ELECTROMAGNETIC FIELDS IN THE TOROIDAL REGION OF LHC-LIKE RINGS

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Abstract

We develop an analytic solution for the electromagnetic field in the toroidal region region between the liner and the vacuum chamber in LHC-like rings.

INTRODUCTION

In a closed ring machine, the region between the external wall of the liner and the internal wall of the surronding vacuum chamber is a toroidal resonator. In the steady regime, the particle bunches circulating in the liner set up stationary field in the toroidal region through the pumping holes. In this paper we outline the procedure for deriving a full analytic solution for the field in the toroidal region, for a multibunch beam. We consider the simplest case of a ring of (axial) length L with circular cross section liner and vacuum chamber. The (external) radius of the liner and the (internal) radius of the vacuum chamber will be denoted as a and b, respectively.

TEM FIELDS IN A TOROIDAL RESONATOR

Under usual conditions, the spectral content of the circulating bunch current is well below the lowest higher order TE and TM cutoff frequency of the coaxial region between the linear and the vacuum chamber. The time-harmonic TEM fields in the toroidal region can be accordingly written (phasor notation, $\exp(j\omega t)$ time factor dropped):

$$\begin{cases} \vec{E} = \dfrac{1}{log(b/a)} \displaystyle\sum_{m=1}^{\infty} \left[V_m \vec{\mathcal{E}}_m^{(e)} + \mathcal{V}_m \vec{\mathcal{E}}_m^{(h)} \right], \\ \vec{H} = \dfrac{1}{log(b/a)} \displaystyle\sum_{m=1}^{\infty} \left[I_m \vec{\mathcal{H}}_m^{(e)} - \mathcal{I}_m \vec{\mathcal{H}}_m^{(h)} \right], \end{cases} \tag{1}$$

where $k_0 = 2\pi/L$, and the (complex) constants V_m, $\mathcal{V}_m$, I_m and $\mathcal{I}_m$ having the dimensions of voltages and currents, respectively, are to be determined. The basis "fields" in (1)

$$\begin{cases} \vec{\mathcal{E}}_n^{(h)} = \dfrac{\vec{u}_r}{r}\, cos(nk_0 z), \\ \vec{\mathcal{H}}_n^{(h)} = -\dfrac{\vec{u}_\phi}{r}\, sin(nk_0 z), \end{cases} \tag{2}$$

$$\begin{cases} \vec{\mathcal{E}}_n^{(e)} = \dfrac{\vec{u}_r}{r}\, sin(nk_0 z), \\ \vec{\mathcal{H}}_n^{(e)} = \dfrac{\vec{u}_\phi}{r}\, cos(nk_0 z), \end{cases} \tag{3}$$

are solutions of

$$\begin{cases} \nabla \times \vec{\mathcal{E}}_n^{(e,h)} = nk_0 \vec{\mathcal{H}}_n^{(e,h)}, \\ \nabla \times \vec{\mathcal{H}}_n^{(e,h)} = nk_0 \vec{\mathcal{E}}_n^{(e,h)}, \end{cases} \tag{4}$$

where $k_0 = \omega_0/c$, describe free-field oscillations with angular frequency $\omega = n\omega_0$, $\omega_0 = 2\pi/T$, $T = L/c$ being the light round-trip time. They satisfy the following conditions

$$\begin{cases} \vec{\mathcal{E}}_n^{(e)} = 0, \\ \vec{\mathcal{H}}_n^{(h)} = 0 \end{cases} \quad at\ z = 0, L, \tag{5}$$

and earn the following orthogonality properties:

$$\begin{cases} \displaystyle\int_V \vec{\mathcal{E}}_n^{(i)} \cdot \vec{\mathcal{E}}_m^{(i)}\, dV = \int_V \vec{\mathcal{H}}_n^{(i)} \cdot \vec{\mathcal{H}}_m^{(i)}\, dV = \\ \displaystyle = \iint_{\partial V} \vec{\mathcal{H}}_n^{(i)} \cdot \vec{\mathcal{H}}_m^{(i)}\, dS = 0,\ m \neq n,\ i = e, h \\ \displaystyle\int_V \vec{\mathcal{E}}_n^{(e)} \cdot \vec{\mathcal{E}}_m^{(h)}\, dV = \int_V \vec{\mathcal{H}}_n^{(e)} \cdot \vec{\mathcal{H}}_m^{(h)}\, dV = \\ \displaystyle = \iint_{\partial V} \vec{\mathcal{H}}_n^{(e)} \cdot \vec{\mathcal{H}}_m^{(h)}\, dS = 0, \quad \forall\, m, n. \end{cases} \tag{6}$$

Moreover, $\forall n$

$$\begin{cases} \displaystyle\int_V \vec{\mathcal{E}}_n^{(i)} \cdot \vec{\mathcal{E}}_n^{(i)}\, dV = \int_V \vec{\mathcal{H}}_n^{(i)} \cdot \vec{\mathcal{H}}_n^{(i)}\, dV = \pi L\, log(b/a), \\ \displaystyle\iint_{\partial V} \vec{\mathcal{H}}_n^{(i)} \cdot \vec{\mathcal{H}}_m^{(i)}\, dS = \pi L\, \frac{a+b}{ab}, \end{cases} \tag{7}$$

where V and ∂V are the coax ring volume and its (complete) boundary, consisting of the outer surface of the liner $(r = a)$ and inner surface of the vacuum chamber $(r = b)$.

