient conditions for asymptotic stability and the well known Halmos theorem concerning Hilbert-Schmidt integral operator. This approach made it

feasible to apply the direct Lyapunov method to essentially nonlinear problems. The Lyapunov function generates a certain kernel operator in the domain of asymptotic stability of the system under analysis. Thus the Lyapunov equation can be reduced to the Fredholm equation [1]. The self-focused and accelerating particle beams are studied using analytic solution of the self-consistent Vlasov equation. In so doing the Lorentz force is substituted with some vector parameter describing the controlling field. As a result we come to some problem on the optimal control solving which we find the charge and the current densities. Substituting the latter into the Maxwell equations and solving this equation we obtain the electromagnetic field  $(E, H)$ .

## PROBLEM STATEMENT. SIMULATION METHOD

In this paper we treat standing wave solutions  $f(t, x, v) = f_0(x, v)e^{i\omega t}$  to the self-consistent Vlasov equation

$$\frac{\partial f}{\partial t} + \partial_x f \cdot v + \partial_v f \cdot \left[ E + \frac{1}{c} v \times H \right] = 0, \quad (1)$$

under the initial condition  $f(0, x, v) = \phi(x, v)$ . Evidently,

$$\partial_x f_0 \cdot v + \partial_v f_0 \cdot \left( E + \frac{1}{c} v \times H \right) = i\omega f_0. \quad (2)$$

The fields  $E, H$  are functions of  $(t, x)$ ,  $x = (x_1, x_2, x_3) \in R^3$ , and satisfy the Maxwell equations:

$$\text{rot}E + \frac{1}{c} \frac{\partial H}{\partial t} = 0, \quad \text{div}E = 4\pi q e^{i\omega t} \int f_0(x, v) dv, \quad (3)$$

$$\text{rot}H - \frac{1}{c} \frac{\partial E}{\partial t} = \frac{4\pi}{c} q e^{i\omega t} \int v f_0(x, v) dv, \quad \text{div}H = 0. \quad (4)$$

In view of the universality of the Maxwell equations we introduce the control parameter  $U = E + \frac{1}{c} v \times H$ . Then, using (2) we deduce the following equation

$$\partial_x f_0 \cdot v + \partial_v f_0 \cdot U = i\omega f_0. \quad (5)$$

### Problem Statement

Let  $\eta = \eta(U)$  be a performance criterion. For example, such criterion may be selected in the case of self-focusing of the longitudinal motion.

Thus, we can find  $(f_0, \omega)$  from the equation(5) . In order to construct a solution to (3), (4) we will treat electric and magnetic fields in the wave form

$$E(t, x) = E_0(x) e^{i\omega t}, \quad H(t, x) = H_0(x) e^{i\omega t}.$$

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Then

$$\begin{aligned} \operatorname{div} E_0 &= 4\pi q \int f_0(x, v) dv, \\ \operatorname{rot} H_0 - \frac{i\omega}{c} &= -\frac{4\pi q}{c} \int v f_0(x, v) dv. \end{aligned} \quad (6)$$

This subject has attracted considerable interest in recent years. Only few cases are known when it is possible to obtain analytic solution in explicit form. Recall the pioneer paper of R.Davidson where the wave propagation is analyzed in symmetric structure of self-consistent equations (6). We treat this problem as a problem of control, to be specific, we seek the solution  $U^0$  from the condition for minimum:  $\eta(U^0) = \min \eta(U)$ , where  $\eta$  is a norm in some functional space.

### Simple example

Let  $x \in R$  and  $\omega = 0$ . Then  $\frac{\partial f}{\partial t} = 0$  and there exists a general solution to the equation

$$\partial_x f_0 \cdot v + \partial_v f_0 \cdot U(x) = 0,$$

which is an arbitrary function of the integral basis  $\Psi(\int U(x)dx - \frac{1}{2}v^2)$ . Now we obtain a unique solution that satisfies the initial condition  $f_0(x, v) = \phi(x, v)$ . Unfortunately, an integral basis can be constructed in rare instances. The approach outlined below makes it possible to overcome this difficulty.

In case of the long-time asymptotics our method is based on the reduction of equation (1) to the Fredholm integral equation (see [2]):

$$\begin{aligned} i\omega f_0 &= \int \int L w(x, v; y, u) f_0(y, u) dy du, \\ x, y &\in \Omega_x \subset R^3, v, u \in \Omega_v \subset R^3, \end{aligned}$$

where  $L$  is a differential operator

$$L = v \frac{\partial}{\partial x} + (E + \frac{1}{c} v \times H) \frac{\partial}{\partial v}$$

and  $w(x, v; y, u)$  is a symmetrical kernel such that

$$\int [w(x, v; x, v)]^2 dx dv < \infty.$$

Here  $\Omega_x$  and  $\Omega_v$  are domains in  $R^3(x)$  and  $R^3(v)$  respectively, on which boundaries  $f(t, x, v) = 0$  vanishes ( $x \in \partial\Omega_x$  or  $v \in \partial\Omega_v$ ), i. e. the beam is restricted in a phase space.