The unknown constants $V_m,\ \mathcal{V}_m,\ I_m,\ \mathcal{I}_m$ can be determined in terms of the source terms, represented by Bethe equivalent electric and magnetic dipoles sitting at the holes connecting the beam pipe to the vacuum chamber, discussed in the next section. The fields (1) obey Maxwell equations:

$$\begin{cases} \nabla \times \vec{E} = -j\omega\mu_0 \vec{H} - j\omega\mu_0 \vec{M}, \\ \nabla \times \vec{H} = j\omega\epsilon_0 \vec{E} + j\omega \vec{P}, \end{cases} \tag{8}$$

We dot-multiply the first equation in (8) by $\vec{\mathcal{H}}_n^{(i)}$, $(i = e, h)$, use the obvious vector identity, $\nabla \cdot (\vec{a} \times \vec{b}) = \nabla \times \vec{a} \cdot \vec{b} - \vec{a} \cdot \nabla \times \vec{b}$, together with equations (4), and the Leontóvich boundary conditions:

$$\hat{n} \times \left[\hat{n} \times \vec{E} - Z_{wall} \vec{H} \right]_{\partial V} = 0 \tag{9}$$

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$\hat{n}$ being the outward unit vector normal to ∂V, and $Z_{wall} = (\omega\mu_0/2\sigma)^{1/2}$ the appropriate wall impedance. Hence ($i = e, h$):

$$\int_V (\nabla \times \vec{E}) \cdot \vec{\mathcal{H}}_n^{(i)}\, dV =$$

$$= \int_{\partial V} Z_{wall}\, \vec{H} \cdot \vec{\mathcal{H}}_n^{(i)} dS + nk_0 \int_V \vec{\mathcal{E}}_n^{(i)} \cdot \vec{E}\, dV =$$

$$= -j\omega\mu_0 \int_V \vec{H} \cdot \vec{\mathcal{H}}_n^{(i)} dV - j\omega\mu_0 \int_V \vec{M} \cdot \vec{\mathcal{H}}_n^{(i)} dV, \qquad (10)$$

whence, using eq.s (1), and (6), (7),

$$\begin{cases} (\kappa - j\alpha) I_n - jnk_0 Y_0 V_n = -\kappa i_n^{(e)}, \\ (\kappa - j\alpha)\mathcal{I}_n + jnk_0 Y_0 \mathcal{V}_n = \kappa i_n^{(h)}, \end{cases} \qquad (11)$$

where:

$$i_n^{(e,h)} = \frac{1}{\pi L} \int_V \vec{M} \cdot \vec{\mathcal{H}}_n^{(e,h)} dV, \qquad (12)$$

and:

$$\kappa = \omega/c, \quad \alpha = Z_{wall} Y_0\, \frac{a+b}{ab\, log(b/a)} \qquad (13)$$

are the TEM propagation and attenuation constant in the toroidal region.

Similarly, we dot-multiply the second equation in (8) by $\vec{\mathcal{E}}_n^{(i)}$, and proceed as before to get ($i = e, h$):

$$\int_V (\nabla \times \vec{H}) \cdot \vec{\mathcal{E}}_n^{(i)}\, dV = nk_0 \int_V \vec{\mathcal{H}}_n^{(i)} \cdot \vec{H}\, dV =$$

$$= j\omega\epsilon_0 \int_V \vec{E} \cdot \vec{\mathcal{E}}_n^{(i)}\, dV + j\omega \int_V \vec{P} \cdot \vec{\mathcal{E}}_n^{(i)}\, dV, \qquad (14)$$

whence,

$$\begin{cases} -jnk_0 Z_0 I_n = \kappa V_n + \kappa v_n^{(e)}, \\ jnk_0 Z_0 \mathcal{I}_n = \kappa \mathcal{V}_n + \kappa v_n^{(h)}, \end{cases} \qquad (15)$$

where:

$$v_n^{(e,h)} = \frac{Z_0 c}{\pi L} \int_V \vec{P} \cdot \vec{\mathcal{E}}_n^{(e,h)} dV. \qquad (16)$$

From (11) and (15) we finally get:

$$\begin{cases} I_n = -\dfrac{\kappa^2 i_n^{(e)} + jnk_0 Y_0 \kappa v_n^{(e)}}{\kappa^2 - j\alpha\kappa - (nk_0)^2}, \\[2ex] V_n = \dfrac{jnk_0 \kappa Z_0 i_n^{(e)} - (\kappa^2 - j\alpha\kappa) v_n^{(e)}}{\kappa^2 - j\alpha\kappa - (nk_0)^2}, \\[2ex] \mathcal{I}_n = \dfrac{\kappa^2 i_n^{(h)} + jnk_0 Y_0 \kappa v_n^{(h)}}{\kappa^2 - j\alpha\kappa - (nk_0)^2}, \\[2ex] \mathcal{V}_n = \dfrac{jnk_0 \kappa Z_0 i_n^{(h)} - (\kappa^2 - j\alpha\kappa) v_n^{(h)}}{\kappa^2 - j\alpha\kappa - (nk_0)^2}. \end{cases} \qquad (17)$$