## SELF-FOCUSING

In this paper we investigate the long-time asymptotics of a longitudinal electron motion in a ring accelerator.

Let  $f_0$  be a solution to the equation

$$L f_0 = 0 \quad (7)$$

for  $f_0(x, v) = \eta_0(x, v)|_{t=0}$ .

If  $f_0 \notin L^2$  then zero belongs to the continuous spectrum.

Assume that  $f$  is a solution of equation (7) under the arbitrary initial value  $\eta(x, v)$ . Obviously, function  $\psi = f - f_0$  is a solution to the equation (7) too.

**Definition.** A solution  $\psi = 0$  of the system  $L\psi = 0$  is called asymptotically stable, if for any  $\varepsilon > 0$  and arbitrary fixed  $t_0 > 0$  there exist a number  $\delta > 0$  such that for any  $t \geq t_0$  and  $\psi_0$  that  $\rho(\psi_0, 0) < \delta$  the inequality holds:  $\rho(\psi(\psi_0, t), 0) < \varepsilon$ , and, moreover,  $\rho(\psi(\psi_0, t), 0) \rightarrow 0$ , as  $t \rightarrow \infty$ .

The case 4D phase space that include, in particular, the most important problem of stationary two-dimensional self-focusing is called critical. It is well known that this case is especially difficult both for analytical and numerical investigation (see, for example, [6]).

Let the motion of a beam be described (in polar coordinates  $(r, \theta)$ ) by the following partial differential equation

$$\frac{\partial f}{\partial t} + v_r \frac{\partial f}{\partial r} + \frac{v_\theta}{r} \frac{\partial f}{\partial \theta} + \left[ \frac{v_\theta^2}{r} - \frac{e}{m} \frac{\partial U}{\partial r} \right] \frac{\partial f}{\partial v_r} - \frac{v_\theta v_r}{r} \frac{\partial f}{\partial \theta} = 0$$

under the condition  $f(r_0, v_{\theta 0}) = \eta(r_0, v_{\theta 0})$ . Here  $\eta$  is a given initial distribution function,  $r_0$  and  $v_{\theta 0}$  – initial values.

We can construct the integral basis for this equation and the function  $f$  can be expressed in terms of this basis as follows:

$$f \left( \frac{r}{r_0} - \frac{v_\theta}{v_{\theta 0}}; 3v^2 - c \frac{r^3}{r_0^2} + 6 \frac{e}{m} U(r); \frac{mv^2}{2} + CU(r) \right).$$

Let us take into consideration a quadratic form

$$[\psi, \psi] = (L\psi, \psi) + (\psi, L\psi).$$

Assume that the following condition holds

$$[\psi, \psi] \leq 0$$

for  $\psi \in D(L)$  and some set of initial values  $\{\eta(x, v)\}$ .

Denote by  $\bar{L}$  the closure of the operator  $L$  in Hilbert space  $L^2$ . We see that the operator  $-\bar{L}$  is positively defined:

$$-[\psi, \psi] = \operatorname{Re}(-\bar{L}\psi, \psi) \geq 0, \quad \psi \in D(\bar{L}).$$

From this it follows that the domain  $D(\bar{L})$  is a dense subset in the space  $L^2$  and the quadratic form  $[\cdot, \cdot]$  is symmetric. In view of the known F.Riesz theorem the operator  $\bar{L}$  is a co-generator of some contraction semigroup  $R_t$  in the Hilbert space:

$$\|R_t \psi\| \leq \alpha \|\psi\|, \quad 0 < \alpha \leq 1,$$

where  $\|\cdot\|$  is a norm in  $L^2$ .

It is easy to show that

$$\|R_t \psi\| \leq \varepsilon(\delta), \quad \text{for } \psi \in L^2: \|\psi\| < \delta,$$

and, moreover,

$$\lim_{t \rightarrow \infty} |R_t \psi| \rightarrow 0.$$

From this we conclude that the search of focusing fields can be reduced to problem on asymptotic stability of solutions  $\psi$ . Whence follows that if

$$\operatorname{Re} \sigma(\bar{L}) < \alpha < 0$$

then the original problem can be solved.

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