THE HOLE COUPLING

In the frame of Bethe's approximation [1], the electromagnetic coupling between the beam field in the liner and the toroidal region through the pumping holes (assumed identical) can be described by the (spectral) source terms

$$\begin{aligned} \vec{P} &= \epsilon_0 \alpha_e \textstyle\sum_p \delta(\vec{r} - \vec{r}_p) \hat{u}_r \hat{u}_r \cdot \vec{E}_i \\ \vec{M} &= \alpha_m \textstyle\sum_p \delta(\vec{r} - \vec{r}_p)(\bar{I} - \hat{u}_r \bar{\hat{u}}_r) \cdot \vec{H}_i, \end{aligned} \qquad (18)$$

where α_e and α_m are the hole electric and magnetic polarizabilities [1], $\vec{E}_i$, $\vec{H}_i$ is the (spectral mate of) the beam field in the liner, discussed in the next section, and $\{\vec{r}_p\}$ are the holes positions. These are the source terms in (8), (12) and (16). Note that in view of the azimuthal invariance of the basis fields in (12) and (16), the azimuthal coordinates of the holes are irrelevant, and can be set to zero. Hence

$$\vec{r}_p = a\hat{u}_r + z_p \hat{u}_z \qquad (19)$$

where, for regularly spaced holes $z_p = p\Lambda$, Λ being the hole spacing.

BUNCHED BEAM FIELD IN LINER

The charge density of a N_b equispaced point-like bunches of equal charge Q circulating on axis along a ring-liner can be written:

$$\rho(\vec{r}, t) = \delta(r)\delta(z - \beta ct\ mod\ (L/N_b)). \qquad (20)$$

The corresponding field can be written:

$$\vec{e}_i(\vec{r}, t) = \frac{N_b Q}{\pi\epsilon_0 L} \frac{\vec{u}_r}{r} \left\{ \frac{1}{2} + \sum_{m=1}^{\infty} Re\left[e^{jmN_b(\omega_b t - k_b z)} \right] \right\}, \qquad (21)$$

where

$$T_b = \frac{L}{\beta c}, \quad \omega_b = \frac{2\pi}{T_b}, \quad k_b = \frac{\omega_b}{\beta c}, \qquad (22)$$

are the bunch circulation time, (angular) frequency and wavenumber, and we used the Fourier representation of the periodic δ-function.

Real world bunches can be better described by a gaussian charge distribution, viz.:

$$\rho(\vec{r}, t) = \delta(r)\delta(z - \beta ct\ mod\ (L/N_b)) * f(z),$$

$$f(z) = (2\pi)^{-1/2} \sigma^{-1}\, e^{-z^2/2\sigma^2}, \qquad (23)$$

where $*$ denotes convolution, σ is the r.m.s. bunch-length, and we assume that $L/N_b \gg \sigma$. Hence[1]

$$\vec{e}_i(\vec{r}, t) = \frac{N_b Q}{\pi\epsilon_0 L} \frac{\vec{u}_r}{r} \cdot$$

$$\cdot \left\{ \frac{1}{2} + \sum_{m=1}^{\infty} Re\left[F(mN_b k_b) e^{jmN_b(\omega_b t - k_b z)} \right] \right\}. \qquad (24)$$

[1] Equation follows taking the $z \to k$ Fourier transform of eq. (23), using Borel theorem, and then switching back to the $z-$domain

where:

$$F(k) = \pi^{1/2} e^{-\sigma^2 k^2/2} \tag{25}$$

is the Fourier transform of the gaussian distribution.

This field is a superposition of time-harmonic forward (counterclockwise) propagating waves, at $\omega = mN_b\omega_b$, with complex (phasor) representation:

$$\vec{E}_{i,m} = \frac{F(mN_bk_b)}{1+\delta_{m0}} \frac{N_bQ}{\pi\epsilon_0 L} \frac{\vec{u}_r}{r} e^{-jmN_bk_bz}, \tag{26}$$

δ_{hk} being the Kronecker function.

CONCLUSIONS

We outlined a general framework for computing the fields in the toroidal region between the liner and the vacuum chamber in ring machines. The main relevant quantities of interest (peak field amplitudes, parasitic losses) can be accordingly readily computed. Numerical results pertinent to LHC will be reported elsewhere. It can be anticipated that these are possibly more accurate than those obtained either for the case of an infinite straight structure [2], or from an impedance boundary condition at the liner's wall which takes consistently into account the presence of the hole-coupled co-axial vacuum chamber [3]. This work has been sponsored in part by INFN.

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